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Continuous Implicit Linear Multistep Methods for the Solution of Initial Value Problems of First-Order Ordinary Differential Equations

B.V. Iyorter[1], T. Luga[2] andS.S. Isah[3]
[1]*Department of Mathematics and Computer Science, College of Science and Education, University of Mkar, Mkar, Nigeria*
[2, 3]*Department of Mathematics/Statistics/Computer Science,College of Science, Federal University of Agriculture, Makurdi, Nigeria.*
Corresponding Author:B.V. Iyorter

Abstract**:** *Continuous implicit linear multistep methods are developed for the solution of initial value problems of first-order ordinary differential equations. The linear multistep methods are derived via interpolation and collocation approach using Laguerre polynomials as basis functions. The discrete form of the continuous methods are evaluated and applied to solve first-order ordinary differential equations. The proposed optimal order methods producedbetter approximations than the Adams-Moulton methods.*

Keywords:*Linear Multistep Methods, Laguerre Polynomial, Collocation, Interpolation, Optimal Order Scheme.*

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I. Introduction

Many phenomena in real life are described in terms of differential equations. Most of these equations are difficult to solve using analytic techniques, hence, numerical approximations are obtained. Some of the numerical methods for solving ordinary differential equations are the Linear Multistep Methods (LMMs). They are not self-starting, hence requires single-step methods like the Runge-Kutta family of methods for starting values.

Linear multistep methods are generally defined as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1}$$

whereα_j and β_j are uniquely determined and $\alpha_0 + \beta_0 \neq 0,\ \alpha_k = 1$ [1].

Discrete schemes are generated from Equation (1) and applied to solve first order Ordinary Differential Equations (ODEs). Continuous collocation and interpolation technique is applied in the development of LMMs of the form

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{k} \beta_j(x) f_{n+j} \tag{2}$$

whereα_j and β_j are expressed as continuous functions of x and are differentiable at least once [2]. Continuous collocation and interpolation technique can also be applied in the derivation of block and hybrid methods. Other techniques for derivation of LMMs include interpolation, numerical integration and Taylor series expansion.

Several researchers have applied different techniques to derive LMMs: [3] derived continuous solvers of IVPs using Chebyshev polynomial in a multistep collocation technique; [4] proposed a two-step continuous multistep method of hybrid type for the direct integration of second order ODEs in a multistep collocation technique; [5] developed a new class of continuous implicit hybrid one-step methods for the solution of second order ODEs using the interpolation and collocation techniques of the power series approximate solution; [6] developed a four step continuous block hybrid method with four non-step points for the direct solution of first-order IVPs; [7] derived a three-step hybrid LMM for solving first order Initial Value Problems (IVPs) of ODEs based on collocation and interpolation technique; [8] adopted the method of collocation of the differential system and interpolation of the approximate solution at grid and off grid points to yield a continuous LMM with constant stepsize; [9] adopted the method of collocation and interpolation of power series approximate solution to generate a continuous LMM; [10] derived continuous LMMs for solving first order ODEs by interpolation and collocation technique using Hermite polynomials as basis functions; [11] proposed a family of single-step, fifth and fourth order continuous hybrid linear multistep methods for solving first order ODEs derived via interpolation and collocation procedure; [12] presented extrapolation-based implicit-explicit second derivative linear multistep methods for solving stiff ODEs.The interpolation and collocation approach was also adopted by [13] to derive a hybrid LMM with multiple hybrid predictors for solving stiff IVPs of ODEs; [14] derived continuous linear multistep methods for solving Volterraintegro-differential equations of second order by interpolation and collocation technique using the shifted Legendre polynomial as basis function with Trapezoidal quadrature and [15] derived a family of hybrid linear multi-step methods type for solution of third order ODEs. In this paper, continuous implicit LMMs for the solution of IVPs of first order ODEs are developedvia collocation and interpolation approach using the Laguerre polynomials as basis functions. The corresponding discrete schemes areevaluated.

II. Methods

Many researchers have proposed LMMs in form of different polynomial functions. [16, 17] proposed continuous LMM in form of a powers series polynomial function of the form

$$y(x) = \sum_{j=0}^{k} \alpha_j x^j . \qquad (3)$$

[18]developed continuous LMMs using Chebyshev polynomial function of the form:

$$y(x) = \sum_{j=0}^{M} \alpha_j T_j \left(\frac{x - x_k}{h}\right),$$

where$T_j(x)$ are some Chebyshev function.

[2]proposed a polynomial function similar to the one in Equation (3) but presented as

$$y(x) = \sum_{j=0}^{k} \alpha_j (x - x_k)^j . \qquad (4)$$

[10]developed continuous LMMs using the Probabilists' Hermite polynomials of the form

$$y(x) = \sum_{j=0}^{k} a_j H_j (x - x_k)$$

where$H_j(x)$ are probabilists' Hermite polynomials.

Several other polynomial functions have been used for the derivation of linear multistep methods, however, in this paper, we will develop continuous LMMs and obtain their corresponding discrete forms using the Laguerre polynomials of the form:

$$y(x) = \sum_{j=0}^{k} a_j L_j (x - x_k)$$

where$L_j(x)$ are Laguerre polynomials generated by the formula [19]:

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n) = \frac{1}{n!}\left(\frac{d}{dx} - 1\right)^n x^n$$

and satisfy the recursive relations

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$$

and

$$xL'_n(x) = nL_n(x) - nL_{n-1}(x)$$

for the solution of first-order IVPs of ODEs of the form:

$$y' = f(x, y(x)), \quad y(x_0) = y_0. \qquad (5)$$

The first seven Laguerre polynomials generated from the recursive relation are:

$L_0 = 1,\quad L_1 = -x + 1,\quad L_2 = \frac{1}{2}(x^2 - 4x + 2),\quad L_3 = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6),$

$L_4 = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24),$

$L_5 = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120),$

$L_6 = \frac{1}{720}(x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720).$

Derivation of the Linear Multistep Methods

We wish to approximate the exact solution $y(x)$ to the IVP in Equation (5) by a polynomial of degree n of the form:

$$y(x) = \sum_{j=0}^{n} a_j L_j(x - x_k), \quad x_k \le x \le x_{k+p} \tag{6}$$

which satisfies the equations

$$\left.\begin{aligned} y'(x) &= f\big(x, y(x)\big), \quad x_k \le x \le x_{k+p} \\ y(x_k) &= y_k. \end{aligned}\right\} \tag{7}$$

Derivation of Three-Step Adams-Moulton Method

To derive the three-step Adams-Moulton method, we let $n = 4$ in Equation (6) and differentiate the obtained function once. This gives:

$$\begin{aligned} y(x) &= a_0 + a_1[-(x - x_k) + 1] + a_2\left[\frac{1}{2}(x - x_k)^2 - 2(x - x_k) + 1\right] \\ &+ a_3\left[-\frac{1}{6}(x - x_k)^3 + \frac{9}{6}(x - x_k)^2 - 3(x - x_k) + 1\right] \\ &\quad + a_4\left[\frac{1}{24}(x - x_k)^4 - \frac{16}{24}(x - x_k)^3 + 3(x - x_k)^2 - 4(x - x_k) + 1\right], \end{aligned} \tag{8}$$

$$\begin{aligned} y'(x) &= -a_1 + a_2[(x - x_k) - 2] + a_3\left[-\frac{1}{2}(x - x_k)^2 + 3(x - x_k) - 3\right] \\ &\quad + a_4\left[\frac{1}{6}(x - x_k)^3 - 2(x - x_k)^2 + 6(x - x_k) - 4\right]. \end{aligned} \tag{9}$$

Interpolating Equation (8) at $x = x_{k+2}$ and collocating Equation (9) at $x = x_k, \ x_{k+1}, \ x_{k+2}, \ x_{k+3}$ give rise to a system of equations which is written in matrix form as follows:

$$\begin{pmatrix} 1 & (-2h+1) & (2h^2 - 4h + 1) & \left(-\frac{4}{3}h^3 + 6h^2 - 6h + 1\right) & \left(\frac{2}{3}h^4 - \frac{16}{3}h^3 + 12h^2 - 8h + 1\right) \\ 0 & -h & -2h & -3h & -4h \\ 0 & -h & (h^2 - 2h) & \left(-\frac{1}{2}h^3 + 3h^2 - 3h\right) & \left(\frac{1}{6}h^4 - 2h^3 + 6h^2 - 4h\right) \\ 0 & -h & (2h^2 - 2h) & (-2h^3 + 6h^2 - 3h) & \left(\frac{4}{3}h^4 - 8h^3 + 12h^2 - 4h\right) \\ 0 & -h & (3h^2 - 2h) & \left(-\frac{9}{2}h^3 + 9h^2 - 3h\right) & \left(\frac{9}{2}h^4 - 18h^3 + 18h^2 - 4h\right) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} y_{k+2} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \\ hf_{k+3} \end{pmatrix}$$

Solving the system of equations by Gaussian elimination, we have that:

$$
\left.
\begin{aligned}
a_0 &= y_{k+2} + \frac{1}{h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k) - \frac{1}{h^2}(f_{k+3} - 4f_{k+2} + 5f_{k+1} - 2f_k) \\
&\quad + \frac{1}{6h}(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k) - \frac{h}{3}(f_{k+2} + 4f_{k+1} + f_k) + f_k \\
a_1 &= -\frac{4}{h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k) + \frac{3}{h^2}(f_{k+3} - 4f_{k+2} + 5f_{k+1} - 2f_k) \\
&\quad - \frac{1}{3h}(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k) - f_k \\
a_2 &= \frac{6}{h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k) - \frac{3}{h^2}(f_{k+3} - 4f_{k+2} + 5f_{k+1} - 2f_k) \\
&\quad + \frac{1}{6h}(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k) \\
a_3 &= -\frac{4}{h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k) + \frac{1}{h^2}(f_{k+3} - 4f_{k+2} + 5f_{k+1} - 2f_k) \\
a_4 &= \frac{1}{h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k)
\end{aligned}
\right\} \qquad (10)
$$

Substituting for $a_j,\ j = 0, 1, 2, 3, 4$ in Equation (8) and collecting like terms yields the continuous method

$$
\begin{aligned}
y(x) &= y_{k+2} + \frac{1}{24h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k)(x - x_k)^4 \\
&- \frac{1}{6h^2}(f_{k+3} - 4f_{k+2} + 5f_{k+1} - 2f_k)(x - x_k)^3 + \frac{1}{12h}(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k)(x - x_k)^2 \\
&+ f_k(x - x_k) - \frac{h}{3}(f_{k+2} + 4f_{k+1} + f_k). \qquad (11)
\end{aligned}
$$

Evaluating Equation (11) at $x = x_{k+3}$, we obtain the discrete scheme:

$$
y_{k+3} = y_{k+2} + \frac{h}{24}(9f_{k+3} + 19f_{k+2} - 5f_{k+1} + f_k). \qquad (12)
$$

Derivation of Three-Step Optimal Order Method

To derive the three-step optimal order method, we shall consider Equation (8) and Equation (9).

Interpolating Equation (8) at $x = x_{k+1}$ and collocating Equation (9) at $x = x_k,\ x_{k+1},\ x_{k+2},\ x_{k+3}$ give rise to the system of equations which is written in matrix form as follows:

$$
\begin{pmatrix}
1 & (-h+1) & \left(\frac{1}{2}h^2 - 2h + 1\right) & \left(-\frac{1}{6}h^3 + \frac{9}{6}h^2 - 3h + 1\right) & \left(\frac{1}{24}h^4 - \frac{16}{24}h^3 + 3h^2 - 4h + 1\right) \\
0 & -h & -2h & -3h & -4h \\
0 & -h & (h^2 - 2h) & \left(-\frac{1}{2}h^3 + 3h^2 - 3h\right) & \left(\frac{1}{6}h^4 - 2h^3 + 6h^2 - 4h\right) \\
0 & -h & (2h^2 - 2h) & (-2h^3 + 6h^2 - 3h) & \left(\frac{4}{3}h^4 - 8h^3 + 12h^2 - 4h\right) \\
0 & -h & (3h^2 - 2h) & \left(-\frac{9}{2}h^3 + 9h^2 - 3h\right) & \left(\frac{9}{2}h^4 - 18h^3 + 18h^2 - 4h\right)
\end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}
=
\begin{pmatrix} y_{k+1} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \\ hf_{k+3} \end{pmatrix}
$$

Solving the system of equations give the same result as Equation (10) above except for a_0 which is given as

$$a_0 = y_{k+1} + \frac{1}{h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k) - \frac{1}{h^2}(f_{k+3} - 4f_{k+2} + 5f_{k+1} - 2f_k) + \frac{1}{6h}(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k) + \frac{h}{24}(-f_{k+3} + 5f_{k+2} - 19f_{k+1} - 9f_k) + f_k$$

Substituting for $a_j,\ j = 0, 1, 2, 3, 4$ in Equation (8) yields the continuous three-step optimal order method:

$$y(x) = y_{k+2} + \frac{1}{120h^4}(f_{k+4} - 4f_{k+3} + 6f_{k+2} - 4f_{k+1} + f_k)(x - x_k)^5 + \frac{1}{48h^3}(-3f_{k+4} + 14f_{k+3} - 24f_{k+2} + 18f_{k+1} - 5f_k)(x - x_k)^4 - \frac{1}{72h^2}(-11f_{k+4} + 56f_{k+3} - 114f_{k+2} + 104f_{k+1} - 35f_k)(x - x_k)^3 + \frac{1}{24h}(-3f_{k+4} + 16f_{k+3} - 36f_{k+2} + 48f_{k+1} - 25f_k)(x - x_k)^2 + f_k(x - x_k) + \frac{h}{90}(f_{k+4} - 4f_{k+3} - 24f_{k+2} - 124f_{k+1} - 29f_k). \tag{13}$$

Evaluating Equation (13) at $x = x_{k+3}$, we obtain the discrete scheme:

$$y_{k+3} = y_{k+1} + \frac{h}{3}(f_{k+3} + 4f_{k+2} + f_{k+1}). \tag{14}$$

Derivation of Four-Step Adams-Moulton Method

To derive the four-step Adams-Moulton method, we let $n = 5$ in Equation (6) and differentiate the obtained function once. This gives:

$$y(x) = a_0 + a_1[-(x - x_k) + 1] + a_2\left[\frac{1}{2}(x - x_k)^2 - 2(x - x_k) + 1\right] + a_3\left[-\frac{1}{6}(x - x_k)^3 + \frac{9}{6}(x - x_k)^2 - 3(x - x_k) + 1\right] + a_4\left[\frac{1}{24}(x - x_k)^4 - \frac{16}{24}(x - x_k)^3 + 3(x - x_k)^2 - 4(x - x_k) + 1\right] + a_5\left[-\frac{1}{120}(x - x_k)^5 + \frac{25}{120}(x - x_k)^4 - \frac{200}{120}(x - x_k)^3 + 5(x - x_k)^2 - 5(x - x_k) + 1\right], \tag{15}$$

$$y'(x) = -a_1 + a_2[(x - x_k) - 2] + a_3\left[-\frac{1}{2}(x - x_k)^2 + 3(x - x_k) - 3\right] + a_4\left[\frac{1}{6}(x - x_k)^3 - 2(x - x_k)^2 + 6(x - x_k) - 4\right] + a_5\left[-\frac{1}{24}(x - x_k)^4 + \frac{25}{30}(x - x_k)^3 - 5(x - x_k)^2 + 10(x - x_k) - 5\right]. \tag{16}$$

Interpolating Equation (15) at $x = x_{k+3}$ and collocating Equation (16) at $x = x_k,\ x_{k+1},\ x_{k+2},\ x_{k+3},\ x_{k+4}$ give rise to a system of equations which is written in matrix form as follows:

$$\begin{pmatrix} 1 & (-3h+1) & \begin{pmatrix} \frac{9}{2}h^2 \\ -6h+1 \end{pmatrix} & \begin{pmatrix} -\frac{9}{2}h^3+\frac{27}{2}h^2 \\ -9h+1 \end{pmatrix} & \begin{pmatrix} \frac{27}{8}h^4-18h^3 \\ +27h^2 \\ -12h+1 \end{pmatrix} & \begin{pmatrix} -\frac{81}{40}h^5+\frac{135}{8}h^4 \\ -45h^3+45h^2 \\ -15h+1 \end{pmatrix} \\ 0 & -h & -2h & -3h & -4h & -5h \\ 0 & -h & (h^2-2h) & \begin{pmatrix} -\frac{1}{2}h^3 \\ +3h^2-3h \end{pmatrix} & \begin{pmatrix} \frac{1}{6}h^4-2h^3 \\ +6h^2-4h \end{pmatrix} & \begin{pmatrix} -\frac{1}{24}h^5+\frac{25}{30}h^4 \\ -5h^3 \\ +10h^2-5h \end{pmatrix} \\ 0 & -h & (2h^2-2h) & \begin{pmatrix} -2h^3 \\ +6h^2-3h \end{pmatrix} & \begin{pmatrix} \frac{4}{3}h^4-8h^3 \\ +12h^2-4h \end{pmatrix} & \begin{pmatrix} -\frac{2}{3}h^5+\frac{20}{3}h^4 \\ -20h^3 \\ +20h^2-5h \end{pmatrix} \\ 0 & -h & (3h^2-2h) & \begin{pmatrix} -\frac{9}{2}h^3 \\ +9h^2-3h \end{pmatrix} & \begin{pmatrix} \frac{9}{2}h^4-18h^3 \\ +18h^2-4h \end{pmatrix} & \begin{pmatrix} -\frac{27}{8}h^5+\frac{45}{2}h^4 \\ -45h^3 \\ +30h^2-5h \end{pmatrix} \\ 0 & -h & (4h^2-2h) & \begin{pmatrix} -8h^3 \\ +12h^2-3h \end{pmatrix} & \begin{pmatrix} \frac{32}{3}h^4-32h^3 \\ +24h^2-4h \end{pmatrix} & \begin{pmatrix} -\frac{32}{3}h^5+\frac{160}{3}h^4 \\ -80h^3 \\ +40h^2-5h \end{pmatrix} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} y_{k+3} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \\ hf_{k+3} \\ hf_{k+4} \end{pmatrix}$$

Solving the system of equations by Gaussian elimination, we have that:

$$\left.\begin{aligned}
a_0 &= y_{k+3} + \frac{1}{h^4}\begin{pmatrix} f_{k+4} - 4f_{k+3} \\ +6f_{k+2} - 4f_{k+1} + f_k \end{pmatrix} + \frac{1}{2h^3}\begin{pmatrix} -3f_{k+4} + 14f_{k+3} \\ -24f_{k+2} + 18f_{k+1} - 5f_k \end{pmatrix} \\
&- \frac{1}{12h^2}\begin{pmatrix} -11f_{k+4} + 56f_{k+3} - 114f_{k+2} \\ +104f_{k+1} - 35f_k \end{pmatrix} + \frac{1}{12h}\begin{pmatrix} -3f_{k+4} + 16f_{k+3} \\ -36f_{k+2} + 48f_{k+1} - 25f_k \end{pmatrix} \\
&+ \frac{h}{80}\begin{bmatrix} 3f_{k+4} - 42f_{k+3} - 72f_{k+2} \\ -102f_{k+1} - 27f_k \end{bmatrix} + f_k \\
a_1 &= -\frac{5}{h^4}\begin{pmatrix} f_{k+4} - 4f_{k+3} \\ +6f_{k+2} - 4f_{k+1} + f_k \end{pmatrix} - \frac{1}{2h^3}\begin{pmatrix} -3f_{k+4} + 14f_{k+3} \\ -24f_{k+2} + 18f_{k+1} - 5f_k \end{pmatrix} \\
&+ \frac{1}{4h^2}\begin{pmatrix} -11f_{k+4} + 56f_{k+3} - 114f_{k+2} \\ +104f_{k+1} - 35f_k \end{pmatrix} - \frac{1}{6h}\begin{pmatrix} -3f_{k+4} + 16f_{k+3} \\ -36f_{k+2} + 48f_{k+1} - 25f_k \end{pmatrix} - f_k \\
a_2 &= \frac{10}{h^4}\begin{pmatrix} f_{k+4} - 4f_{k+3} \\ +6f_{k+2} - 4f_{k+1} + f_k \end{pmatrix} + \frac{3}{h^3}\begin{pmatrix} -3f_{k+4} + 14f_{k+3} \\ -24f_{k+2} + 18f_{k+1} - 5f_k \end{pmatrix} \\
&- \frac{1}{4h^2}\begin{pmatrix} -11f_{k+4} + 56f_{k+3} - 114f_{k+2} \\ +104f_{k+1} - 35f_k \end{pmatrix} + \frac{1}{12h}\begin{pmatrix} -3f_{k+4} + 16f_{k+3} \\ -36f_{k+2} + 48f_{k+1} - 25f_k \end{pmatrix} \\
a_3 &= -\frac{10}{h^4}\begin{pmatrix} f_{k+4} - 4f_{k+3} \\ +6f_{k+2} - 4f_{k+1} + f_k \end{pmatrix} - \frac{2}{h^3}\begin{pmatrix} -3f_{k+4} + 14f_{k+3} \\ -24f_{k+2} + 18f_{k+1} - 5f_k \end{pmatrix} \\
&+ \frac{1}{12h^2}\begin{pmatrix} -11f_{k+4} + 56f_{k+3} - 114f_{k+2} \\ +104f_{k+1} - 35f_k \end{pmatrix} \\
a_4 &= \frac{5}{h^4}\begin{pmatrix} f_{k+4} - 4f_{k+3} \\ +6f_{k+2} - 4f_{k+1} + f_k \end{pmatrix} + \frac{1}{2h^3}\begin{pmatrix} -3f_{k+4} + 14f_{k+3} \\ -24f_{k+2} + 18f_{k+1} - 5f_k \end{pmatrix} \\
a_5 &= -\frac{1}{h^4}(f_{k+4} - 4f_{k+3} + 6f_{k+2} - 4f_{k+1} + f_k)
\end{aligned}\right\} \quad (17)$$

Substituting for a_j, $j = 0, 1, 2, 3, 4, 5$ in Equation (15) yields the continuous method

$$\begin{aligned}
y(x) &= y_{k+3} + \frac{1}{120h^4}(f_{k+4} - 4f_{k+3} + 6f_{k+2} - 4f_{k+1} + f_k)(x - x_k)^5 \\
&+ \frac{1}{48h^3}(-3f_{k+4} + 14f_{k+3} - 24f_{k+2} + 18f_{k+1} - 5f_k)(x - x_k)^4 \\
&- \frac{1}{72h^2}(-11f_{k+4} + 56f_{k+3} - 114f_{k+2} + 104f_{k+1} - 35f_k)(x - x_k)^3 \\
&+ \frac{1}{24h}(-3f_{k+4} + 16f_{k+3} - 36f_{k+2} + 48f_{k+1} - 25f_k)(x - x_k)^2 + f_k(x - x_k) \\
&- \frac{h}{80}(3f_{k+4} - 42f_{k+3} - 72f_{k+2} - 102f_{k+1} - 27f_k). \qquad (18)
\end{aligned}$$

Evaluating Equation (18) at $x = x_{k+4}$, we obtain the discrete scheme:

$$y_{k+4} = y_{k+3} + \frac{h}{720}(251f_{k+4} + 646f_{k+3} - 264f_{k+2} + 106f_{k+1} - 19f_k). \qquad (19)$$

Derivation of Four-Step Optimal Order Method

To derive the four-step optimal order method, we shall consider Equation (15) and Equation (16). Interpolating Equation (15) at $x = x_{k+2}$ and collocating Equation (16) at $x = x_k,\ x_{k+1},\ x_{k+2},\ x_{k+3},\ x_{k+4}$ give rise to the system of equations which is written in matrix form as follows:

$$\begin{pmatrix} 1 & (-2h+1) & \begin{pmatrix}2h^2-4h\\+1\end{pmatrix} & \begin{pmatrix}-\frac{4}{3}h^3+6h^2\\-6h+1\end{pmatrix} & \begin{pmatrix}\frac{2}{3}h^4-\frac{16}{3}h^3\\+12h^2-8h+1\end{pmatrix} & \begin{pmatrix}-\frac{4}{15}h^5+\frac{10}{3}h^4\\-\frac{40}{3}h^3+20h^2\\-10h+1\end{pmatrix} \\ 0 & -h & -2h & -3h & -4h & -5h \\ 0 & -h & (h^2-2h) & \begin{pmatrix}-\frac{1}{2}h^3+3h^2\\-3h\end{pmatrix} & \begin{pmatrix}\frac{1}{6}h^4-2h^3\\+6h^2-4h\end{pmatrix} & \begin{pmatrix}-\frac{1}{24}h^5\\+\frac{25}{30}h^4\\-5h^3\\+10h^2-5h\end{pmatrix} \\ 0 & -h & (2h^2-2h) & \begin{pmatrix}-2h^3+6h^2\\-3h\end{pmatrix} & \begin{pmatrix}\frac{4}{3}h^4-8h^3\\+12h^2-4h\end{pmatrix} & \begin{pmatrix}-\frac{2}{3}h^5+\frac{20}{3}h^4\\-20h^3\\+20h^2-5h\end{pmatrix} \\ 0 & -h & (3h^2-2h) & \begin{pmatrix}-\frac{9}{2}h^3+9h^2\\-3h\end{pmatrix} & \begin{pmatrix}\frac{9}{2}h^4-18h^3\\+18h^2-4h\end{pmatrix} & \begin{pmatrix}-\frac{27}{8}h^5+\frac{45}{2}h^4\\-45h^3\\+30h^2-5h\end{pmatrix} \\ 0 & -h & (4h^2-2h) & \begin{pmatrix}-8h^3+12h^2\\-3h\end{pmatrix} & \begin{pmatrix}\frac{32}{3}h^4-32h^3\\+24h^2-4h\end{pmatrix} & \begin{pmatrix}-\frac{32}{3}h^5\\+\frac{160}{3}h^4\\-80h^3\\+40h^2-5h\end{pmatrix} \end{pmatrix} \begin{pmatrix}a_0\\a_1\\a_2\\a_3\\a_4\\a_5\end{pmatrix} = \begin{pmatrix}y_{k+2}\\hf_k\\hf_{k+1}\\hf_{k+2}\\hf_{k+3}\\hf_{k+4}\end{pmatrix}$$

Solving the system of equations give the same result as in Equation (17) above except for a_0 which is given as

$$a_0 = y_{k+2} + \frac{1}{h^4}\begin{pmatrix}f_{k+4}-4f_{k+3}\\+6f_{k+2}-4f_{k+1}+f_k\end{pmatrix} + \frac{1}{2h^3}\begin{pmatrix}-3f_{k+4}+14f_{k+3}\\-24f_{k+2}+18f_{k+1}-5f_k\end{pmatrix}$$
$$-\frac{1}{12h^2}\begin{pmatrix}-11f_{k+4}+56f_{k+3}-114f_{k+2}\\+104f_{k+1}-35f_k\end{pmatrix} + \frac{1}{12h}\begin{pmatrix}-3f_{k+4}+16f_{k+3}\\-36f_{k+2}+48f_{k+1}-25f_k\end{pmatrix}$$
$$+\frac{h}{90}\begin{bmatrix}f_{k+4}-4f_{k+3}-24f_{k+2}\\-124f_{k+1}-29f_k\end{bmatrix} + f_k$$

Substituting for $a_j,\ j = 0, 1, 2, 3, 4, 5$ in Equation (15) yields the continuous five-step optimal order method:

$$y(x) = y_{k+2} + \frac{1}{120h^4}(f_{k+4}-4f_{k+3}+6f_{k+2}-4f_{k+1}+f_k)(x-x_k)^5$$
$$+\frac{1}{48h^3}(-3f_{k+4}+14f_{k+3}-24f_{k+2}+18f_{k+1}-5f_k)(x-x_k)^4$$
$$-\frac{1}{72h^2}(-11f_{k+4}+56f_{k+3}-114f_{k+2}+104f_{k+1}-35f_k)(x-x_k)^3$$
$$+\frac{1}{24h}(-3f_{k+4}+16f_{k+3}-36f_{k+2}+48f_{k+1}-25f_k)(x-x_k)^2 + f_k(x-x_k)$$
$$+\frac{h}{90}(f_{k+4}-4f_{k+3}-24f_{k+2}-124f_{k+1}-29f_k). \quad (20)$$

Evaluating Equation (20) at $x = x_{k+4}$, we obtain the discrete scheme:

$$y_{k+4} = y_{k+2} + \frac{h}{90}(29f_{k+4}+124f_{k+3}+24f_{k+2}+4f_{k+1}-f_k). \quad (21)$$

Derivation ofFive-Step Adams-Moulton Method

To derive the five-step Adams-Moulton method, we let $n = 6$ in Equation (6) and differentiate the obtained function once. This gives:

$$
\begin{aligned}
y(x) &= a_0 + a_1[-(x-x_k)+1] + a_2\left[\frac{1}{2}(x-x_k)^2 - 2(x-x_k) + 1\right] \\
&+ a_3\left[-\frac{1}{6}(x-x_k)^3 + \frac{9}{6}(x-x_k)^2 - 3(x-x_k) + 1\right] \\
&+ a_4\left[\frac{1}{24}(x-x_k)^4 - \frac{16}{24}(x-x_k)^3 + 3(x-x_k)^2 - 4(x-x_k) + 1\right] \\
&+ a_5\left[-\frac{1}{120}(x-x_k)^5 + \frac{25}{120}(x-x_k)^4 - \frac{200}{120}(x-x_k)^3 + 5(x-x_k)^2 - 5(x-x_k) + 1\right] \\
&+ a_6\left[\frac{1}{720}(x-x_k)^6 - \frac{1}{20}(x-x_k)^5 + \frac{5}{8}(x-x_k)^4 - \frac{10}{3}(x-x_k)^3 + \frac{15}{2}(x-x_k)^2 - 6(x-x_k) + 1\right], \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
y'(x) &= -a_1 + a_2[(x-x_k) - 2] + a_3\left[-\frac{1}{2}(x-x_k)^2 + 3(x-x_k) - 3\right] \\
&+ a_4\left[\frac{1}{6}(x-x_k)^3 - 2(x-x_k)^2 + 6(x-x_k) - 4\right] \\
&+ a_5\left[-\frac{1}{24}(x-x_k)^4 + \frac{25}{30}(x-x_k)^3 - 5(x-x_k)^2 + 10(x-x_k) - 5\right] \\
&+ a_6\left[\frac{1}{120}(x-x_k)^5 - \frac{1}{4}(x-x_k)^4 + \frac{5}{2}(x-x_k)^3 - 10(x-x_k)^2 + 15(x-x_k) - 6\right]. \qquad (23)
\end{aligned}
$$

Interpolating Equation (22) at $x = x_{k+4}$ and collocating Equation (23) at $x = x_k,\ x_{k+1},\ x_{k+2},\ x_{k+3},\ x_{k+4},\ x_{k+5}$ give rise to the system of equations which is written in matrix form as follows:

$$
\begin{pmatrix}
1 & (-4h+1) & (8h^2-8h+1) & \left(-\frac{32}{3}h^3+24h^2-12h+1\right) & \left(\frac{32}{3}h^4-\frac{128}{3}h^3+48h^2-16h+1\right) & \left(-\frac{128}{15}h^5+\frac{160}{3}h^4-\frac{320}{3}h^3+80h^2-20h+1\right) & \left(\frac{256}{45}h^6-\frac{256}{5}h^5+160h^4-\frac{640}{3}h^3+120h^2-24h+1\right) \\
0 & -h & -2h & -3h & -4h & -5h & -6h \\
0 & -h & (h^2-2h) & \left(-\frac{1}{2}h^3+3h^2-3h\right) & \left(\frac{1}{6}h^4-2h^3+6h^2-4h\right) & \left(-\frac{1}{24}h^5+\frac{25}{30}h^4-5h^3+10h^2-5h\right) & \left(\frac{1}{120}h^6-\frac{1}{4}h^5+\frac{5}{2}h^4-10h^3+15h^2-6h\right) \\
0 & -h & (2h^2-2h) & \left(-2h^3+6h^2-3h\right) & \left(\frac{4}{3}h^4-8h^3+12h^2-4h\right) & \left(-\frac{2}{3}h^5+\frac{20}{3}h^4-20h^3+20h^2-5h\right) & \left(\frac{4}{15}h^6-4h^5+20h^4-40h^3+30h^2-6h\right) \\
0 & -h & (3h^2-2h) & \left(-\frac{9}{2}h^3+9h^2-3h\right) & \left(\frac{9}{2}h^4-18h^3+18h^2-4h\right) & \left(-\frac{27}{8}h^5+\frac{45}{2}h^4-45h^3+30h^2-5h\right) & \left(\frac{81}{40}h^6-\frac{81}{4}h^5+\frac{135}{2}h^4-90h^3+45h^2-6h\right) \\
0 & -h & (4h^2-2h) & \left(-8h^3+12h^2-3h\right) & \left(\frac{32}{3}h^4-32h^3+24h^2-4h\right) & \left(-\frac{32}{3}h^5+\frac{160}{3}h^4-80h^3+40h^2-5h\right) & \left(\frac{128}{15}h^6-64h^5+160h^4-160h^3+60h^2-6h\right) \\
0 & -h & (5h^2-2h) & \left(-\frac{25}{2}h^3+15h^2-3h\right) & \left(\frac{125}{6}h^4-50h^3+30h^2-4h\right) & \left(-\frac{625}{24}h^5+\frac{625}{6}h^4-125h^3+50h^2-5h\right) & \left(\frac{625}{24}h^6-\frac{625}{4}h^5+\frac{625}{2}h^4-250h^3+75h^2-6h\right)
\end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}
=
\begin{pmatrix} y_{k+4} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \\ hf_{k+3} \\ hf_{k+4} \\ hf_{k+5} \end{pmatrix}
$$

Solving the system of equations by Gaussian elimination, we have that:

$$
\left.
\begin{aligned}
a_0 &= y_{k+4} + \frac{1}{h^5}\begin{pmatrix} f_{k+5} - 5f_{k+4} + 10f_{k+3} \\ -10f_{k+2} + 5f_{k+1} - f_k \end{pmatrix} - \frac{1}{h^4}\begin{pmatrix} 2f_{k+5} - 11f_{k+4} + 24f_{k+3} \\ -26f_{k+2} + 14f_{k+1} - 3f_k \end{pmatrix} + \frac{1}{4h^3}\begin{pmatrix} 7f_{k+5} - 41f_{k+4} + 98f_{k+3} \\ -118f_{k+2} + 71f_{k+1} - 17f_k \end{pmatrix} \\
&\quad - \frac{1}{12h^2}\begin{pmatrix} 10f_{k+5} - 61f_{k+4} + 156f_{k+3} \\ -214f_{k+2} + 154f_{k+1} - 45f_k \end{pmatrix} + \frac{1}{60h}\begin{pmatrix} 12f_{k+5} - 75f_{k+4} + 200f_{k+3} \\ -300f_{k+2} + 300f_{k+1} - 137f_k \end{pmatrix} \\
&\quad - h\begin{bmatrix} \frac{1}{450359962737050}f_{k+5} + \frac{14}{45}f_{k+4} \\ + \frac{64}{45}f_{k+3} + \frac{8}{15}f_{k+2} + \frac{64}{45}f_{k+1} + \frac{14}{45}f_k \end{bmatrix} + f_k \\
a_1 &= -\frac{6}{h^5}\begin{pmatrix} f_{k+5} - 5f_{k+4} + 10f_{k+3} \\ -10f_{k+2} + 5f_{k+1} - f_k \end{pmatrix} + \frac{5}{h^4}\begin{pmatrix} 2f_{k+5} - 11f_{k+4} + 24f_{k+3} \\ -26f_{k+2} + 14f_{k+1} - 3f_k \end{pmatrix} - \frac{1}{h^3}\begin{pmatrix} 7f_{k+5} - 41f_{k+4} + 98f_{k+3} \\ -118f_{k+2} + 71f_{k+1} - 17f_k \end{pmatrix} \\
&\quad + \frac{1}{4h^2}\begin{pmatrix} 10f_{k+5} - 61f_{k+4} + 156f_{k+3} \\ -214f_{k+2} + 154f_{k+1} - 45f_k \end{pmatrix} - \frac{1}{30h}\begin{pmatrix} 12f_{k+5} - 75f_{k+4} + 200f_{k+3} \\ -300f_{k+2} + 300f_{k+1} - 137f_k \end{pmatrix} - f_k \\
a_2 &= \frac{15}{h^5}\begin{pmatrix} f_{k+5} - 5f_{k+4} + 10f_{k+3} \\ -10f_{k+2} + 5f_{k+1} - f_k \end{pmatrix} - \frac{10}{h^4}\begin{pmatrix} 2f_{k+5} - 11f_{k+4} + 24f_{k+3} \\ -26f_{k+2} + 14f_{k+1} - 3f_k \end{pmatrix} + \frac{3}{2h^3}\begin{pmatrix} 7f_{k+5} - 41f_{k+4} + 98f_{k+3} \\ -118f_{k+2} + 71f_{k+1} - 17f_k \end{pmatrix} \\
&\quad - \frac{1}{4h^2}\begin{pmatrix} 10f_{k+5} - 61f_{k+4} + 156f_{k+3} \\ -214f_{k+2} + 154f_{k+1} - 45f_k \end{pmatrix} + \frac{1}{60h}\begin{pmatrix} 12f_{k+5} - 75f_{k+4} + 200f_{k+3} \\ -300f_{k+2} + 300f_{k+1} - 137f_k \end{pmatrix} \\
a_3 &= -\frac{20}{h^5}\begin{pmatrix} f_{k+5} - 5f_{k+4} + 10f_{k+3} \\ -10f_{k+2} + 5f_{k+1} - f_k \end{pmatrix} + \frac{10}{h^4}\begin{pmatrix} 2f_{k+5} - 11f_{k+4} + 24f_{k+3} \\ -26f_{k+2} + 14f_{k+1} - 3f_k \end{pmatrix} - \frac{1}{h^3}\begin{pmatrix} 7f_{k+5} - 41f_{k+4} + 98f_{k+3} \\ -118f_{k+2} + 71f_{k+1} - 17f_k \end{pmatrix} \\
&\quad + \frac{1}{12h^2}\begin{pmatrix} 10f_{k+5} - 61f_{k+4} + 156f_{k+3} \\ -214f_{k+2} + 154f_{k+1} - 45f_k \end{pmatrix} \\
a_4 &= \frac{15}{h^5}\begin{pmatrix} f_{k+5} - 5f_{k+4} + 10f_{k+3} \\ -10f_{k+2} + 5f_{k+1} - f_k \end{pmatrix} - \frac{5}{h^4}\begin{pmatrix} 2f_{k+5} - 11f_{k+4} + 24f_{k+3} \\ -26f_{k+2} + 14f_{k+1} - 3f_k \end{pmatrix} + \frac{1}{4h^3}\begin{pmatrix} 7f_{k+5} - 41f_{k+4} + 98f_{k+3} \\ -118f_{k+2} + 71f_{k+1} - 17f_k \end{pmatrix} \\
a_5 &= -\frac{6}{h^5}\begin{pmatrix} f_{k+5} - 5f_{k+4} + 10f_{k+3} \\ -10f_{k+2} + 5f_{k+1} - f_k \end{pmatrix} + \frac{1}{h^4}\begin{pmatrix} 2f_{k+5} - 11f_{k+4} + 24f_{k+3} \\ -26f_{k+2} + 14f_{k+1} - 3f_k \end{pmatrix} \\
a_6 &= \frac{1}{h^5}(f_{k+5} - 5f_{k+4} + 10f_{k+3} - 10f_{k+2} + 5f_{k+1} - f_k)
\end{aligned}
\right\} \quad (24)
$$

Substituting for $a_j,\ j = 0, 1, 2, 3, 4, 5, 6$ in Equation (22) yields the continuous method

$$
\begin{aligned}
y(x) &= y_{k+4} + \frac{1}{720h^5}(f_{k+5} - 5f_{k+4} + 10f_{k+3} - 10f_{k+2} + 5f_{k+1} - f_k)(x - x_k)^6 \\
&- \frac{1}{120h^4}(2f_{k+5} - 11f_{k+4} + 24f_{k+3} - 26f_{k+2} + 14f_{k+1} - 3f_k)(x - x_k)^5 \\
&+ \frac{1}{96h^3}(7f_{k+5} - 41f_{k+4} + 98f_{k+3} - 118f_{k+2} + 71f_{k+1} - 17f_k)(x - x_k)^4 \\
&- \frac{1}{72h^2}(10f_{k+5} - 61f_{k+4} + 156f_{k+3} - 214f_{k+2} + 154f_{k+1} - 45f_k)(x - x_k)^3 \\
&+ \frac{1}{120h}(12f_{k+5} - 75f_{k+4} + 200f_{k+3} - 300f_{k+2} + 300f_{k+1} - 137f_k)(x - x_k)^2 + f_k(x - x_k) \\
&- h\left[\frac{1}{450359962737050}f_{k+5} + \frac{14}{45}f_{k+4} + \frac{64}{45}f_{k+3} + \frac{8}{15}f_{k+2} + \frac{64}{45}f_{k+1} + \frac{14}{45}f_k\right]. \qquad (25)
\end{aligned}
$$

Evaluating Equation (25) at $x = x_{k+5}$, we obtain the discrete scheme:

$$
y_{k+5} = y_{k+4} + \frac{h}{1440}(475f_{k+5} + 1427f_{k+4} - 798f_{k+3} + 482f_{k+2} - 173f_{k+1} + 27f_k). \quad (26)
$$

Derivation ofFive-Step Optimal Order Method

The optimal order scheme is an implicit multistep method similar to the Adams-Moulton method. To derive the five-step optimal order method, we shall consider the system of equations in Equation (22) and Equation (23). Interpolating Equation (22) at $x = x_{k+3}$ and collocating Equation (23) at $x = x_k,\ x_{k+1},\ x_{k+2},\ x_{k+3},\ x_{k+4},\ x_{k+5}$ give rise to a system of equations which is written in matrix form as follows:

$$
\begin{pmatrix}
1 & (-3h+1) & \begin{pmatrix}\frac{9}{2}h^2-6h\\ +1\end{pmatrix} & \begin{pmatrix}-\frac{9}{2}h^3+\frac{27}{2}h^2\\ -9h+1\end{pmatrix} & \begin{pmatrix}\frac{27}{8}h^4-18h^3\\ +27h^2-12h\\ +1\end{pmatrix} & \begin{pmatrix}-\frac{81}{4}h^5+\frac{135}{8}h^4\\ -45h^3+45h^2\\ -15h+1\end{pmatrix} & \begin{pmatrix}\frac{81}{80}h^6-\frac{243}{20}h^5\\ +\frac{405}{8}h^4\\ -90h^3+\frac{135}{2}h^2\\ -18h+1\end{pmatrix}\\
0 & -h & -2h & -3h & -4h & -5h & -6h\\
0 & -h & (h^2-2h) & \begin{pmatrix}-\frac{1}{2}h^3+3h^2\\ -3h\end{pmatrix} & \begin{pmatrix}\frac{1}{6}h^4-2h^3\\ +6h^2-4h\end{pmatrix} & \begin{pmatrix}-\frac{1}{24}h^5+\frac{25}{30}h^4\\ -5h^3+10h^2\\ -5h\end{pmatrix} & \begin{pmatrix}\frac{1}{120}h^6-\frac{1}{4}h^5\\ +\frac{5}{2}h^4\\ -10h^3+15h^2\\ -6h\end{pmatrix}\\
0 & -h & (2h^2-2h) & \begin{pmatrix}-2h^3+6h^2\\ -3h\end{pmatrix} & \begin{pmatrix}\frac{4}{3}h^4-8h^3\\ +12h^2-4h\end{pmatrix} & \begin{pmatrix}-\frac{2}{3}h^5+\frac{20}{3}h^4\\ -20h^3+20h^2\\ -5h\end{pmatrix} & \begin{pmatrix}\frac{4}{15}h^6-4h^5\\ +20h^4\\ -40h^3+30h^2\\ -6h\end{pmatrix}\\
0 & -h & (3h^2-2h) & \begin{pmatrix}-\frac{9}{2}h^3+9h^2\\ -3h\end{pmatrix} & \begin{pmatrix}\frac{9}{2}h^4-18h^3\\ +18h^2-4h\end{pmatrix} & \begin{pmatrix}-\frac{27}{8}h^5+\frac{45}{2}h^4\\ -45h^3+30h^2\\ -5h\end{pmatrix} & \begin{pmatrix}\frac{81}{40}h^6-\frac{81}{4}h^5\\ +\frac{135}{2}h^4\\ -90h^3+45h^2\\ -6h\end{pmatrix}\\
0 & -h & (4h^2-2h) & \begin{pmatrix}-8h^3+12h^2\\ -3h\end{pmatrix} & \begin{pmatrix}\frac{32}{3}h^4-32h^3\\ +24h^2-4h\end{pmatrix} & \begin{pmatrix}-\frac{32}{3}h^5+\frac{160}{3}h^4\\ -80h^3+40h^2\\ -5h\end{pmatrix} & \begin{pmatrix}\frac{128}{15}h^6-64h^5\\ +160h^4\\ -160h^3+60h^2\\ -6h\end{pmatrix}\\
0 & -h & (5h^2-2h) & \begin{pmatrix}-\frac{25}{2}h^3\\ +15h^2-3h\end{pmatrix} & \begin{pmatrix}\frac{125}{6}h^4-50h^3\\ +30h^2-4h\end{pmatrix} & \begin{pmatrix}-\frac{625}{24}h^5+\frac{625}{6}h^4\\ -125h^3+50h^2\\ -5h\end{pmatrix} & \begin{pmatrix}\frac{625}{24}h^6-\frac{625}{4}h^5\\ +\frac{625}{2}h^4\\ -250h^3+75h^2\\ -6h\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}a_0\\ a_1\\ a_2\\ a_3\\ a_4\\ a_5\\ a_6\end{pmatrix}
=
\begin{pmatrix}y_{k+3}\\ hf_k\\ hf_{k+1}\\ hf_{k+2}\\ hf_{k+3}\\ hf_{k+4}\\ hf_{k+5}\end{pmatrix}
$$

Solving the system of equations give the same result as Equation (24) above except for a_0 which is given as

$$
a_0 = y_{k+3} + \frac{1}{h^5}\begin{pmatrix}f_{k+5}-5f_{k+4}+10f_{k+3}\\ -10f_{k+2}+5f_{k+1}-f_k\end{pmatrix} - \frac{1}{h^4}\begin{pmatrix}2f_{k+5}-11f_{k+4}+24f_{k+3}\\ -26f_{k+2}+14f_{k+1}-3f_k\end{pmatrix}
+\frac{1}{4h^3}\begin{pmatrix}7f_{k+5}-41f_{k+4}+98f_{k+3}\\ -118f_{k+2}+71f_{k+1}-17f_k\end{pmatrix} - \frac{1}{12h^2}\begin{pmatrix}10f_{k+5}-61f_{k+4}+156f_{k+3}\\ -214f_{k+2}+154f_{k+1}-45f_k\end{pmatrix}
+\frac{1}{60h}\begin{pmatrix}12f_{k+5}-75f_{k+4}+200f_{k+3}\\ -300f_{k+2}+300f_{k+1}-137f_k\end{pmatrix} + \frac{h}{160}\begin{bmatrix}-3f_{k+5}+21f_{k+4}-114f_{k+3}\\ -114f_{k+2}-219f_{k+1}-51f_k\end{bmatrix} + f_k.
$$

Substituting for $a_j,\ j = 0,1,2,3,4,5,6$ in Equation (22) yields the continuous five-step optimal order method:

$$
\begin{aligned}
y(x) = {} & y_{k+3} + \frac{1}{720h^5}(f_{k+5}-5f_{k+4}+10f_{k+3}-10f_{k+2}+5f_{k+1}-f_k)(x-x_k)^6\\
& -\frac{1}{120h^4}(2f_{k+5}-11f_{k+4}+24f_{k+3}-26f_{k+2}+14f_{k+1}-3f_k)(x-x_k)^5\\
& +\frac{1}{96h^3}(7f_{k+5}-41f_{k+4}+98f_{k+3}-118f_{k+2}+71f_{k+1}-17f_k)(x-x_k)^4\\
& -\frac{1}{72h^2}(10f_{k+5}-61f_{k+4}+156f_{k+3}-214f_{k+2}+154f_{k+1}-45f_k)(x-x_k)^3\\
& +\frac{1}{120h}(12f_{k+5}-75f_{k+4}+200f_{k+3}-300f_{k+2}+300f_{k+1}-137f_k)(x-x_k)^2 + f_k(x-x_k)\\
& +\frac{h}{160}(-3f_{k+5}+21f_{k+4}-114f_{k+3}-114f_{k+2}-219f_{k+1}-51f_k).
\end{aligned} \tag{27}
$$

Evaluating Equation (27) at $x = x_{k+5}$, we obtain the discrete scheme:

$$
y_{k+5} = y_{k+3} + \frac{h}{90}(28f_{k+5}+129f_{k+4}+14f_{k+3}+14f_{k+2}-6f_{k+1}+f_k). \tag{28}
$$

III. Result

In this section, the derived LMMs are applied to solve two IVPs in ODEs. The associated absolute errors with the methods are also computed. The Runge-Kutta methods and Adams-Bashforth methods of the same orders are respectively used to obtain the starting values and as predictor methods to the implicit schemes of Adams-Moulton and the proposed Optimal Order.

Example 1

Solve the IVP

$$y' = 0.5(1-y), \quad y(0) = 0.5, \quad 0 \le x \le 1, \; h = 0.1$$

Exact solution: $y(x) = 1 - 0.5\exp(-0.5x)$.

Example 2

Solve the IVP

$$y' = xe^{3x} - 2y, \quad y(0) = 0, \quad 0 \le x \le 1, \quad h = 0.1$$

Exact solution: $y(x) = \frac{1}{5}xe^{3x} - \frac{1}{25}e^{3x} + \frac{1}{25}e^{-2x}$.

The error is defined as

$$|y(x) - y_n(x)|.$$

where $y(x)$ is the exact solution and $y_n(x)$ is the approximate solution.

Table 1: Result of example 1 and associated errors by the derived three-step implicit methods

x-value	Exact Solution $y(x)$	3-Step ADM Approximation $y_n(x)$	3-Step Optimal Order Approximation $y_n(x)$	Error in 3-Step ADM Scheme	Error in 3-step Optimal Order Scheme
0.1	0.5243852877	0.5243852865	0.5243852866	1.2913 E-009	1.2913 E-009
0.2	0.5475812910	0.5475812885	0.5475812885	2.4567 E-009	2.4567 E-009
0.3	0.5696460118	0.5696460133	0.5696460123	1.4791 E-009	5.2350 E-010
0.4	0.5906346235	0.5906346294	0.5906346234	5.9881 E-009	7.7083 E-011
0.5	0.6105996085	0.6105996194	0.6105996120	1.0960 E-008	3.5080 E-009
0.6	0.6295908897	0.6295909060	0.6295908929	1.6361 E-008	3.2795 E-009
0.7	0.6476559551	0.6476559772	0.6476559625	2.2082 E-008	7.3330 E-009
0.8	0.6648399770	0.6648400050	0.6648399843	2.8060 E-008	7.2979 E-009
0.9	0.6811859242	0.6811859584	0.6811859359	3.4226 E-008	1.1689 E-008
1.0	0.6967346701	0.6967347107	0.6967346819	4.0523 E-008	1.1715 E-008

Table 2: Result of example 1 and associated errors by the derived four-step implicit methods

x-value	Exact Solution $y(x)$	4-Step ADM Approximation $y_n(x)$	4-Step Optimal Order Approximation $y_n(x)$	Error in 4-Step ADM Scheme	Error in 4-step Optimal Order Scheme
0.1	0.5243852877	0.5243852878	0.5243852878	2.7050 E-011	2.7050 E-011
0.2	0.5475812910	0.5475812910	0.5475812910	5.1461 E-011	5.1461 E-011
0.3	0.5696460118	0.5696460119	0.5696460119	7.3427 E-011	7.3427 E-011
0.4	0.5906346235	0.5906346234	0.5906346234	6.1620 E-011	3.2934 E-011
0.5	0.6105996085	0.6105996082	0.6105996084	2.2362 E-010	3.6869 E-011
0.6	0.6295908897	0.6295908893	0.6295908895	4.0740 E-010	1.6805 E-010
0.7	0.6476559551	0.6476559545	0.6476559549	6.1309 E-010	1.9065 E-010
0.8	0.6648399770	0.6648399761	0.6648399766	8.3323 E-010	3.4013 E-010
0.9	0.6811859242	0.6811859231	0.6811859238	1.0666 E-009	3.7257 E-010
1.0	0.6967346701	0.6967346688	0.6967346696	1.3093 E-009	5.3519 E-010

Table 3: Result of example 1 and associated errors by the derived five-step implicit methods

x-value	Exact Solution $y(x)$	5-Step ADM Approximation $y_n(x)$	5-Step Optimal Order Approximation $y_n(x)$	Error in 5-Step ADM Scheme	Error in 5-step Optimal Order Scheme
0.1	0.5243852877	0.5243852877	0.5243852877	1.0000 E-014	1.0000 E-014
0.2	0.5475812910	0.5475812910	0.5475812910	1.9000 E-014	1.9000 E-014
0.3	0.5696460118	0.5696460118	0.5696460118	2.7000 E-014	2.7000 E-014
0.4	0.5906346235	0.5906346235	0.5906346235	3.5000 E-014	3.5000 E-014
0.5	0.6105996085	0.6105996085	0.6105996085	4.8820 E-012	3.3390 E-012
0.6	0.6295908897	0.6295908897	0.6295908897	1.1006 E-011	4.6020 E-012
0.7	0.6476559551	0.6476559552	0.6476559551	1.8145 E-011	8.9640 E-012
0.8	0.6648399770	0.6648399770	0.6648399770	2.6405 E-011	1.1243 E-011
0.9	0.6811859242	0.6811859242	0.6811859242	3.5241 E-011	1.6274 E-011
1.0	0.6967346701	0.6967346702	0.6967346702	4.4817 E-011	1.9169 E-011

Table 4: Result of example 2 and associated errors by the derived three-step implicit methods

x-value	Exact Solution $y(x)$	3-Step ADM Approximation $y_n(x)$	3-Step Optimal Order Approximation $y_n(x)$	Error in 3-Step ADM Scheme	Error in 3-step Optimal Order Scheme
0.1	0.0057520540	0.0057546313	0.0057546313	2.577340 E-006	2.577340 E-006
0.2	0.0268128018	0.0268187705	0.0268187706	5.968755 E-006	5.968755 E-006
0.3	0.0711445276	0.0711841477	0.0711610346	3.962004 E-005	1.650691 E-005
0.4	0.1507778355	0.1508998405	0.1508360880	1.220050 E-004	5.825254 E-005
0.5	0.2836165219	0.2838684425	0.2837168194	2.519206 E-004	1.002976 E-004
0.6	0.4960195656	0.4964776000	0.4962177134	4.580343 E-004	1.981477 E-004
0.7	0.8264808698	0.8272342465	0.8267777897	7.533767 E-004	2.969199 E-004
0.8	1.3308570264	1.3320365782	1.3313440774	1.179552 E-003	4.870510 E-004
0.9	2.0897743970	2.0915536652	2.0904711425	1.779268 E-003	6.967455 E-004
1.0	3.2190993190	3.2217254915	3.2201567313	2.626172 E-003	1.057412 E-003

Table 5: Result of example 2 and associated errors by the derived four-step implicit methods

x-value	Exact Solution $y(x)$	4-Step ADM Approximation $y_n(x)$	4-Step Optimal Order Approximation $y_n(x)$	Error in 4-Step ADM Scheme	Error in 4-Step Optimal Order Scheme
0.1	0.0057520539	0.0057518112	0.0057518112	2.427671 E-007	2.427671 E-007
0.2	0.0268128018	0.0268122411	0.0268122411	5.607499 E-007	5.607499 E-007
0.3	0.0711445277	0.0711435312	0.0711435312	9.965007 E-007	9.965007 E-007
0.4	0.1507778355	0.1507882290	0.1507845599	1.039348 E-005	6.724378 E-006
0.5	0.2836165219	0.2836555306	0.2836371233	3.900874 E-005	2.060143 E-005
0.6	0.4960195656	0.4961024246	0.4960597062	8.285896 E-005	4.014056 E-005
0.7	0.8264808698	0.8266365802	0.8265583436	1.557103 E-004	7.747383 E-005
0.8	1.3308570264	1.3311113888	1.3309723576	2.543624 E-004	1.153312 E-004
0.9	2.0897743970	2.0901755375	2.0899659578	4.011405 E-004	1.915608 E-004
1.0	3.2190993190	3.2196983346	3.2193643909	5.990156 E-004	2.650719 E-004

Table 6: Result of example 2 and associated errors by the derived five-step implicit methods

x-value	Exact Solution $y(x)$	5-Step ADM Approximation $y_n(x)$	5-Step Optimal Order Approximation $y_n(x)$	Error in 5-Step ADM Scheme	Error in 5-step Optimal Order Scheme
0.1	0.0057520539	0.0057520538	0.0057520538	1.659050 E-010	1.659050 E-010
0.2	0.0268128018	0.0268128015	0.0268128015	3.904440 E-010	3.904440 E-010
0.3	0.0711445277	0.0711445270	0.0711445270	7.089740 E-010	7.089740 E-010
0.4	0.1507778355	0.1507778343	0.1507778343	1.171000 E-009	1.171000 E-009
0.5	0.2836165219	0.2836200778	0.2836190200	3.555894 E-006	2.498091 E-006
0.6	0.4960195656	0.4960329446	0.4960275733	1.337893 E-005	8.007629 E-006
0.7	0.8264808698	0.8265090241	0.8264953678	2.815426 E-005	1.449799 E-005
0.8	1.3308570264	1.3309117322	1.3308873108	5.470577 E-005	3.028437 E-005
0.9	2.0897743970	2.0898615162	2.0898148450	8.711918 E-005	4.044794 E-005
1.0	3.2190993190	3.2192401912	3.2191748260	1.408722 E-004	7.550700 E-005

IV. Discussion of Result

Six implicit LMMs are derived and applied to solve first-order IVPs in ODEs. Table 1 and Table 4 present results of the three-step Adams-Moulton and the proposed optimal order methods for problems 1 and 2 respectively; Table 2 and Table 5 present results ofthefour-step Adams-Moulton and proposed optimal order methods for problems 1 and 2 respectively while Table 3 and Table 6 present results ofthefive-step Adams-Moulton and the proposedoptimal order methods for problems 1 and 2 respectively. From the results obtained, the proposed optimal order methods produced better approximations than the Adams-Moulton methods.

V.Conclusion

In this paper, six implicit LMMs are derived through collocation and interpolation technique using Laguerre polynomials as basis functions. The corresponding discrete schemes are obtained and applied to solve two IVPs of first order ODEs. Results by the proposed methods are comparablewiththeAdams-Moulton methods of the same order.

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