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On the foundations of Completely Continuous, Compact and Relatively Compact Operators

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Abstract: *This paper shows that a completely continuous operator is continuous but every continuous operator is not completely continuous whereas continuous operator of finite rank is completely continuous operator.*
Keywords: *Compact operator, Reiszoerator, Completely Continuous operator, Bounded operator*.

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I. Introduction

The theory of compact operators, Riesz operators, completely continuous operators and relatively compact operators plays a very important role in functional analysis, a branch of mathematics.A compact operator is a linear operator K from a Branch space X to another Branch space Y, s.t. the image under K of any bounded subset of X is relatively compact subset of Y.

e.g. the identity operator is a compact operator iff the space is finite dimensimal.

Riesz operators are those bounded operators that have a Riesz spectral theory i.e. the spectral theory of compact operators. The Italian mathematician.**Riesz (1918)** who had first time developed the theory of compact operators has shown in his theory.

Every Riesz operator on a Hilbert space can be decompose into the sum of compact and quasi-nilpotent. In general, this decomposition is not unique.The origin of the theory of compact operators is in the theory of integral equations, whose integral operators supply concrete examplewhich gives rise to a compact operator K on function space, compactness property is shown by equicontinuity.

The compact operators on topological vector space are linear operators maps the bounded sets to pre-compact sets and only bounded operator that has finite rank is a compact operator. The compact operator operators are simplest class of operators among all operators on Hilbert spaces. The latest modified definition of compact operator on Hilbert space can be brought to the notice as follows:

A linear operator K:X→Y from a pre-Hilbert space X to a Hilbert space Y is compact if it maps the unit ball in X to a pre-compact set in Y.Equivalently: K is compact iff it maps bounded sequence in X to sequences in Y with convergent subsequences.

A compact operator is always a Risez operator. The Risez operator can be decompose into the sum of operator but there are certain spaces on which Risez operator can not be decompose and therefore every Risez operator need not be compact operator.

Fredholm has proved in his work **(1900, 1903)** that the alternative theorem is valid for contain class of linear integral equation.

$$x(s) - \int_a^b K(s,t)x((t)dt = y(s)$$

these equations are known as fredholm integral equations. Here K(S,t) is a continuous function on [a×b]×[a,b] and is called the kernel of integral equation, y(s) is continuous on [a,b] and therefore y∈C[a,b].

Let E and F are normed linear spaces over the scalar field ϕ. We denote the set of all continuous linear map of E into F by L(E,F) the set of all completely continuous linear map S of E into F by R(E,F).

Theorem:(1.1)Arzelz-Ascoli theorem:
If F is a bounded and equicontinuous family inC[a,b] then every sequence of element of F contains a convergent subsequences. "From **Arzela – Ascoli's** theorem it follows that the image sequence K_{x_1}, K_{x_2} ……… of a bounded sequence in C[a,b) contains a convergent subsequence. So, Riesz has constructed the theory of Fredholm integral on this properly ofK."

Definition:(2.1)A subset M of a topological space R is called relatively compact if M is contained in a compact subset of R.

Definition: (2.2) A subset M of a topological R is called compact, if every open covering of M contains a finite sub covering.

Definition:(2.3)A subset M of a topological vector zero vector space is called bounded if corresponding to every zero nbd U, there exists a ,α> 0 s.t.α U ⊃ M.
Definition:**(2.4)**A map T∈L(E, F) is called bounded if there exists a nbd of 0 in E whose image T(U) is a bounded subset of F.
Definition:**(2.5)**If E and F are normed linear spaces, K is linear map of E into F, then K is completely continuous iff there exists a nbd U of origin in E s.t. K(U) is relatively compact.
Definition:(2.6) If E and F are normed linear spaces then a map K∈L(E,F) is completely continuous if every bounded sequence {xn} in E, the image sequence $\{K_{x_n}\}$ in F has a convergent subsequence.
Theorem:(1.2)A completely continuous linear operator K is continuous.

Proof: If K is not continuous, K is not bounded, sup $\|Kx_n\| = \infty,$ hence there exists a sequence {xn} with $\|x_n\| \leq 1 and \|Kx_n\| \to \infty$ which contradicts that K is completely continuous. But every continuous operator is not completely continuous.
For example: The identity map I on an infinite dimensional normed linear space is continuous but not completely continuous. This can be shown by applying the Riesz lemma.
Theorem:(1.3)A continuous operator K of infinite rank is completely continuous.

Proof: If {xn} is a bounded sequence in E, then the image sequence $\{Kx_n\}$ in F is bounded. Now, since B(K) i.e., the image space of K is finite dimensional, hence B(K) is complete. Henceapplying**Bolzano-Weirestrass** theorem it follows that there exists a convergent subsequence $\{K_{x_n}\}$.

Theorem:(1.4) If E and F are normed linear spaces, K is a linear map of E into F and $S = \{x \in E : \|x\| \leq 1\}$ the closed unit ball of E, then the following properties are equivalent.

a) K is completely continuous
b) K(S) is relatively compact
c) $\overline{K(S)}$ is compact

Proof:(a) ⇒ (b) : If K∈R(E,F) and {yn) a sequence from K(S), then three exists xn, n∈Nwith yn = K.xn, xn∈ S. Since {xn} is bounded sequence, there exists a subsequence $\{K_{x_n}\}$ such that $\{y_{x_n}\} = \{K.x_{n_k}\}$ converges to some y∈F, then K(S) is relatively compact. For equivalence of (c) and (b) we have to prove the additional proposition. If K is completely continuous the U = S gives the desire result.
Conversely, if there exists such a zero nbd in E, then there exists U⊃ρ.S = {x∈E, ||x||<ρ}. Then also K(ρ.S) ∈K(U), hence K(ρS) = ρK(s) lies in a compact set $M.\frac{1}{\rho}.M$ is also compact, as K(s) = $\frac{1}{\rho}$ M, K(s) in relatively compact and K is completely continuous.

Some properties of operators:

1. The sum and scalar multiple of completely continuous operator are completely continuous.
2. The product of completely continuous operator with a continuous operator is completely continuous.
3. A completely continuous operator is continuous
4. A continuous operator of finite rank is completely continuous.
5. A compact operator is continuous.
6. Sum and scalar multiple of compact operator is compact.
7. The product of compact operator with a continuous operator is compact.
8. The product of bounded operator with a continuous operator is bounded.
9. Every finite dimensional continuous operator is compact.

II. Conclusion

The compact and completely continuous operators in normed linear space are same but in topological vector space this is not true. The compact operators in topological vector space has rich properties than completely continuous operators. A completely continuous linear operators is continuous but every continuous operator is not completely continuous, whereas continuous operator with finite rank is completely continuous. A linear operator K of E into F(where E and F are normed linear space) with closed space unit ball of E, then completely continuous, relatively compact and compact operators are equivalent.

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