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Isometrically Isomorphism between Two Banach Spaces Measured By Hausdorff Distance

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Abstract: *In this paper, we study that two Banach spaces are isometrically iso- morphic if Hausdorff distance between them measures zero.*
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I. Introduction

Given a metric space (a set and a scheme for assigning distances between elements of the set), an isometry is a transformation which maps elements to the same or another metric space such that the distance between the image elements in the new metric space is equal to the distance between the elements in the original metric space. In a two-dimensional or three-dimensional Euclidian space, two geometric figures are congruent if they are related by an isometry, the isometry that relates them is either a rigid motion (translation or rotation), or composition of a rigid motion and a reflection.

In this paper, we study new notions of distance called Metric space. A metric space is a set X with function of two variables which measures the distance between two points. Hausdorff distance, named after Felix Hausdorff, gives the largest length out of the set of all distances between each point of a set to the closest point of a second set. Given any metric space, we find that the Hausdorff distance defines a metric on the space of all nonempty, compact subsets of the metric space.

Early days, in 1953 Beckman worked on isometry of Euclidean spaces. In 1993, Huttenlocher came to knew an application of Hausdorff distance as image comparison. The biologist Yau has studied DNA sequence representation without degeneracy in 2003. Further, in 2008 Yau and his group studied a protein map and its applications. Recently, Kun Tian and group worked on two dimensional Yau-Hausdorff distance with applications on comparison of DNA and protein sequences in 2015. Based on above observations, in this paper we worked for comparison of two DNA's structure and if they are isometrically isomorphic then we conclude that they have same structure.

II. Preliminaries

The concepts in this section should be familiar to anyone who has taken a course in real analysis. Therefore, we expect the reader to be familiar with the following definitions when applied to the metricspace(R,d),where$d(x,y)=|x-y|$.However, withtheexclusionofsomeexamples,forthe majority of this paper we will be working in a general metric space. Thus our definitions will be given with respect to any metric space(X, d).

Deftnition 2.1. *Metric space (X, d) consists of a set X and a function $d : X \times X \to R$ that satisfies the following four properties.*

(1) $d(x, y) \geq 0$ *for all* $x, y \in X$.

(2) $d(x, y) = 0$ *if and only if* $x = y$.

(3) $d(x,y)=d(y,x)$*forall*$x,y\in X$.

(4) $d(x,y) \leq d(x,z)+d(z,y)$ for all $x,y,z \in X$.
The function *d*, which gives the distance between two points in *X*, is called a *metric*.

Definition 2.2. *Let $v \in X$ and let $r > 0$. Open ball centered at v with radius r is defined by $B_d(v,r) = \{x \in X : d(x,v) < r\}$.*

Definition 2.3. *A set $E \subseteq X$ is Bounded in (X,d) if there exist $x \in X$ and $M > 0$ such that $E \subseteq B_d(x,M)$.*

Definition 2.4. *A set $K \subseteq X$ is Totally bounded if for each $\epsilon > 0$ there is a finite subset $\{x_i : 1 \leq i \leq n\}$ of K such that $K \subseteq \bigcup_{i=1}^{n} B_d(x_i, \epsilon)$.*

For the following definitions, let $\{x_n\}$ be a sequence in a metric space (X,d).

Definition 2.5. *The sequence $\{x_n\}$ converges to $x \in X$ if for each $\epsilon > 0$ there exists a positive integer N such that $d(x_n,x) < \epsilon$, for all $n \geq N$. We say $\{x_n\}$ converges if there exists a point $x \in X$ such that $\{x_n\}$ converges to x.*

Definition 2.6. *The sequence $\{x_n\}$ is a Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer N such that $d(x_n,x_m) < \epsilon$ for all $m,n \geq N$.*

Definition 2.7. *A metric space (X,d) is complete if every Cauchy sequence in (X,d) converges to a point in X.*

Definition 2.8. *A set $K \subseteq X$ is sequentially compact in (X,d) if each sequence in K has a subsequence that converges to a point in K.*

Definition 2.9. *Norm $\|.\|$ on a linear space X is a mapping X to R satisfying*

(1) $\|x\| \geq 0$ *for all* $x \in X$.

(2) $\|x\| = 0$ *if and only if* $x = 0$.

(3) $\|\lambda x\| = |\lambda|\|x\|$ *for all* $\lambda \in R$ *and* $x \in X$.

(4) (Triangle inequality) $\|x+y\| \leq \|x\| + \|y\|$ *for all* $x,y \in X$.

A normed linear space $(X, \|.\|)$ is a linear space X equipped with a norm $\|.\|$.

Definition 2.10. *A complete normed linear space is called a Banach space.*

Corollary 2.1. [6] *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space (X,d). If $\{x_n\}$ converges to x and $\{y_n\}$ converges to y, then $\{d(x_n,y_n)\}$ converges to $d(x,y)$.*

Corollary 2.2. [6] *If $\{z_k\}$ is a sequence in a metric space (X,d) with the property that $d(z_k,z_{k+1}) < \frac{1}{2^k}$ for all k, then $\{z_k\}$ is a Cauchy sequence.*

Lemma 2.1. *Let (X,d) be a metric space and let A be a closed subset of X. If $\{a_n\}$ converges to x and $a_n \in A$ for all n, then $x \in A$.*

Corollary 2.2. [6] *If $\{z_k\}$ is a sequence in a metric space (X, d) with the property that $d(z_k, z_{k+1}) < \frac{1}{2^k}$ for all k, then $\{z_k\}$ is a Cauchy sequence.*

Lemma 2.1. *Let (X, d) be a metric space and let A be a closed subset of X. If $\{a_n\}$ converges to x and $a_n \in A$ for all n, then $x \in A$.*

Proof. Suppose $\{a_n\}$ is a sequence that converges to x and $a_n \in A$ for all n. There are two cases to consider. If there exists a positive integer n such that $a_n = x$, then it is clear $x \in A$. If there does not exist a positive integer n such that $a_n = x$, then x is a limit point of A by Theorem 8.49 in [6]. Since A is closed, $x \in A$. □

Definition 2.11. *Two normed spaces X and Y are called isomorphic if there is a linear bijection $T : X \to Y$ & $T^{-1} : Y \to X$ such that both T and T^{-1} are continuous. T is called an isomorphism (i.e., an isomorphism between normed spaces is a homeomorphism that preserves the linear structure).*

Definition 2.12. *An injection $f : X \to Y$ (i.e., one-to-one) between two normed spaces X and Y is called an norm-preserving if $(x \in X)$ $\|f(x)\| = \|x\|$. If the image of f is Y then the two spaces are called isometric and f is called an isometry.*

Definition 2.13. *Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed linear vector spaces. A surjection $T : X \to Y$ is called an Isometrically Isomorphic between X and Y if $\|Tx\|_Y = \|x\|_X, \quad \forall x \in X$.*

Definition 2.14. *The two norms $\|.\|$ and $\|.\|'$ on the same linear space is said to be equivalent if the identity mapping under normed linear space X is topological isomorphism of $(X, \|.\|)$ onto $(X, \|.\|')$.*

III. Construction of the Hausdorff Metric

We now define the Hausdorff metric on the set of all nonempty, compact subsets of a metric space. Let (X, d) be a complete metric space and let κ be the collection of all nonempty compact subsets of X. Note that κ is closed under finite union and nonempty intersection. For $x \in X$ and $A, B \in \kappa$, define

$$r(x, B) = inf\{d(x, b) : b \in B\} \qquad and \qquad \rho(A, B) = sup\{r(a, B) : a \in A\}.$$

Note that r is nonnegative and exists by the completeness axiom, since $d(a, b) \geq 0$ by the definition of a metric space. Since r exists and is nonnegative, then both $\rho(A, B)$ and $\rho(B, A)$ exist and are nonnegative. In addition, we define the *Hausdorff distance* between sets A and B in κ as

$$h(A, B) = max\{\rho(A, B), \rho(B, A)\}.$$

Before proving that h defines a metric on the set κ, let us consider a few examples to get a grasp on how these distances work. Consider the following example of closed interval sets in (R, d), where $d(x, y) = |x - y|$.

Example 1. *Let $A = [0, 10]$ and let $B = [12, 21]$.*

We find that $r(x, B)$ is going to be the infimum of the set of distances from each $a \in A$ to the closest point in B. As an example of one of these distances, consider $a = 2$. Then $r(2, B) = inf\{d(2, b) : b \in B\} = d(2, 12) = 10$. We can note that for each $a \in A$, the closest point in B that gives the smallest distance will always be $b = 12$. Therefore, we find that $\rho(A, b) = sup\{d(a, 12) : a \in A\}$. The point $a = 0$ in A maximizes this distance. Therefore $\rho(A, B) = d(0, 12) = |12 - 0| = 12$.

Now that we have gained a knowledge on how r, ρ, and h work in a few special cases, we refer some basic properties of r and ρ.

IV. Hausdorff Metric Space

Normed linear space is a Hausdorff metric space equipped with the metric $d(x, y) = \|x - y\|$. A metric in a linear space defines a norm if it satisfies translational invariant ($d(x - z, y - z) = d(x, y)$) and homogeneity ($d(\lambda x, 0) = \lambda d(x, 0)$). Given a complete metric space (X, d), we have now construction of new metric space (κ, h) from the nonempty, compact subsets of X using the Hausdorff distance. The following theorem shows Hausdorff distance defines a metric on κ.

Theorem 4.2. [8] *The set κ with the Hausdorff distance h define a metric space (κ, h).*

Proof. To prove that (κ, h) is a metric space, we need to verify the following four properties.

(1) $h(A, B) \geq 0$ for all $A, B \in \kappa$.

(2) $h(A, B) = 0$ if and only if $A = B$.

(3) $h(A, B) = h(B, A)$ for all $A, B \in \kappa$.

(4) $h(A, B) \leq h(A, C) + h(C, B)$ for all $A, B, C \in \kappa$.

To prove the first property, since $\rho(A, B)$ and $\rho(B, A)$ are nonnegative, it follows that $h(A, B) \geq 0$ for all $A, B \in \kappa$.

For the second property, suppose $A = B$. Therefore $A \subseteq B$ and $B \subseteq A$. By Property (2) of Theorem 3.1 we find that $\rho(A, B)$ and $\rho(B, A) = 0$, and thus $h(A, B) = 0$. Now suppose $h(A, B) = 0$. This implies $\rho(A, B) = \rho(B, A) = 0$. We see that $A \subseteq B$ and $B \subseteq A$ and it follows that $A = B$.

The third property can be proved from the symmetry of the definition since

$$\begin{aligned} h(A, B) &= max\{\rho(A, B), \rho(B, A)\} \\ &= max\{\rho(B, A), \rho(A, B)\} \\ &= h(B, A). \end{aligned}$$

The final property follows from the definition of ρ and h and from property (8) of Theorem 3.1. We find that

$$\rho(A, B) \leq \rho(A, C) + \rho(C, B).$$

Similarly,

$$\rho(B, A) \leq \rho(B, C) + \rho(C, A).$$

Therefore, $h(A, B) = max\{\rho(A, B), \rho(B, A)\} \leq h(A, C) + h(C, B)$. □

Therefore we know that h defines a metric on κ. Hence it defines Hausdorff metric space (κ, h). In the next section, we will look at example of what this metric space might look like, and then one may proceed to prove if the metric space (X, d) is complete, then the metric space (κ, h) which is induced by Hausdorff distance is also complete.

Example 2. *Let (R, d_0) be the complete metric space, where d_0 is the discrete metric,*

$$d_0(x,y) = \begin{cases} 0, & when \quad x = y. \\ 1, & when \quad x \neq y. \end{cases}$$

Since κ is the set of all nonempty, compact subsets of (R, d_0), we find that κ is the set of all nonempty finite subsets of R. The infinite sets are not in κ because they are not totally bounded and are thus not compact. Furthermore, we may notice that

$$r(x,B) = inf\{d_0(x,b) : b \in B\} = d_0(x,y) = \begin{cases} 0, & when \quad x \in B. \\ 1, & when \quad x \notin B. \end{cases}$$

Therefore,

$$\rho(A,B) = sup\{r(a,B) : a \in A\} = \begin{cases} 0, & when \quad a \in B. \\ 1, & when \quad a \notin B. \end{cases}$$

So it follows that

$$h(A,B) = \begin{cases} 0, & when \quad A = B. \\ 1, & when \quad A \neq B. \end{cases}$$

Therefore we have a metric space with the set κ of the discrete subsets of R with the Hausdorff metric as the discrete metric. It is easy to verify that our newly created space is not totally bounded. However, we know all discrete metric spaces are complete, so (κ, h) is complete. Therefore, the space

(κ, h) of finite sets with the discrete metric is an example of our Hausdorff induced metric space (κ, h).

To illustrate our notion of completeness, now briefly consider a sequence of nonempty compact sets that converges to the unit circle in R^2. This is an example a converging Cauchy sequence in the Hausdorff induced metric space that converges to a set also in the space.

5. Proving that the Hausdorff Metric Space (κ, h) is Complete

As previously stated, to be a complete metric space, every Cauchy sequence in (κ, h) must converge to a point in κ. Therefore, in order to prove that the metric space (κ, h) is complete, we will choose an arbitrary Cauchy sequence $\{A_n\}$ in κ and show that it converges to some $A \in \kappa$. define A to be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$ for all n. We will eventually show that the set A is an appropriate candidate. However, we must begin with some important theorems regarding A. Given a set $A \in \kappa$ and a positive number ϵ, we define the set $A + \epsilon$ by $\{x \in X : r(x, A) \leq \epsilon\}$. We need to show that this set is closed for all possible choices of A and ϵ. To do this, we will begin by choosing an arbitrary limit point of the set, $A + \epsilon$, and then showing that it is contained in the set.

Proposition 1. *$A + \epsilon$ is closed for all possible choices of $A \in \kappa$ and $\epsilon > 0$.*

However, the following theorem gives us an alternative way of proving convergence.

Theorem 5.3. [8] *Suppose that $A, B \in \kappa$ and that $\epsilon > 0$. Then $h(A,B) \leq \epsilon$ if and only if $A \subseteq B + \epsilon$ and $B \subseteq A + \epsilon$.*

Extension Lemma: Let $\{A_n\}$ be a Cauchy sequence in κ and let $\{n_k\}$ be an increasing sequence of positive integers. If $\{x_{n_k}\}$ is a Cauchy sequence in X for which $x_{n_k} \in A_{n_k}$ for all k, then there exists a Cauchy sequence $\{y_n\}$ in X such that $y_n \in A_n$ for all n and $y_{n_k} = x_{n_k}$ for all k.

The following lemma makes use of the extension lemma to guarantee that A is closed and nonempty. We will need this fact in proving that A is in κ, since we must show that A is a nonempty, compact subset of κ. This lemma gives us that A is closed and nonempty. Since closed and totally bounded sets are compact, it remains to show that A is totally bounded.

Lemma 5.2. [6] *Let $\{A_n\}$ be a sequence in κ and let A be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$ for all n. If $\{A_n\}$ is a Cauchy sequence, then the set A is closed and nonempty.*

With the previous lemma, to prove $A \in \kappa$, it only remains to show that A is totally bounded. The following lemma will allow us to do so.

Lemma 5.3. [6] *Let $\{D_n\}$ be a sequence of totally bounded sets in X and let A be any subset of X. If for each $\epsilon > 0$, there exists a positive integer N such that $A \subseteq D_N + \epsilon$, then A is totally bounded.*

It gives the foundation to prove complete metric space (X, d), we constructed the metric space (κ, h) from the nonempty compact subsets of X using the Hausdorff metric. After examining important theorems and results, we can now state that

Theorem 5.4. [8] *If (X, d) is complete, then (κ, h) is complete.*

Proof. Let $\{A_n\}$ be a Cauchy sequence in κ, and define A to be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$ for all n. We must prove that $A \in \kappa$ and $\{A_n\}$ converges to A.

By Lemma 5.2, the set A is closed and nonempty. Let $\epsilon > 0$. Since $\{A_n\}$ is Cauchy sequence then there exists a positive integer N such that $h(A_n, A_m) < \epsilon$ for all $m, n \geq N$. $A_m \subseteq A_n + \epsilon$ for all $m > n \geq N$. Let $a \in A$, then we want to show $a \in A_n + \epsilon$. Fix $n \geq N$, by definition of the set A, there exists a sequence $\{x_i\}$ such that $x_i \in A_i$ for all i and $\{x_i\}$ converges to a. By Proposition 1 we know that $A_n + \epsilon$ is closed. Since $x_i \in A_n + \epsilon$ for each i, then it follows that $a \in A_n + \epsilon$. This shows that $A \subseteq A_n + \epsilon$. By Lemma 5.3, the set A is totally bounded. Additionally, we know A is complete, since it is a closed subset of a complete metric space. Since A is nonempty, complete and totally bounded, then A is compact and thus $A \in \kappa$.

Let $\epsilon > 0$, to show that $\{A_n\}$ converges to $A \in \kappa$, we need to show that there exists a positive integer N such that $h(A_n, A) < \epsilon$ for all $n \geq N$. To do this, we know that $A \subseteq A_n + \epsilon$ and $A_n \subseteq A + \epsilon$. From the first part of our proof, we know there exists N such that $A \subseteq A_n + \epsilon$ for all $n \geq N$.

To prove $A_n \subseteq A+\epsilon$ let $\epsilon > 0$. Since $\{A_n\}$ is a Cauchy sequence, we can choose a positive integer N such that $h(A_m, A_n) < \frac{\epsilon}{2}$ for all $m, n \geq N$. Since $\{A_n\}$ is a Cauchy sequence in κ, there exists a strictly increasing sequence $\{n_i\}$ of positive integers such that $n_1 > N$ and such that $h(A_m, A_n) < \epsilon 2^{-i-1}$ for all $m, n > n_i$.
We can use property (3) of Theorem 3.1 to get the following:

$$\begin{aligned} Since \;\; A_n &\subseteq A_{n_1} + \frac{\epsilon}{2}, \;\; \exists \; x_{n_1} \in A_{n_1} \ni d(y, \;\; x_{n_1}) \leq \frac{\epsilon}{2}, \\ since \;\; A_{n_1} &\subseteq A_{n_2} + \frac{\epsilon}{4}, \;\; \exists \; x_{n_2} \in A_{n_2} \ni d(x_{n_1}, x_{n_2}) \leq \frac{\epsilon}{4}, \\ since \;\; A_{n_2} &\subseteq A_{n_3} + \frac{\epsilon}{8}, \;\; \exists \; x_{n_3} \in A_{n_3} \ni d(x_{n_2}, x_{n_3}) \leq \frac{\epsilon}{8}, \cdots , \end{aligned}$$

by continuing this process we are able to obtain a sequence $\{x_{n_i}\}$ such that for all positive integers i then $x_{n_i} \in A_{n_i}$ and $d(x_{n_i}, x_{n_i+1}) \leq \epsilon 2^{-i-1}$. By Corollary 2.2, we find x_{n_i} is a Cauchy sequence, so by the extension lemma the limit of the sequence a is in A. Additionally we find that

$$\begin{aligned} d(y, x_{n_i}) &\leq d(y, x_{n_1}) + d(x_{n_1}, x_{n_2}) + d(x_{n_2}, x_{n_3}) + \cdots + d(x_{n_{i-1}}, x_{n_i}) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots + \frac{\epsilon}{2^i} < \epsilon. \end{aligned}$$

Since $d(y, x_{n_i}) \leq \epsilon$ for all i, it follows that $d(y, a) \leq \epsilon$ and therefore $y \in A + \epsilon$. Thus we know that there exists N such that $A_n \subseteq A + \epsilon$, so it follows that $h(A_n, A) < \epsilon$ for all $n \geq N$ and thus $\{A_n\}$ converges to $A \in \kappa$. Therefore, if (X, d) is complete, then (κ, h) is complete. □

6. Isometrically Isomorphic Banach spaces

Theorem 6.5. [15] *Let X and Y be two normed linear space over field F and $T : X \to Y$ be a linear operator and T is onto, then T is topologically isomorphism if and only if there exists $k_1 > 0$ and $k_2 > 0$ such that $k_1\|x\| \leq \|T_x\| \leq k_2\|x\|$ for all $x \in X$.*

Theorem 6.6. [15] *Every Hausdorff metric space is isometric to dense subset of a complete metric space.*

Theorem 6.7. *Two Banach spaces are isometrically isomorphic if Hausdorff distance between them measures zero.*

Proof. Let X and Y be two normed linear space over field F and $T : X \to Y$ be a linear operator then T is said to be bounded if and only if T maps bounded set in X into bounded set in Y(by definition 2.16).

Further, T is topologically isomorphism if and only if $\exists$ k_1 and $k_2 > 0 \ni k_1\|x\| \leq \|Tx\| \leq k_2\|x\|$ for all $x \in X$(Theorem 6.5).

Two norms $\|.\|$ and $\|.\|'$ are equivalent if the identity mapping $I_X : (X, \|.\|) \to (X, \|.\|')$ is topologically isomorphic then by definition (2.17) it follows

$\Rightarrow k_1\|x\| \leq \|I_X x\|' \leq k_2\|x\|$

$\Rightarrow k_1\|x\| \leq \|x\|' \leq k_2\|x\| \qquad \forall x \in X.$

Hence, by the completion theorem(Theorem 6.6), Hausdorff distance measures zero. □

Conclusion

In molecular biology the Hausdorff distance has been successfully applied to protein structure alignment. A protein backbone consists of amino acids linked by peptide bonds. It can be modeled as a polygonal chain in $R3$ with the amino acids modeled as vertices and peptide bonds as edges. A natural distance measure for aligning, i.e., matching protein backbones is the Hausdorff distance. The Hausdorff distance measures the similarity of polygonal curves based on the distances between vertices and taking into account the order of the vertices given by the edges. Protein structure alignment by matching under the Hausdorff distance has been successfully applied to protein data(DNA).

In this paper, we are interested in the theoretical complexity of shape matching of DNA. According to our result (Theorem (6.7)) without loss of generality, we considered each DNA as Banach and as distance measures zero between two DNA structures then they are identical and hence we analyzed the complexity of the decision and computation problem for the Hausdorff distance between DNA's structure.

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