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Some Issues on Linear Numeration Systems

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Abstract: *Recurrence sequence of bases can be used to construct number systems representations where algorithm of addition can be easily derived from definition of sequence.*

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I. Introduction

Starting from A. Rènyi[1] numerous mathematicians and computer scientists studied non-standard number systems with an algebraic number as base[2,3,4,5,6,7]. There are numerous fine and general results on computational properties such number systems. Here we try to show that slightly violating uniformity of base and transferring to linear systems[7,8] we get easy algorithmically realizable number systems. This work was inspired by numerous works like referenced above and possibilities to set control isomorphic to overflows in such systems in parallel and distributed computing[9,10]. This work financially supported by the Russian Federation represented by Ministry of Education and Science (grant id: RFMEFI61319X0092).

II. Basic definitions and known facts

Recurrence sequences defined here are a simple generalization of standard definition of linear recurrence sequences. The notion of linear numeration systems seems to be very poor studied and the term itself seems to be inadequate: exponential systems are a simple case of `linear`. Maybe recurrent systems would be more adequate name.

Definition 1 (Recurrence sequences) Infinite sequence is a function a from integer numbers $\{i \in Z | L < i < U\}$ where $L = -\infty$ or $U = +\infty$ or both into (natural, integer, rational, algebraic, complex) numbers. Resp. they are called natural, integer and so on. Recurrence sequence such that all members excluding $a_L, \cdots, a_{L+k-1}$ if $L > -\infty$ or $a_{U-k+1}, \cdots, a_U$ if $U < +\infty$ satisfy a linear equation $a_{i+1} = c_1 \cdot a_i + \cdots + c_k \cdot a_{i-k+1}$. If both $L = -\infty$ and $U = +\infty$ then sequence is full, if only $L = -\infty$ it is lower and if only $U = +\infty$ it is upper. Segments of members $a_L, \cdots, a_{L+k-1}$ in upper sequence and $a_{U-k+1}, \cdots, a_U$ in lower one are called *initial data*.

Usually numeration systems use exponential bases and there are some isolated examples of other bases, e.g. superexponential, factorial. Here we give a general notion of numeration system with fixed finite number of digits.

Definition 2 (Additive numeration systems) Let there is an infinite sequence of non-zero numbers such that holds $|a_i < a_{i+1}|$. Its members are called *bases*. Let there is a finite collection of *digits* D containing 0.A number x is representable in system $\langle a, D\rangle$ when there is a sequence of digits d_iI such that $x = \sum_{i>l}^{i<u} d_i \cdot a_i$. If each (real, complex, rational, integer, natural) number is representable in $\langle a, D\rangle$ then system is full for that class of numbers. A system is adequate for (rational, integer, natural) numbers if each such number id representable with sequence of digits where only finite number of digits are non-zero. A numeration system is *normalized* if $a_0 = 1$. It has *standard digits* if they are consecutive integers $\{-k, -k+1, \cdots .0, 1, \cdots, l\}$.
Below we use term `effective' to represent results valid in each notion of computability satisfying the demand that finite information on the result demands finite information about arguments (Brouwer's principle).

Example 1. There is an example of ancient additive systems dull foe positive reals and adequate for positive rational numbers: aliquote (or Egyptian) one[11]. The sequence of bases is infinite downwards:

$$a_{-i} = \frac{1}{i}; i \in (-\infty, 0).$$

Digits are {0,1}.

Each rational positive number is finitely representable in such system and moreover in each lower segment of this system. This representation can be computed primitive-recursively. Each algebraic positive number is represented by primitive-recursive sequence of digits. There is a primitive recursive functional computing for every positive real numbers given by effective sequence of digit in some usual exponential system result of arithmetical operations on these numbers represented in the Egyptian system. This is somewhat striking because it is well known that arithmetical operations on real system in standard representation are non-computable w.r..t. each notion of computability[12].
All results for Egyptian representations are generalized for all numbers of corresponding type if we extend numbers to $\{-1,0,1\}$ or make each odd member of the bases sequence negative[12].

Example 2. First system with unusual digits maybe is system of D.Knuth[13]. As a system with unusual sets of digits consider the system of C. Frougny[14]. He proposed the exponential system with base $\frac{p}{q}. p > q$ and numbers $\left\{0, \frac{1}{q}, \cdots, \frac{p-1}{q}\right\}$, All natural numbers are finitely representable in this system.

Definition 3. (**Linear enumeration system)** [7] Linear enumeration system is a normalized additive system with standard digits where its bases are infinite upwards, start with 0 and form a recurrence sequence where all c_i are non-negative natural numbers and initial conditions are
$a_{-k+1} = \cdots = a_{-1} = 0;\ \ a_0 = 1.$

These systems were studied very poorly from the practical point of view. Mathematicians interested in representation of reals[7,8] but not every linear system is good for this purpose.
Example 3. Each exponential system can be viewed as linear one with generator $a_{i+1} = c_1 \cdot a_i$. Here c_1 is usually called simply its base. The famous Fibonacci system[15] is given by recurrent equality $a_{i+1} = 1 \cdot a_i + 1 \cdot a_{i-1}$ and has digits $\{0,1\}$. Its direct generalizations n-bonacci[16,17,18] systems are given by similar generators $a_{i+1} = 1 \cdot a_i + \cdots + 1 \cdot a_{i-n+1}$.
Generator $a_{i+1} = 2 \cdot a_i - 1 \cdot a_{i-1}$ forms system φ with bases 1,2,3,4 … Generator $a_{i+1} = 3 \cdot a_i + 2 \cdot a_{i-1}$ forms system β with bases 1,3,11,39,150,… Generator $a_{i+1} = 5 \cdot a_i - 6 \cdot a_{i-1}$ forms a suequence of bases $\alpha 23{:}\ 1, 5, 19, 65 \cdots$

There is a simple result[7] on sufficient set of digits for such systems.
Proposition 1. Each natural number is representable in a linear system with generating equation $a_{i+1} = c_1 \cdot a_i + \cdots + c_k \cdot a_{i-k+1},\ c_i > 0$ and set of digits $\{0, 1, \cdots, C\}$. Here C is the maximum of c_i. Each integer number is representable with set of digits $\{-l, \cdots, C-l, l < C\}$.

Proof. A trivial induction using greedy algorithm for search of a representation.

In each non-exponential recurrent numeration system the same number can have different representations.

III. Addition in linear systems

One of crucial algorithmic questions to decide whether representation could be useful is possibility to perform various operations. Here we study problem how simply add numbers in a linear system.

The main problem to compute sum is to deal with `overflows': parts of sums of two digits which induce additions in some places upper and lower the initial. In standard systems overflows go only up but in non-standard they spread in both directions.

Example 4. When we add two 1 in some place of Fibonacci systems then using equality
$2 \cdot a_n = a_{n+1} + a_{n-2}$
we have a `standard' overflow going into next position and non-standard one going downwards two positions. In n-bonacci systems downward overflow go n positions deeper.

Theorem 1. Let in our generator all $c_i \geq c_{i+1}$.and are positive, Then we can add two digits having overflows going up to 1 position an equal to 1 and down no more than k positions and to be correct digits.
Proof. Consider $(C+1) \cdot a_n$. We can deal this overflow by the following transformations using recurrent equality:

$$(C+1) \cdot a_n = C \cdot a_n + a_n = a_{n+1} + a_n - c_2 \cdot a_{n-1} - c_3 \cdot a_{n-2} - \cdots - c_k \cdot a_{n-k+1} =$$
$$a_{n+1} + (c_1 - c_2) \cdot a_{n-1} + (c_2 - c_3) \cdot a_{n-2} + \cdots + (c_{k-1} - c_k) \cdot a_{n-k+1} + c_k \cdot a_{n-k}.$$

According to monotony of c_i all delegated downward overflows are small and positive. According to terminology used in theory of parallel addition for exponential systems we have local function with memory 1 and anticipation k (i-th `digit' of the sum is computed using k upper results and 1 result below).

Of course these overflows can induce new overflows. Downwards progressing overflows are evaporated in the worst case by going into digits with negative positions where according to initial conditions $a_i = 0$.Thus we have easily automatically generated algorithm of addition. It demands $L \cdot n^2$ elementary additions of digits where n is number of digits in arguments. A constant L can be computed by bilinear form C and k. Because this trivial algorithm can be optimized we will not compute a constant for it. This algorithm can be computed as partially parallel but stages of pervasive overflows are to be processed consequently.

Now we give two examples what can happen if our conditions are violated.

Example 5. Consider the system β from Example 3 with $a_{i+1} = 3 \cdot a_i + 2 \cdot a_{i-1}$. Then we have the following rule for addition: $4 \cdot a_n = a_{n+1} + a_{n-1} + 2 \cdot a_{n-2}$. For system $a_{i+1} = 2 \cdot a_i + 3 \cdot a_{i-1}$ we get
$4 \cdot a_n = a_{n+1} + 2 \cdot a_n - 3 \cdot a_{n-1} = a_{n+1} + a_{n-1} + 6 \cdot a_{n-2}$ and overflow is out of digit diapason.
If we violate demands of positivity then there can be more severe things. For example consider
$a_{i+1} = 3 \cdot a_i - 2 \cdot a_{i-1}$. Here we get $4 \cdot a_n = a_{n+1} + 5 \cdot a_{n-1} - 2 \cdot a_{n-2}$.

Nevertheless in some cases violating do not lead to severe consequences. Consider system φ from Example 2. Here
$2 \cdot a_n = a_{n+1} +\cdot a_n + a_{n-1}$. This is because of algebraic properties of recurrence sequences[19]. At XIX century P. L. Chebyshev et al. investigated them thoroughly and proved the following. Let Ch be so called characteristic equation $x^k - c_1 \cdot x^{k-1} - c_2 \cdot x^{k-2} - \cdots - c_k = 0$. If its roots δ_i are all different then we have $a_n = \sum_{i=1}^{k} B_i \cdot \delta_i^n$. Here constants B_i are solutions of the linear system

$$\begin{cases} B_1 + \cdots + B_k = 1 \\ B_1 \cdot \delta_1^{-i} + \cdots + B_k \cdot \delta_k^{-i} = 0 \\ (i = 1 \cdots k-1) \end{cases}$$

If there are m equal to δ_j roots then the corresponding member of the sum is $P_m(i) \cdot \delta_j^n$ where $P_m(i)$ is polynomial of power m. We can see that characteristic equation for φ is simply $(x-1)^2 = 0$.

For system β we have the characteristic equation $x^2 - 3 \cdot x - 2 = 0$ and its members are

$$a_n = \frac{1}{\sqrt{11}} \cdot \left(\left(\frac{3}{2} + \frac{\sqrt{11}}{2} \right)^n - \left(\frac{3}{2} - \frac{\sqrt{11}}{2} \right)^n \right)$$

You can see that this theoretically valuable representation is almost useless for practical computations. It shows that recurrent number systems can be in some sense considered as composition of several exponential systems. Thus the main related works are on exponential systems with algebraic base.

IV. Connections with related works

There is a stream of works on number system with an algebraic base. The main problem here is how to compute addition.
The general characteristic of systems where addition can be computed using only a local segment of digits for each elementary step is the following[20].
A complex number β, $|\beta| > 1$, has the strong representation of zero property if

$$0 = b_k \cdot \beta^k + \cdots + b_1 \cdot \beta + b_0 + b_{-1} \cdot \beta^{-1} + \cdots + b_{-h} \cdot \beta^{-h}$$

where $b_0 > 2 \cdot \sum_{i \neq 0} |b_i|$. Then there is an algorithm for addition in the system with base β taking into account memory k and anticipation h.
So slightly violating demand of uniformity of bases we get an easier for practical application number representations.

V. Conclusion

Further problems include optimized algorithms of addition in linear systems and their extension onto real numbers.

References

[1]. Rènyi A. Representations for real numbers and their ergodic properties. Acta Math. Acad. Sci. Hungary. vol 8, 1957, 477–493
[2]. Avizienis A. Signed-Digit Numbe Representations for Fast Parallel Arithmetic. IRE Trans. Electronic 1961, 381–400
[3]. Knuth D. An Imaginary Number System. Communications of the ACM, 1960, vol. 3, pp. 245-247

[4]. Frougny Ch., Pavelka M., Pelantová E. and Svobodová M. On-line algorithms for multiplication and division in real and complex numeration systems, *Discrete Mathematics and Theoretical Computer Science*, vol. 21 no. 3 (2019)

[5]. Frougny Ch., Pelantova E., Svobodova M. Minimal Digit Sets for Parallel Addition in Non-Standard Numeration Systems. Journal of Integer Sequences. 2013. Vol. 16, № 13.2.17

[6]. Parhami B. On the implementation of arithmetic support functions for generalized signed-digit number systems. IEEE Trans. Computers 42 (1993), 379–384.

[7]. Shallit J. Numeration systems, linear recurrences and regular sets. Inform. Comut. 18 (1994), 331-347

[8]. Frougny Ch., Solomyak B. On representation of integers in linear numeration systems. Ergodic Theory of Z^d actions, *London Math. Soc. Lecture Note Ser.* 228 (1996) 345–368

[9]. Nikolai Nepejvoda. Local computations in system theory and VLDB. Program systems: Theory and applications, 2016, 7:4(31), pp. 145–160. (In Russian). URL: http://psta.psiras.ru/read/psta2016_4_145-160.pdf

[10]. Nikolai Nepejvoda. Using overflows to control parallel and distributed computations. Program systems: Theory and applications, 2017, 8:3(34), pp. 87–107. (In Russian). URL: http://psta.psiras.ru/read/psta2017_3_87-107.pdf

[11]. Creyaufmüller W. Aliquot Sequences, 2016, URL: http://www.aliquot.de/aliquote.htm

[12]. Nikolai Nepejvoda. Additive representations of numbers: some remarks., 2017, 8:4(35), pp. 101–115. (In Russian). URL: http://psta.psiras.ru/read/psta2017_4_101-115.pdf

[13]. Shvorin A. B. Parallel Addition of Real Numbers in Overlaying Numeration Systems. Program systems: Theory and applications, 6:2 (2015), pp. 101–117 (in Russian), URL: http://psta.psiras.ru/read/psta2015_2_101-117.pdf

[14]. Akiyama S., Frougny Ch. and Sakarovitch J. Powers of rationals modulo 1 and rational base number systems, Israël J. Math., 168 (2008) 53–91

[15]. L. Pisano. Fibonacci's Liber Abaci: A Translation into Modern English of the Book of Calculation, Springer, 2002.

[16]. D. Zh. Hui. The formula of t-step Fibonacci sequence, 2008 (in Chinese, in English), URL: http://bbs.emath.ac.cn/forum.php, 667.4.

[17]. M. Barr. Mathematical Amusements. The Sketch, 1913, pp. 32.

[18]. M. Barr. Parameters of beauty. Architecture (NY), 60 (1929), pp. 325.

[19]. J. von zur Gathen and J. Gerhard. Modern Computer Algebra. Cambridge University Press, Cambridge, third edition, 2013.

[20]. Akiyama S., Drungilas P., Jankauskas J. Height reducing problem on algebraic integers. Functiones et Approximatio Commentarii Mathematici 47 (1), 105-119, 2012

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