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# Algebraic Properties of Fuzzy Chromatic Polynomials

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***Abstract:*** *The concept of fuzzy chromatic polynomial $P_\alpha^f(G,k)$ of a fuzzy graph $G$ based on $\alpha$-cuts of $G$ was introduced by Mamo and Srinivasa Rao. The authors studied the concept on a fuzzy graph whose vertex set is crisp and fuzzy. In this paper, we present some properties of fuzzy chromatic polynomials of fuzzy graphs by relating to their coefficients and chromatic roots. Related results are proved.*
***Keywords:*** *Fuzzy graph; Fuzzy chromatic polynomials; Coefficients; Chromatic roots*

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## I. Introduction

In 1912, Birkhoff [1] introduced the concept of the chromatic polynomial on maps in the hope of proving the Four-Color Theorem. Whitney [2] extended the concept from maps to arbitrary graphs. After that, Read [3] studied chromatic polynomials and their properties. Several scholars have published their works on coefficients and chromatic roots of chromatic polynomials (see, [4, 5, 6, 7, 8, 9, 10, 11] ). Lately, Zhang and Dong [12] studied some properties of chromatic polynomials of hypergraphs.

Many real-world problems cannot be properly modeled by the classical graph since the problem contains uncertain information. To handle such a situation, Rosenfeld [13] developed a fuzzy graph theory based on Zadeh's [14] fuzzy sets. A chromatic polynomial in a fuzzy graph, called fuzzy chromatic polynomial. Recently, Mamo and Srinivasa Rao [15] studied the concept of fuzzy chromatic polynomial, $P_\alpha^f(G,k)$ of a fuzzy graph $G$ based on $\alpha$-cuts of $G$. They defined $P_\alpha^f(G,k)$ as the chromatic polynomial of its crisp graph $G_\alpha$. One of the significant result obtained by these authors is that the degree of $P_\alpha^f(G,k)$ is less than or equal to the number of vertices of the fuzzy graph. In this article, we study some properties of fuzzy chromatic polynomials based on their coefficients and roots. The necessary conditions of $P_\alpha^f(G,k)$ proved.

The remainder of the paper is organized as follows. In section 2, we review basic definitions and main results on the fuzzy chromatic polynomial. In section 3, we present algebraic properties of fuzzy chromatic polynomials. Finally, the paper is concluded in Section 4.

## II. Preliminaries

In this section, we review the basic definition of the fuzzy graph and main results on the fuzzy chromatic polynomials. Definitions of the fuzzy graph and related concepts are taken from [16, 17].

**Definition 2.1** A fuzzy graph $G=(V,\sigma,\mu)$ is a triple consisting of a nonempty set $V$ together with a pair of functions $\sigma: V \to [0,1]$ and $\mu: V\,x\,V \to [0,1]$ such that for all $x,y \in V, \mu(x,y) \le \sigma(x) \wedge \sigma(y)$. Here, the fuzzy set $\sigma$ is called the fuzzy vertex set of $G$ and $\mu$ the fuzzy edge set of $G$.

We consider a fuzzy graph $G$ is simple, connected, finite and undirected. *For* notational convenience, we use simply $G$ *or* $G=(\sigma,\mu)$ to represent the fuzzy graph $G=(V,\sigma,\mu)$.

**Definition 2.2** The fuzzy graph $H=(P,\tau,\upsilon)$ is called a fuzz subgraph of $G=(V,\sigma,\mu)$ induced by $P$ if $P \subseteq V, \tau(x)=\sigma(x)$ for all $x \in P$ and $\upsilon(x,y)=\mu(x,y)$ for all $x,y \in P$.

**Definition 2.3** We denote the underlying crisp graph of a fuzzy graph $G$ by $G^*=(\sigma^*,\mu^*)$ where $\sigma^*=\{u \in V : \sigma(u)>0\}$ and $\mu^*=\{(u,v) \in V\,x\,V : \mu(u,v)>0\}$.

**Definition 2.4** The level set of fuzzy set $\sigma$ is defined as $L_\sigma=\{\alpha/\sigma(u)=\alpha \text{ for some } u \in \sigma^*\}$ and the level set of $\mu$ is defined as $L_\mu=\{\alpha/\mu(u,v)=\alpha \text{ for some } (u,v) \in \mu^*\}$. The fundamental set (or level set) of the fuzzy graph $G=(V,\sigma,\mu)$ is defined as $L=L_\sigma \cup L_\mu$.

**Definition 2.5** For each $\alpha \in I=L \cup \{0\}$, $G_\alpha$ denotes the $\alpha$-cut of the fuzzy graph $G$ which is the crisp graph $G_\alpha=(\sigma_\alpha,\mu_\alpha)$ where $\sigma_\alpha=\{u \in V/\sigma(u) \ge \alpha\}$ and $\mu_\alpha=\{(u,v) \in V\,x\,V/\mu(u,v) \ge \alpha\}$.

Mamo and Srinivasa Rao [15] introduced the concept of the fuzzy chromatic polynomial of a fuzzy graph based on α-cuts of the fuzzy graph which are crisp graphs and they defined as follows.

**Definition 2.6** For a fuzzy graph $G$, the fuzzy chromatic polynomial of $G$ is denoted by $P_\alpha^f(G,k)$ and defined as the chromatic polynomial of its crisp graphs $G_\alpha$, for $\alpha \in I$.

That is, $P_\alpha^f(G,k) = P(G_\alpha, k),\ \forall\alpha \in I$.
Some of the results in [15] are as follows.
**Theorem 2.1** Let $G$ be a fuzzy graph with n vertices and $G^*$ be its underlying crisp graph. If $\alpha \in I = L \cup \{0\}$ and $\beta = \min(L)$, then

$$P_\alpha^f(G,k) = \begin{cases} P(K_n,k) & if\ \ \alpha = 0 \\ P(G^*,k) & if\ \alpha = \beta \\ P(G_\alpha,k) & if\ \beta < \alpha \le 1 \end{cases}$$

where, $K_n$ is a complete crisp graph with n vertices.
**Theorem 2.2** Let $G$ be a complete fuzzy graph with n vertices. If $\alpha \in I = L \cup \{0\}$ and $\beta = \min(L)$ then

$$P_\alpha^f(G,k) = \begin{cases} P(K_n,k) & if\ \ \alpha = 0 \text{ and } \beta \\ P(G_\alpha,k) & if\ \beta < \alpha \le 1 \end{cases}$$

**Lemma 2.1** Let $G = (V,\mu)$ be a fuzzy graph with crisp vertices and fuzzy edges. Then the degree of $P_\alpha^f(G,k) = |V|$, for all $\alpha \in I$.
**Theorem 2.3** Let $G = (V,\sigma,\mu)$ be a fuzzy graph. Then the degree of $P_\alpha^f(G,k) \le |V|,\ \forall\alpha \in I$.

## III. Properties of Fuzzy Chromatic Polynomials

In this section, we discuss some properties of fuzzy chromatic polynomials regarding their coefficients and roots.
**Theorem 3.1** Let $G$ be a fuzzy graph and $G_\alpha, \alpha \in I$ be $\alpha$-cuts of $G$. If $G_\alpha$ for some $\alpha \in I$, has connected components $G_1, G_2, \dots, G_r$ then $P_\alpha^f(G,k) = P(G_1,k)P(G_2,k) \dots P(G_r,k)$.
*Proof:* Let $G_\alpha, \alpha \in I$ be $\alpha$-cuts of $G$. Suppose $G_\alpha$, for some $\alpha \in I$ has $r$ connected components say $G_1, G_2, \dots, G_r$. By Theorem 2 of [3], $P(G_\alpha,k) = P(G_1,k)P(G_2,k) \dots P(G_r,k)$. Hence, the result follows from Definition 2.6.

### 3.1 Coefficients of $P_\alpha^f(G,k)$

In this subsection, we will discuss the coefficients of fuzzy chromatic polynomials. Let $G = (\sigma,\mu)$ be a fuzzy graph with $|\sigma^*| = n$ and $|\mu^*| = m$
**Theorem 3.1.1** The leading coefficient of $P_\alpha^f(G,k)$ for all $\alpha \in I$ is 1.
*Proof:* Let G be a fuzzy graph and $G_\alpha, \alpha \in I$ be $\alpha$-cuts of G. Since $G_\alpha$ is a crisp graph for all $\alpha \in I$ and by Theorem 8 of [3], the leading coefficient of $P(G_\alpha,k)$ is 1. Hence, the result immediately holds by Definition 2.6.
**Theorem 3.1.2** The constant term in $P_\alpha^f(G,k)$ for all $\alpha \in I$ is zero.
*Proof:* Let $G_\alpha, \alpha \in I$ be $\alpha$-cuts of the fuzzy graph $G$. By Theorem 9 of [3], $P(G_\alpha,k)$ has no constant term for all $\alpha \in I$. Hence, the constant term in $P_\alpha^f(G,k)$ for all $\alpha \in I$ is zero from Definition 2.6.
**Theorem 3.1.3** The coefficient of $k^{n-1}$ in $P_\alpha^f(G,k)$ for all $\alpha \in I$ is the negative of the number of edges in $G_\alpha$.
*Proof:* Let $G$ be a fuzzy graph with $|\sigma^*| = n$. Let $G_\alpha, \alpha \in I$ be $\alpha$-cuts of the fuzzy graph $G$. By Theorem 11 of [3], the coefficient of $k^{n-1}$ in $P(G_\alpha,k)$ is $-|E(G_\alpha)|$. Hence, the result immediately holds by Definition 2.6.
**Theorem 3.1.4** The coefficient of $k$ in $P_\alpha^f(G,k)$ is nonzero if and only if $G_\alpha$ is connected for some $\alpha \in I$.
*Proof:* Let $G_\alpha, \alpha \in I$ be $\alpha$-cuts of the fuzzy graph $G$. Suppose $G_\alpha$ is disconnected for some $\alpha \in I$. We need to show the coefficient of $k$ in $P_\alpha^f(G,k)$ is zero. By Theorem 3.1, the chromatic polynomial of $G_\alpha$ is the product of the chromatic polynomial of its connected components. If we have at least two terms, each being divisible by $k$, then their product is divisible by $k^2$. Hence, the coefficient of $k$ in $P(G_\alpha,k)$ is zero. From Definition 2.6, the coefficient of $k$ in $P_\alpha^f(G,k)$ is also zero.
Conversely, suppose $G_\alpha$ is connected for some $\alpha \in I$. This show that all coefficient of $P(G_\alpha,k)$ except the constant one is nonzero. Specifically, the coefficient of k in $P(G_\alpha,k)$ is nonzero. From Definition 2.6, the coefficient of $k$ in $P_\alpha^f(G,k)$ is nonzero.
**Theorem 3.1.5** The coefficients of $P_\alpha^f(G,k)$ for all $\alpha \in I$ are alternate in sign.
*Proof:* Let $G_\alpha, \alpha \in I$ be $\alpha$-cuts of the fuzzy graph $G$. By Theorem 10 of [3], the coefficients of $P(G_\alpha,k)$ for $\alpha \in I$ are alternate in sign. Hence, the result holds from Definition 2.6.
**Theorem 3.1.6** If $G_\alpha$ is a connected $\alpha$-cuts graph of G then the absolute value of the coefficient of $k^r$ in $P_\alpha^f(G,k)$ is not less than $\binom{p-1}{r-1}$.
*Proof:* Let $G_\alpha$ be connected $\alpha$-cuts of the fuzzy graph $G$ with $|V(G_\alpha)| = p$. By Theorem 14 of [3], the absolute value of the coefficient of $k^r$ in $P(G_\alpha,k)$ is not less than $\binom{p-1}{r-1}$. since $P_\alpha^f(G,k) = P(G_\alpha,k)$ for $\alpha \in I$ by Definition 2.6 and $G_\alpha$ is connected, the result immediately holds.

**Corollary 3.1.1** The smallest number $r$ such that $k^r$ has a nonzero coefficient in $P_\alpha^f(G,k)$ is the number of components of $G_\alpha$.

### 3.2 Chromatic Roots of $P_\alpha^f(G,k)$

In this subsection, we discuss some properties of fuzzy chromatic polynomials concerning their chromatic roots.

**Definition 3.2.1** Let $G$ be a fuzzy graph and $k \in \mathbb{R}$. $k$ is said to be chromatic root of $P_\alpha^f(G,k)$ if $P_\alpha^f(G,k) = 0$ for $\alpha \in I$.

Since the evaluations of the fuzzy chromatic polynomial $P_\alpha^f(G,k)$ of a fuzzy graph for each $\alpha \in I$ count the number of colorings of a crisp graph $G_\alpha$, it follows that the numbers $0, 1, \dots, \chi_\alpha - 1$ are always chromatic roots of $P_\alpha^f(G,k)$.

**Theorem 3.2.1** Let $G$ be a fuzzy graph. The multiplicity of 0 as a root of $P_\alpha^f(G,k)$ equals the number of connected components in $G_\alpha$ of $G$.

*Proof:* Let $G_\alpha, \alpha \in I$ be $\alpha$-cuts of the fuzzy graph $G$. We know that the multiplicity of 0 as a root of $P(G_\alpha, k)$ for all $\alpha \in I$ equals the number of connected components of $G_\alpha$. By Definition 2.6, the result immediately holds.

**Theorem 3.2.2** Let $G$ be a fuzzy graph. The $P_\alpha^f(G,k)$ has no real roots in the interval $(-\infty, 0)$.

*Proof:* Let $G_\alpha, \alpha \in I$ be $\alpha$-cuts of the fuzzy graph G. By [18], the $P(G_\alpha, k)$ for all $\alpha \in I$ has no negative real roots. Hence the result follows from Definition 2.6.

**Theorem 3.2.3** Let $G$ be a fuzzy graph. The $P_\alpha^f(G,k)$ has no real roots in the interval $(0, 1)$.

*Proof:* Let $G_\alpha, \alpha \in I$ be $\alpha$-cuts of the fuzzy graph G. By [19], the $P(G_\alpha, k)$ for all $\alpha \in I$ has no real roots between 0 and 1. Hence the result follows from Definition 2.6.

**Theorem 3.2.4** Let $G$ be a fuzzy graph. The $P_\alpha^f(G,k)$ has no real roots in the interval $\left(1, \frac{32}{27}\right]$.

*Proof:* Let $G_\alpha, \alpha \in I$ be $\alpha$-cuts of the fuzzy graph $G$. The $P(G_\alpha, k)$ for all $\alpha \in I$, has no real roots in the interval $\left(1, \frac{32}{27}\right]$ by Theorem 5 of [7]. Hence the result follows from Definition 2.6.

**Theorem 3.2.5** Let $G$ be a fuzzy graph. The real roots of all $P_\alpha^f(G,k)$ are dense in $[\frac{32}{27}, \infty)$.

*Proof:* Let $G_\alpha, \alpha \in I$ be $\alpha$-cuts of the fuzzy graph $G$. The real roots of all $P(G_\alpha, k)$ for all $\alpha \in I$, are dense in $[\frac{32}{27}, \infty)$ by Theorem 2.5 of [20]. Hence the result follows from Definition 2.6.

**Theorem 3.2.6** Let $G$ be a fuzzy graph. The roots of all $P_\alpha^f(G,k)$ are dense in the complex plane.

*Proof:* Let $G$ be a fuzzy graph and $G_\alpha$ be $\alpha$-cuts of the fuzzy graph $G$. By [21], the roots of all chromatic polynomials are dense in the complex plane. Specifically, the roots of $P(G_\alpha, k)$ for all $\alpha \in I$ are dense in the complex plane. From Definition 2.6 the result immediately holds.

## IV. Conclusion

In this paper, some properties of fuzzy chromatic polynomials are discussed. Some results on coefficients and roots of fuzzy chromatic polynomials are proved based on the definition of a fuzzy chromatic polynomial and previously proved results of the chromatic polynomial of a crisp graph.

## References

[1]. G.D. Birkhoff, "A determinant formula for the number of ways of coloring a map," *Annals of Mathematics*, 1912, 14: 42-46.
[2]. H. Whitney, "The colouring of graphs," *Annals of Mathematics*, 1932, 33: 688-718.
[3]. R.C. Read, " An introduction to chromatic polynomials," *Journal of Combinatorial Theory*, 1968, 4: 52-71.
[4]. C.H. Meredith, "Coefficients of Chromatic Polynomials," *Journal of Combinatorial Theory (B)*, 1972, 13: 14-17.
[5]. E. J. Farrell, "On chromatic coefficients," *Discrete Mathematics*, 1980, 29: 257-264.
[6]. E.J. Farrell, "Chromatic roots-some observations and conjectures," *Discrete Mathematics*, 1980, 29: 161-167.
[7]. B. Jackson, "A zero- free interval for chromatic polynomials of graphs," *Combinatorics, probability and computing*, 1993, 2(3): 325-336.
[8]. D.R. Woodall, "A zero-free interval for chromatic polynomials," *Discrete Mathematics*, 1992, 101: 333-341.
[9]. J.I. Brown, "On the roots of chromatic polynomials," *Journal of combinatorial Theory, Series B*, 1998, 72: 251-256.
[10]. H. Bielak, "Roots of chromatic polynomials," *Discrete Mathematics*, *2001*, 231: 97-102.
[11]. P.J. Cameron and K. Morgan, "Algebraic properties of chromatic roots," *The Electronic Journal of Combinatorics,2017*, 24(1): 1-21.
[12]. R. Zhang and F. Dong, "Properties of chromatic polynomials of hypergraphs not held for chromatic polynomials of graphs," *European Journal of Combinatorics, 2017*, 64: 138-151.
[13]. Rosenfeld, " Fuzzy graphs," in Fuzzy Sets and Their Applications, L. A. Zadeh, K.S. Fu, and M. Shimura, Eds., *Academic Press*, pp. 77-95, 1975.
[14]. L. A. Zadeh, "Fuzzy sets," *Information and Computation,1965*, 8(3): 338-353.
[15]. A.A. Mamo and R.V.N. Srinivasa Rao, "Fuzzy chromatic polynomial of fuzzy graphs with crisp and fuzzy vertices using α-cuts," *Advances in Fuzzy Systems*, *2019*, 2019(Article ID 5213020): 11 pages.
[16]. J.N. Mordeson and P.S. Nair, *Fuzzy Graphs and Fuzzy Hypergraphs*, Heidelberg: Physica-Verlag, Germany, 2000.
[17]. S. Mathew, J.N. Mordeson, and D.S. Malik, " Fuzzy graphs," in Fuzzy graph Theory, vol. 363, pp. 13-83, Springer International Publishing AG, 2018.

[18]. R.C. Read and W. T. Tutte, Chromatic polynomials. *In Selected Topics in Graph Theory*, Volume 3 (L. W. Beineke and R.J. Wilson, eds), New York: Academic Press, pp. 15-42, 1988

[19]. W.T. Tutte, "Chromails," in *Lecture Notes in Mathematics*, vol. 411, New York, Springer-Verlag, pp. 243-266, 1974.

[20]. C.Thomassen, "The zero-free intervals for chromatic polynomials of graphs," *Combinatorics, Probability and Computing, 1997,* 6: 497-506.

[21]. A.D. Sokal, "Chromatic roots are dense in the whole complex plane," *Combinatorics, Probability, and Computing, 2004,* 13(2): 221-261.


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