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Mathematical Eco-Epidemic Model on Prey-Predator System

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Abstract: *This paper aims to study the Spread of disease in prey-predator system with treatment given to infected prey and predator population only .For these problem a system of five ordinary differential Equations has been proposed . Positivity, Boundedness of model equation has been analyzed. Existence of the solution has been checked using* Derrick and Groosman theorem. *Stability of all possible Equilibrium points of the model has been done. Local and global stability of disease free and endemic equilibrium points are performed. Numerical simulations are presented to clarify analytical results.*

Keywords: *Mathematical eco-epidemiology, predator-prey system, Local Stability, Routh –Hurwitz Criterion, Reproduction Number, Simulation Study.*

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I. Introduction

However, in ecosystem, the interaction between the predator and prey is a nonlinear and complex process [1]. Mathematical study of such predator-prey populations has attracted attentions of ecologists and mathematicians from several years back [7]. As a result many mathematical model Equations have been developed, and these models have become a vital tools in analyzing the interaction of different prey-predator populations [3,7].There are Many a predator-prey interaction of species can be observed in ecological system throughout the world, such as a fox-rabbit relation, Cat -rate relation, Engel-hen relation are some of them[8]. In a normal life, predator - prey species in a given ecosystem exhibit regular cycles of abundance or population increase and decrease [7].

Ecological prey-predator systems suffer from various infectious diseases [1, 7]. These diseases sometimes play a significant role in regulating size of prey-predator population [1, 7]. Within a predator-prey population, it is often to see that a parasite spreads between prey to prey, prey to predator, and predator to predator and all the populations becomes disease affected. See for example rabbit fever and deer fly fever [13, 17]. The prey populations could be affected due to the presence of both parasites and predators [8].

Modified prey-predator models with disease have been introduced, example, the disease in prey [18], predators consume only infected preys [5], predators avoid infected prey [12], the disease in predators only [14], predators consume both Susceptible and infected preys but with Cosner type functional response [18].

In this present paper, we are going to study an eco-epidemiological mathematical model with treatment and disease infection in both Prey and Predator population with the assumption that diseases can be transmitted between susceptible predator and infected predator population by contact**.** This is an extension of the study of eco-epidemiological model which was studied by [1].

II. Model Formulation

The Prey-predator system contains two classes of populations. Let us denote the prey population density by $F(t)$ since it is source of food for predator population. Let $X(t)$ denotes susceptible prey, $W(t)$ denotes infected prey then the total population of the prey is given by

$$F(t) = X(t) + W(t) \tag{1}$$

Let $P(t)$ denotes population density of the predator, and Y(t) denotes susceptible predator, $z(t)$ denotes infected predator. Then the total population of the predator is given by

$$P(t) = Y(t) + Z(t) \tag{2}$$

Furthermore let the infected prey and infected predator population under treatment be denoted by $T(t)$. Then the total population under treatment is given by

$$T(t) = W(t) + Z(t). \tag{3}$$

Also, the total number of prey and predator populations is given by

$$N(t) = F(t) + P(t) = X(t) + W(t) + Y(t) + Z(t)$$

(4)

In formulating the present model, the following assumptions have been made use of

(i) When there is disease, the prey and predator population consists of two subclasses each: $X(t)$ susceptible prey, $W(t)$ infected prey,Y(t) susceptible predator, $z(t)$ infected predator [1].

(ii) When there is no diseases, prey population grows logistically with intrinsic growth rate $r > 0$ and environmental carrying capacity $k > 0$.

(iii) Only susceptible prey $X(t)$ can reproduce, and infected prey removed with death rate μ_2 before predation, or before going to treatment and reproducing. However infected population $W(t)$ Contributes with $X(t)$ to population growth towards carrying capacity k.

(iv) Diseases spread among the prey populations and can be transmitted to predator during predation. Moreover the disease is not genetically inherited and infected populations can only recover through treatment [1].

(v) Susceptible prey becomes infected, when it comes in contact with the infected prey and this contact process is assumed to follow simple mass action kinetics with convolution rate β [1]**.**

(vi) Susceptible predators become infected predator, when it comes in contact with the infected predator and this contact process is assumed to follow simple mass action kinetics with convolution rate α.

(vii) The predator population $Y(t)$ and $Z(t)$ suffer due to the death rate μ_3 and μ_4 respectively.

(viii) The predation functional response of the predator towards susceptible prey as well as infected prey are assumed to follow Michaelis – Menten kinetics and modeled using a Holling type II functional form with p_1, p_2, p_3 be predation coefficients and m be half-saturated constant **.** Consumed prey is converted into predator with efficiency q [1, 8].

(ix) Infected prey and infected predators have (i) treatment rates of a_1 and a_2 respectively (ii) removed without immunity(recovery rates) at rate of b_1 and b_2 respectively and (iii) death at rate of both population under treatment μ_1

Table 1 Notation and Description of model Variables

Variables	Descriptions
X(t)	Population size of susceptible prey
W(t)	Population size of infected prey
Y(t)	Population size of susceptible predator
Z(t)	Population size of infected predator
$T(t)$	Population size of both infected prey and infected predator under treatment

Table 2 Notations and Description of model parameters

Parameter	Description of parameter
r	Intrinsic growth rate of susceptible prey
k	Carrying capacity of susceptible prey
α	Convolution rate of susceptible predator to be infected predator
β	Convolution rate of susceptible prey to be infected prey
a_1	Treatment rate of infected prey
a_2	Treatment rate of infected predator
b_1	Recovery rate of infected prey
b_2	Recovery rate of infected predator
p_1	Predation coefficient of susceptible prey due to susceptible predator
p_2	Predation coefficients of infected prey due to predators(susceptible predator & infected predator)
p_3	Predation coefficient of susceptible prey due to infected predator.
q	Efficiency of predation
m	Half-saturation constant
μ_1	Death rate of both infected prey and infected predator under treatment
μ_2	Death rate of infected prey
μ_3	Death rate of susceptible predator
μ_4	Death rate of infected predator

According to the above assumptions, the description of variables and parameters the present model will have the flow diagram given in Figure 1.

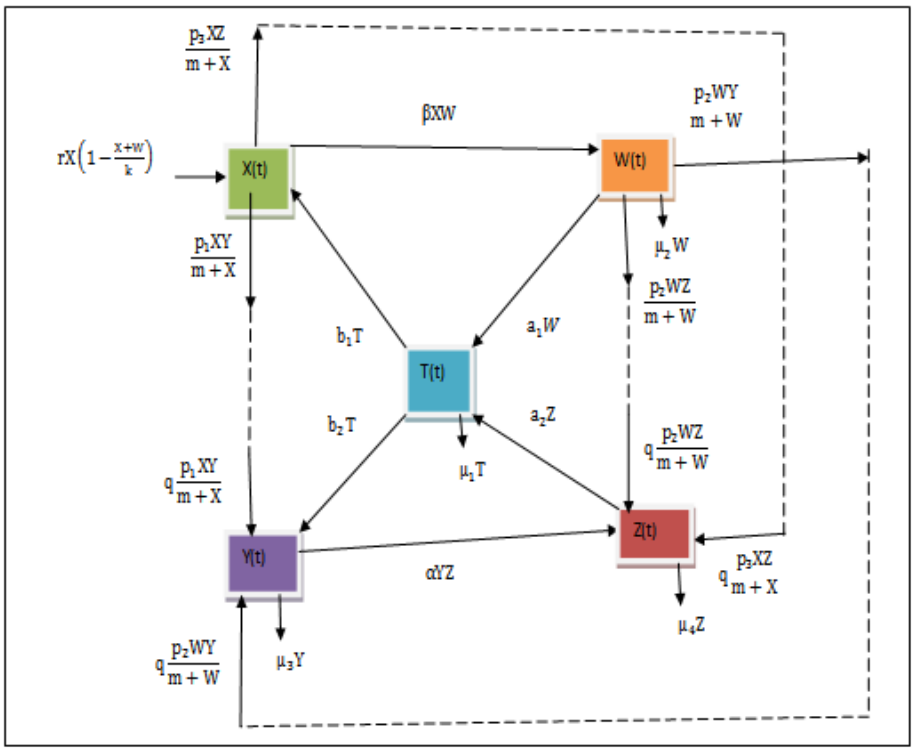

Figure 1 Model Flow Chart

From the flow chart the model will be governed by the following equations

$$dX/dt = rX\{1 - [(X + W)/k]\} + b_1T - \beta XW - [p_1XY/(m + X)] - [p_3XZ/(m + X)] \quad (5)$$

$$dW/dt = \beta XW - a_1W - \mu_2W - [p_2WY/(m + W)] - [p_2WZ/(m + W)] \quad (6)$$

$$dY/dt = [qp_1XY/(m + X)] + [qp_2WY/(m + W)] + b_2T - \alpha YZ - \mu_3Y \quad (7)$$

$$dZ/dt = [qp_2WZ/(m + W)] + [qp_3XZ/(m + W)] + \alpha YZ - a_2Z - \mu_4Z \quad (8)$$

$$dT/dt = a_1W + a_2Z - \mu_1T - b_1T - b_2T \quad (9)$$

The initial conditions here are $X(0) = X_o \geq 0$, $W(0) = W_0 \geq 0$, $Y(0) = Y_0 \geq 0$, $Z(0) = Z_0 \geq 0$, $T(0) = T_o \geq 0$, $p_1, p_2, p_3 > 0$ and $0 < q \leq 1$.

III. Model Analysis

Model (5) – (9) will be analyzed qualitatively to get insight into its dynamical features which will give better understanding on the effect of treatment of an infected prey and infected predator populations. In this section, we are going to analysis the following features of the model: Positivity, Boundedness and Existence of solutions, Trivial Equilibrium point, Axial Equilibrium point, Disease free equilibrium points/boundary equilibrium points, Endemic equilibrium points, Global stability of disease free equilibrium point and Local stability of endemic equilibrium point. All these concepts are presented and discussed in the following sub-sections.

3.1 Positivity of solutions

For model (5) - (9) to be epidemiologically meaningful and well posed, it is necessary to prove that all solutions of system with positive initial data will remain positive for all times $t > 0$. This will be established by the following theorem [1]:

Theorem 1 Positivity Let $X(0) > 0$, $W(0) > 0$, $Y(0) > 0$, $Z(0) > 0$, $T(0) > 0$. Then the solutions $X(t)$, $W(t)$, $Y(t)$, $Z(t)$, $T(t)$ of system equations (5) – (9) are positive $\forall t \geq 0$.

Proof: Positivity of the model variables is shown separately for each of the model variables $X(t)$, $W(t)$, $Y(t)$, $Z(t)$ and $T(t)$.

Positivity of $X(t)$: The model equation (5) given by $dX/dt = rX\{1 - [(X + W)/k]\} + b_1T - \beta XW - [p_1XY/(m + X)] - [p_3XZ/(m + X)]$ can be expressed without loss of generality, after eliminating the positive terms $(rX + b_1T)$ which are appearing on the right hand side, as an inequality as $dX/dt \geq -\{[rX^2 + rW]/k + \beta WX + [(p_1Y + p_3Z)X/(m + X)]\}$. This inequality can also be written $(dX/\beta WX) \geq dX/\{[rX^2 + rW]/k + \beta WX + [(p_1Y + p_3Z)X/(m + X)]\} \geq -dt$. Then we have $dX/\beta WX \geq -dt$ using separation of variable method and on applying integration, the solution of the foregoing differentially inequality can be obtained as $X(t) \geq [1/\beta W]\exp(-\beta Wt)$. Recall that an exponential function is always non–negative irrespective of the sign of the exponent, Hence, it can be concluded that $X(t) \geq 0$.

Positivity of $W(t)$: The equation (6) given by $dW/dt = \beta XW - a_1W - \mu_2W - [p_2WY/(m + W)] - [p_2WZ/(m + W)]$ can be expressed without loss of generality, after eliminating the positive term βXW which are appearing on the right hand side, as an inequality as $dW/dt \geq -\{(a_1 + u_2)W + [p_2\,W(Y + Z)/(m + W)]\}$. This inequality can be written as $(dW/a_1W) \geq dW/\{(a_1 + u_2)W + [p_2\,W(Y + Z)]/(m + W)\} \geq -dt$ hence $dW/a_1W \geq -dt$ using variables separable method and on applying integration, the solution of the foregoing differential inequality can be obtained as $W(t) \geq \exp(-a_1t)$. Recall that an exponential function is always non–negative irrespective of the sign of the exponent, i.e., the exponential function $\exp(-a_1t)$ is a non-negative quantity. Hence, it can be concluded that $W(t) \geq 0$.

Positivity of $Y(t)$: The model equation (7) given by $dY/dt = [qp_1XY/(m + X)] + [qp_2WY/(m + W)] + b_2T - \alpha YZ - \mu_3Y$ can be expressed without loss of generality, after eliminating the positive term $qp_1XY/[m + X] + qp_2WY/[m + W] + b_2T$ which are appearing on the right hand side, as an inequality as $dY/dt \geq -\alpha YZ - \mu_3Y$ This inequality can be written as $dY/dt \geq -(\alpha Z + \mu_3)Y$ hence $dY/Y \geq -(\alpha Z + \mu_3)dt$. Using variables separable method and on applying integration, the solution of the foregoing differentially inequality can be obtained as $Y(t) \geq \exp[-(\alpha Z + \mu_3)t]$. Recall that an exponential function is always non–negative irrespective of the sign of the exponent, i.e., the exponential function $\exp[-(\alpha z + \mu_3)t]$ is a non-negative quantity. Hence, it can be concluded that $Y(t) \geq 0$.

Positivity of $Z(t)$: The model equation (8) given by $dZ/dt = [qp_2WZ/(m + W)] + [qp_3XZ/(m + W)] + \alpha YZ - a_2Z - \mu_4Z$ can be expressed without loss of generality, eliminating the positive term $q\,p_2WZ/[m + W] + q\,p_3XZ/[m + W] + \alpha YZ$ which are appearing on the right hand side, as an inequality as $dZ/dt \geq -(a_2 + \mu_4)Z$. Using variables separable method and on applying integration, the solution of the foregoing differentially inequality can be obtained as $Z(t) \geq \exp[-(a_2 + \mu_4)t]$ Recall that an exponential function is always non–negative irrespective of the sign of the exponent, i.e., the exponential function $\exp[-(a_2 + \mu_4)t]$ is a non-negative quantity. Hence, it can be concluded that $Z(t) \geq 0$.

Positivity of $T(t)$: The model equation (9) given by $dT/dt = a_1W + a_2Z - \mu_1T - b_1T - b_2T$ can be expressed without loss of generality, after eliminating the positive term $(a_1W + a_2Z)$ which are appearing the right hand side, as an inequality as $dT/dt \geq -(\mu_1 + b_1 + b_2)T$. Using variables separable method and on applying integration, the solution of the foregoing differentially inequality can be obtained as $T(t) \geq \exp[-(\mu_1 + b_1 + b_2)t]$. Recall that an exponential function is always non–negative irrespective of the sign of the exponent, i.e., the exponential function $\exp[-(\mu_1 + b_1 + b_2)t]$ is a non-negative quantity. Hence, it can be concluded that $T(t) \geq 0$.

Thus, the model variables $X(t)$, $W(t)$, $Y(t)$, $Z(t)$ and $T(t)$ representing population sizes of various types of prey and predator are positive quantities and will remain in $\mathbb{R}_+^5$ for all t.

3.2 Boundedness of the Model

In the theoretical eco-epidemiology, the boundedness of the system implies that the system is biologically valid and well behaved. Then, we first show the biological validity of the model by providing the Boundedness of the solution of the model (5) – (9) by the following theorem [1]:

Theorem 2 Boundedness *All* solutions of the model (5) – (9) are uniformly bounded.

Proof: To show that each population size is bounded if and only if the total population size is bounded. Hence, it is sufficient to prove that the total population size $N = X(t) + W(t) + Y(t) + Z(t) + T(t)$ is bounded for all t. Now, summation of all the five model equations (5) – (9) $dN(t)/dt = (dX/dt) + (dW/dt) + (dY/dt) + (dZ/dt)$ gives $[dN(t)/dt + \eta N(t)] \leq rX + [qp_1XY/(m + X)] + [qp_2WY/(m + W)] + [qp_2WZ/(m + W)] + [qp_3XZ/(m + X)] + \eta N(t) = \mu$. It can be shown that all feasible solutions are uniformly bounded in a proper subset $\Omega \in \mathbb{R}_+^5$ where the feasible region Ω is given by $\Omega = \{(X,\ W,\ Y,\ Z,\ T) \in \mathbb{R}_+^5;\ N \leq (\mu/\eta)\}$. Without loss of generality, after eliminating the negative terms which are appearing on the right hand side, the foregoing equation can be expressed as an inequality as $dN(t)/dt \leq [\mu - \eta N(t)]$. Equivalently this inequality can be expressed as a linear ordinary differential inequality as general solution upon solving as $0 \leq N(X, W, Y, T) \leq [\mu/\eta][1 - \exp(-nt)] + N(0)\exp(-nt)$. But, the term $N(0)$ denotes the initial values of the

respective variable i.e., $N(t) = N(0)$ at $t = 0$. Thus, the particular solution can be expressed as $N(t) \leq [\mu/\eta][1 - \exp(-nt)] + N(0)\exp(-nt)$. Further, it can be observed that $N(t) \to (\mu/\eta)$ as $t \to \infty$. That is, the total population size $N(t)$ takes off from the value $N(0)$ at the initial time $t = 0$ and ends up with the bounded value (μ/η) as the time t grows to infinity. Thus it can be concluded that $N(t)$ is bounded as $0 \leq N(t) \leq (\mu/\eta)$. Therefore, (μ/η) is an upper bound of $N(t)$. Hence, feasible solution of the system of model equations (5) – (9) remains in the positively invariant region Ω. Thus, the system is biologically meaningful and mathematically well posed in the domain Ω. Further, it is sufficient to consider the dynamics of the populations represented by the model system (5) – (9) in that domain. This proves the theorem. Therefore, it can be summarized the result of Theorem2 as "the model variables $X(t)$, $W(t)$, $Y(t)$, $Z(t)$ and $T(t)$ are bounded for all t.

Theorem 3 Existence Solutions of the model equations (5) – (9) together with the initial conditions $X(0) > 0$, $W(0) \geq 0$, $Y(0) \geq 0$, $Z(0) \geq 0, T(0) \geq 0$ exist in $\mathbb{R}_+^5$ i.e., the model variables $X(t)$, $W(t)$, $Y(t)$, $Z(t)$ and $T(t)$ exist for all t and will remain in $\mathbb{R}_+^5$.

Proof: Let the system of equation (5) – (9) be as follows:

$f_1 = rX\{1 - [(X + W)/k]\} + b_1T - \beta XW - [p_1XY/(m + X)] - [p_3XZ/(m + X)]$

$f_2 = \beta XW - a_1W - \mu_2W - [p_2WY/(m + W)] - [p_2WZ/(m + W)]$

$f_3 = [qp_1XY/(m + X)] + [qp_2WY/(m + W)] + b_2T - \alpha YZ - \mu_3Y$

$f_4 = [qp_2WZ/(m + W)] + [qp_3XZ/(m + W)] + \alpha YZ - a_2Z - \mu_4Z$

$f_5 = a_1W + a_2Z - \mu_1T - b_1T - b_2T$

According to Derrick and Groosman theorem, let Ω denote the region $\Omega = \{(X, W, Y, Z, T) \in \mathbb{R}+5;\ N \leq \mu\eta$. Then equations (5) – (9) have a unique solution if $\partial fi\partial xj,\ \forall\ i, j=1, 2, 3, 4, 5$ are continuous and bounded in Ω.. Here $x_1 = X$, $x_2 = W$, $x_3 = Y$, $x_4 = Z$, $x_5 = T$. The continuity and the boundedness can be verified as follows:

For f_1:

$|(\partial f_1)/(\partial X)| = |r(1 - [X + W]/k) - rX/k - \beta W - m(p_1Y + p_3Z)/(m + X)^2| < \infty$

$|(\partial f_1)/(\partial W)| = |-(r + k\beta)X/k| < \infty$

$|(\partial f_1)/(\partial Y)| = |-p_1X/[m + X]| < \infty$

$|(\partial f_1)/(\partial Z)| = |-p_3X/[m + X]| < \infty.$

$|(\partial f_1)/(\partial T)| = |b_1| < \infty$

For f_2:

$|(\partial f_2)/(\partial X)| = |\beta W| < \infty$

$|(\partial f_2)/(\partial W)| = |\beta X - (a_1 + \mu_2) - mp_2Y/(m + W)^2| < \infty$

$|(\partial f_2)/(\partial Y)| = |-p_2W/[m + W]| < \infty$

$|(\partial f_2)/(\partial Z)| = 0 < \infty$

$|(\partial f_2)/(\partial T)| = 0 < \infty$

For f_3:

$|(\partial f_3)/(\partial X)| = |mqp_1Y/(m + X)^2| < \infty$

$|(\partial f_3)/(\partial W)| = 0 < \infty$

$|(\partial f_3)/(\partial Y)| = |qp_1X/[m + X] - \mu_3 - \alpha Z| < \infty$

$|(\partial f_3)/(\partial Z)| = 0 < \infty$

$|(\partial f_3)/(\partial T)| = b_2$

For f_4:

$|(\partial f_4)/(\partial X)| = |mqp_3Z/(m + X)^2| < \infty$

$|(\partial f_4)/(\partial W)| = |mqp_2Z/(m + W)^2| < \infty$

$|(\partial f_4)/(\partial Y)| = 0 < \infty$

$|(\partial f_4)/(\partial Z)| = |qp_2W/[m + W] + qp_3X/[m + X] - (a_2 + \mu_4)| < \infty$

$|(\partial f_4)/(\partial T)| = 0 < \infty$

For f_5:

$|(\partial f_5)/(\partial X)| = 0 < \infty$

$|(\partial f_5)/(\partial W)| = a_1$

$|(\partial f_5)/(\partial Y)| = 0 < \infty$

$|(\partial f_5)/(\partial Z)| = |a_2| < \infty$

$|(\partial f_5)/(\partial T)| = |-(\mu_1 + b_1 + b_2)| < \infty$

Thus, all the partial derivatives $(\partial f_i)/(\partial x_j)$, $\forall\ i,\ j = 1, 2, 3, 4, 5$ exist, continuous and bounded in Ω.. Hence, by Derrick and Groosman theorem, a solution for the model (5) – (9) exists and is unique.

3.3 Equilibrium Points

Disease free equilibrium point of model (5) – (9) is obtained by solving $dX/dt = dW/dt = dY/dt = dZ/dt = dT/dt = 0$. Model (5) – (9) possesses the following equilibrium points: (i) Trivial equilibrium point $E_T(0,\ 0,\ 0,\ 0,\ 0)$ (ii) Axial equilibrium point $E_A(k,\ 0,\ 0,\ 0,\ 0)$ (iii) Boundary equilibrium points: $E_{B1}(\bar{X},\ 0,\ \bar{Y},\ 0,\ 0)$ -disease free equilibrium point and $E_{B2}(X^*,\ W^*,\ 0,\ 0,\ T^*)$ predator free equilibrium point, and prey free equilibrium point $E_{B3}(0,\ 0,\ Y^*,\ Z^*,\ T^*)$, (iv) co-existence equilibrium point/endemic equilibrium point or positive equilibrium point $E^* = (X^*,\ W^*,\ Y^*,\ Z^*,\ T^*)$ equilibrium points are presented in [1].

1.3.1 Stability Analysis of Equilibrium Points

The local stability can be established by linearization of the model equations using Jacobian matrix.

Let $F(X, W, Y, Z, T) = rX\{1 - [(X + W)/k]\} + b_1T - \beta XW - [p_1XY/(m + X)] - [p_3XZ/(m + X)]$

$G(X, W, Y, Z, T) = \beta XW - a_1W - \mu_2W - [p_2WY/(m + W)] - [p_2WZ/(m + W)]$

$H(X, W, Y, Z, T) = [qp_1XY/(m + X)] + [qp_2WY/(m + W)] + b_2T - \alpha YZ - \mu_3Y$

$I(X, W, Y, Z, T) = [qp_2WZ/(m + W)] + [qp_3XZ/(m + W)] + \alpha YZ - a_2Z - \mu_4Z$

$J(X, W, Y, Z, T) = a_1W + a_2Z - \mu_1T - b_1T - b_2T$

The Jacobian matrix of the foregoing functions is given by

$$B(X, W, Y, Z) = \begin{pmatrix} F_X & F_W & F_Y & F_Z & F_T \\ G_X & G_W & G_Y & G_Z & G_T \\ H_X & H_W & H_Y & H_Z & H_T \\ I_X & I_W & I_Y & I_Z & I_T \\ J_X & J_W & J_Y & J_Z & J_T \end{pmatrix}$$

Here the components of the matrix $B(X, W, Y, Z)$ are partial derivatives and they represent the following parametric expressions:

$F_X = r[1 - (X + W)/k] - (rX/k) - \beta W - [m(p_1Y + p_3Z)/(m + X)^2]$

$F_W = -[(r + k\beta)X]/k$

$F_Y = -p_1X/(m + X)$

$F_Z = -p_3X/(m + X)$

$F_T = b_1$

$G_X = \beta W$

$G_W = \beta X - (a_1 + \mu_2) - [mp_2Y/(m + W)^2]$

$G_Y = -p_2W/(m + W)$

$G_Z = G_T = 0$

$H_X = [mqp_1Y/(m + X)^2]$

$H_Y = qp_1X/(m + X) - \mu_3 - \alpha Z$

$H_Z = H_W = 0$

$H_T = b_2$

$I_X = mqp_3Z/(m + X)^2$

$I_W = mqp_2Z/(m + W)^2$

$I_z = [qp_2W/(m + W)] + [qp_3X/(m + X)] - (a_2 + \mu_4)$

$I_T = I_Y = 0$

$J_X = J_Y = 0$

$J_W = a_1$

$J_Z = a_2$

$J_T = -(\mu_1 + b_1 + b_2)$

Theorem 4 [Trivial Equilibrium Point] The trivial equilibrium point $E_T = (0 \quad 0 \quad 0 \quad 0 \quad 0)$ is unstable.
Proof: Consider the Next generation matrix at E_T and it takes the form as

$$B(E_T) = \begin{pmatrix} r & 0 & 0 & 0 & b_1 \\ 0 & -(a_1+\mu_2) & 0 & 0 & 0 \\ 0 & 0 & -\mu_3 & 0 & b_2 \\ 0 & 0 & 0 & -(a_2+\mu_4) & 0 \\ 0 & a_1 & 0 & a_2 & -(\mu_1+b_1+b_2) \end{pmatrix}$$

Now, the eigenvalues of $B(E_T)$ are found by solving the corresponding characteristic equation $\det[B(E_T) - \lambda I5=0$ as follows:

$$\begin{vmatrix} r-\lambda & 0 & 0 & 0 & b_1 \\ 0 & -(a_1+\mu_2)-\lambda & 0 & 0 & 0 \\ 0 & 0 & -\mu_3-\lambda & 0 & b_2 \\ 0 & 0 & 0 & -(a_2+\mu_4)-\lambda & 0 \\ 0 & a_1 & 0 & a_2 & -(\mu_1+b_1+b_2)-\lambda \end{vmatrix} = 0$$

The characteristic equation for the given model at trivial equilibrium point thus takes the form as $(r - \lambda - a1 + \mu 2 - \lambda - \mu 3 - \lambda - a2 + \mu 4 - \lambda - \mu 1 + b1 + b2 - \lambda = 0$. The eigenvalues are then obtained to be

$$\lambda_1 = r, \quad \lambda_2 = -(a_1+\mu_2), \quad \lambda_3 = -\mu_3, \quad \lambda_4 = -(a_2+\mu_4), \quad \lambda_5 = -(\mu_1+b_1+b_2)$$

Here four eigenvalues are negative and one eigenvalue is positive so the trivial equilibrium point is not stable.

Theorem 5 [**Axial Equilibrium Point**] Axial equilibrium point $E_A = (k, \ 0, \ 0, \ 0, \ 0)$ is stable if the following three conditions hold (i) $\beta k - (a_1+\mu_2) < 0$ (ii) $qp_1 k - \mu_3(m+k) < 0$ and (iii) $qp_3 k - (a_2 + \mu_4)(m+k) < 0$.
Proof: Consider the Jacobian matrix at axial equilibrium point $B(E_A)$

$$B(E_A) = \begin{bmatrix} -r & -(r+k\beta) & -p_1 k/[m+k] & -p_3 k/[m+k] & b_1 \\ 0 & \beta k - (a_1+\mu_2) & 0 & 0 & 0 \\ 0 & 0 & [qp_1 k/(m+k)] - \mu_3 & 0 & b_2 \\ 0 & 0 & 0 & [qp_3 k/(m+k)] - (a_2+\mu_4) & 0 \\ 0 & a_1 & 0 & a_2 & -(\mu_1+b_1+b_2) \end{bmatrix}$$

To find eigenvalues of $B(E_A)$ take $\det[B(E_A) - \lambda I_5] = 0$ and solve as follows:

$$\begin{vmatrix} -r-\lambda & -(r+k\beta)-\lambda & -p_1 k/[m+k] & -p_3 k/[m+k] & b_1 \\ 0 & \beta k - (a_1+\mu_2) - \lambda & 0 & 0 & 0 \\ 0 & 0 & [qp_1 k/(m+k)] - \mu_3 - \lambda & 0 & b_2 \\ 0 & 0 & 0 & [qp_3 k/(m+k)] - (a_2+\mu_4) - \lambda & 0 \\ 0 & a_1 & 0 & a_2 & -(\mu_1+b_1+b_2)-\lambda \end{vmatrix} = 0$$

Thus, $\{-r-\lambda\}\{\beta k - (a_1+\mu_2) - \lambda\}\{[qp_1 k/(m+k)] - \mu_3 - \lambda\}\{[qp_3 k/(m+k)] - (a_2+\mu_4) - \lambda - \mu 1 + b1 + b2 - \lambda = 0$ is characteristic equation of the model at axial equilibrium point and the eigenvalues are obtained as

$\lambda_1 = -r$
$\lambda_2 = -(\mu_1 + b_1 + b_2)$
$\lambda_3 = \beta k - (a_1+\mu_2)$
$\lambda_4 = [qp_1 k - \mu_3(m+k)]/(m+k)$
$\lambda_5 = [qp_3 k - (a_2+\mu_4)(m+k)]/(m+k)$

Therefore the axial equilibrium point E_A will have stable manifold in the direction of I if (i) $\beta k - (a_1+\mu_2) < 0$ (ii) $qp_1 k - \mu_3(m+k) < 0$ and (iii) $qp_3 k - (a_2+\mu_4)(m+k) < 0$, otherwise it is unstable.

Theorem 6 [Disease- Free Equilibrium Point] Disease – free equilibrium point $E_{B1} = (\bar{X},\ 0,\ \bar{Y},\ 0,\ 0)$ is stable if $I_Z(E_{B1}) < 0$, $G_W(E_{B1}) < 0$, $F_X(E_{B1}) + H_Y(E_{B1}) > 0$ and $F_X(E_{B1}) * H_Y(E_{B1}) + [mp_1{}^2k^2/(m+k)^3] > 0$.

Proof: Consider the next generation matrix at disease free equilibrium point $(\bar{X},\ 0,\ Y,\ 0,\ 0)$

$$B(E_{B1}) = \begin{bmatrix} F_X(E_{B1}) & -\mathrm{r} - \beta\mathrm{k} & -\mathrm{p_1k}/(\mathrm{m}+\mathrm{k}) & -\mathrm{p_3k}/(\mathrm{m}+\mathrm{k}) & \mathrm{b_1} \\ 0 & G_W(E_{B1}) & 0 & 0 & 0 \\ \mathrm{mqp_1k}/(\mathrm{m}+\mathrm{k})^2 & 0 & H_Y(E_{B1}) & -\alpha\mathrm{k} & \mathrm{b_2} \\ 0 & 0 & 0 & I_Z(E_{B1}) & 0 \\ 0 & \mathrm{a_1} & 0 & \mathrm{a_2} & J_T(E_{B1}) \end{bmatrix}$$

Here in the matrix $B(E_{B1})$ some elements are denoted by notations and their parametric expressions are given as follows:

$F_X(E_{B1}) = -\mathrm{r} - [\mathrm{p_1k}/(\mathrm{m}+\mathrm{k})^2]$
$G_W(E_{B1}) = \beta\mathrm{k} - (\mathrm{a_1} + \mu_2) - (\mathrm{p_2k}/\mathrm{m})$
$H_Y(E_{B1}) = [\mathrm{qp_1k}/(\mathrm{m}+\mathrm{k})] - \mu_3$
$I_Z(E_{B1}) = [\mathrm{qp_3k}/(\mathrm{m}+\mathrm{k})] - (\mathrm{a_2} + \mu_4) + \alpha\mathrm{k}$
$J_T(E_{B1}) = -(\mu_1 + \mathrm{b_1} + \mathrm{b_2})$

To find eigenvalues of such matrix, compute $det(\mathrm{B(E_{B1})} - \lambda\mathrm{I_5}) = 0$ using fourth row

$$\begin{vmatrix} F_X(E_{B1}) - \lambda & -\mathrm{r} - \beta\mathrm{k} & -\mathrm{p_1k}/[\mathrm{m}+\mathrm{k}] & -\mathrm{p_3k}/[\mathrm{m}+\mathrm{k}] & \mathrm{b_1} \\ 0 & G_W(E_{B1}) - \lambda & 0 & 0 & 0 \\ \mathrm{mqp_1k}/(\mathrm{m}+\mathrm{k})^2 & 0 & H_Y(E_{B1}) - \lambda & -\alpha\mathrm{k} & \mathrm{b_2} \\ 0 & 0 & 0 & I_Z(E_{B1}) - \lambda & 0 \\ 0 & \mathrm{a_1} & 0 & \mathrm{a_2} & J_T(E_{B1}) - \lambda \end{vmatrix} = 0$$

$$[I_Z(E_{B1}) - \lambda] \begin{vmatrix} F_X(E_{B1}) - \lambda & -\mathrm{r} - \beta\mathrm{k} & -\mathrm{p_1k}/[\mathrm{m}+\mathrm{k}] & \mathrm{b_1} \\ 0 & G_W(E_{B1}) - \lambda & 0 & 0 \\ \mathrm{mqp_1k}/(\mathrm{m}+\mathrm{k})^2 & 0 & H_Y(E_{B1}) - \lambda & \mathrm{b_2} \\ 0 & \mathrm{a_1} & 0 & J_T(E_{B1}) - \lambda \end{vmatrix} = 0$$

Now again use second row to compute determinant

$$[I_Z(E_{B1}) - \lambda][G_W(E_{B1}) - \lambda] \begin{vmatrix} F_X(E_{B1}) - \lambda & -\mathrm{p_1k}/[\mathrm{m}+\mathrm{k}] & \mathrm{b_1} \\ \mathrm{mqp_1k}/(\mathrm{m}+\mathrm{k})^2 & H_Y(E_{B1}) - \lambda & \mathrm{b_2} \\ 0 & 0 & J_T(E_{B1}) - \lambda \end{vmatrix} = 0$$

Then use third row to find determinant

$$[I_Z(E_{B1}) - \lambda][G_W(E_{B1}) - \lambda][J_T(E_{B1}) - \lambda] \begin{vmatrix} F_X(E_{B1}) - \lambda & -\mathrm{p_1k}/[\mathrm{m}+\mathrm{k}] \\ \mathrm{mqp_1k}/(\mathrm{m}+\mathrm{k})^2 & H_Y(E_{B1}) - \lambda \end{vmatrix} = 0$$

Finally, the characteristic equation is given by

$$[I_Z(E_{B1}) - \lambda][G_W(E_{B1}) - \lambda][J_T(E_{B1}) - \lambda] * \{[F_X(E_{B1}) - \lambda][H_Y(E_{B1}) - \lambda] + [mp_1{}^2k^2/(m+k)^3]\} = 0$$

Then the first three Eigenvalues are obtained as

$$\lambda_1 = I_Z(E_{B1}),\quad \lambda_2 = G_W(E_{B1}),\quad \lambda_3 = J_T(E_{B1})$$

Remaining two roots are the solutions of the quadratic equation $[F_X(E_{B1}) - \lambda][H_Y(E_{B1}) - \lambda] + [mp_1{}^2k^2/(m+k)^3] = 0$. Using Routh Hurwitz criterion stability the disease free equilibrium point at $E_{B1} = (\bar{X},\ 0,\ \bar{Y},\ 0,\ 0)$ will be asymptotically stable if $F_X(E_{B1}) + H_Y(E_{B1}) > 0$, $F_X(E_{B1}) * H_Y(E_{B1}) + [mp_1{}^2k^2/(m+k)^3] > 0$, provided that λ_1, $\lambda_2 < 0$ and it is know that $\lambda_3 < 0$.

Theorem 7 Endemic Equilibrium Point The coexistence equilibrium point $E = (X^*,\ W^*,\ Y^*,\ Z^*,\ T^*)$ is stable if $\lambda^5 + A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$ has to be stable and correspondingly the following conditions are satisfied otherwise not stable: (i) $A > 0$ (ii) $AB - C > 0$ (iii) $ABC + AE - A^2D - C^2 > 0$ (iv) $H_4 = (CD - BE)(AB - C) - (AD - E)^2 > 0$ (v) $DH_4 > 0$

Proof: Consider Jacobian matrix at coexistence equilibrium point $(X^*,\ W^*,\ Y^*,\ Z^*,\ T^*)$

Let $B(X^*,\ W^*,\ Y^*,\ Z^*,\ T^*) = \begin{vmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ c_6 & c_7 & c_8 & c_9 & c_{10} \\ c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{16} & c_{17} & c_{18} & c_{19} & c_{20} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \end{vmatrix}$. Here the notations used as the matrix elements represent the following parametric expressions:

$c_1 = r[1-(2X+W)/k] - \beta W - [m(p_1Y+p_3Z)/(m+X)^2],\ c_2 = [-(r+k\beta)X/k],\ c_3 = -[p_1X/(m+X)],$
$c_4 = -[p_3X/(m+X)],\ c_5 = b_1,\ c_6 = \beta W,$
$c_7 = \beta X - (a_1+\mu_2) - [mp_2Y/(m+W)^2],\ c_8 = -p_2W/(m+W),\ c_9 = c_{10} = 0,\ c_{11} =$
$[mqp_1Y/(m+X)^2],\ c_{12} = 0,\ c_{13} = [qp_1X/(m+X)] - \mu_3 - \alpha Z,\ c_{14} = \alpha Y,\ c_{15} = b_2,\ c_{16} =$
$[mqp_3Z/(m+X)^2],\ c_{17} = [mqp_2Z/(m+W)^2],\ c_{18} = 0,\ c_{19} = qp_2W/[m+W] + qp_3X/[m+X] -$
$(a_2+\mu_4), c_{20} = c_{21} = 0,\ c_{22} = a_1, c_{23} = 0,\ c_{24} = a_2,\ c_{25} = -(\mu_1+b_1+b_2).$

Then evaluation of the determinant of the characteristic equation $|B-\lambda I| = 0$ gives a fifth order algebraic equation of the form $\lambda^5 + A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$. Using the Hurwitz criterion the coexistence will be stable if the following conditions are satisfied: (i) $A > 0$ (ii) $B - C > 0$ (iii) $ABC + AE - A^2D - C^2 > 0$ (iv) $H_4 = (CD-BE)(AB-C) - (AD-E)^2 > 0$ (v) $D * H_4 > 0$. Otherwise it is not stable.

1.3.2 Global Stability Analysis

Here, the global stability analysis of the system of model equations (5) – (9) around the positive equilibrium point $E = (X^*, W^*, Y^*, Z^*, T^*)$ or the coexistence equilibrium is conducted. Consider the following theorem on the Lyapunov function L.

Theorem 8 Global Stability Suppose $L = (X-X^*)^2/2 + \alpha_1(W-W^*)^2/2 + \alpha_2(Y-Y^*)^2/2 + \alpha_3(Z-Z^*)^2/2 + \alpha_4(T-T^*)^2/2$ where $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$ are all positive ,and chosen properly such that $dL/dt = 0$ where $E = (X^*, W^*, Y^*, Z^*, T^*)\ L = (X,\ W,\ Y,\ Z,\ T) > 0, \forall (X, W, Y, Z, T)/\{E\}$ and $dL/dt \le 0\ \ \forall (X, W, Y, Z, T) \in \Gamma^+,\ dL/dt = 0\ \forall (X^*, W^*, Y^*, Z^*, T^*) \in \Gamma^+$ .This implies E^* of the system is Lyapunove stable and $dL/dt < 0$, $\forall (X, W, Y, Z, T) \in \Gamma^+$ near E^*This implies E^* is globally stable [1].

Proof: $L = (X-X^*)^2/2 + \alpha_1(W-W^*)^2/2 + \alpha_2(Y-Y^*)^2/2 + \alpha_3(Z-Z^*)^2/2 + \alpha_4(T-T^*)^2/2$

$dL/dt = (X-X^*)\,dX/dt + \alpha_1(W-W^*)\,dW/dt + \alpha_2(Y-Y^*)\,dY/dt + \alpha_3(Z-Z^*)\,dZ/dt + \alpha_4(T-T^*)\,dT/dt$ (10)

Now substitute the model equation (5) - (9) into equation (10)

$$dL/dt = (X-X^*)\{rX\{1-[(X+W)/k]\} + b_1T - \beta XW - [p_1XY/(m+X)] - [p_3XZ/(m+X)]\}$$
$$+\alpha_1(W-W^*)\{\beta XW - a_1W - \mu_2W - [p_2WY/(m+W)] - [p_2WZ/(m+W)]\}$$
$$+\alpha_2(Y-Y^*)\{[qp_1XY/(m+X)] + [qp_2WY/(m+W)] + b_2T - \alpha YZ - \mu_3Y\}$$
$$+\alpha_3(Z-Z^*)\{[qp_2WZ/(m+W)] + [qp_3XZ/(m+W)] + \alpha YZ - a_2Z - \mu_4Z\}$$
$$+\alpha_4(T-T^*)[a_1W + a_2Z - \mu_1T - b_1T - b_2T]$$

Take out $X,\ W,\ Y,\ Z,\ T$ and put as change

$$dL/dt = (X-X^*)(X-X^*)\{r(1-[(X+W)/k]) + [b_1T/X] - \beta W - [p_1Y/(m+X)] - [p_3Z/(m+X)]\}$$
$$+\alpha_1(W-W^*)(W-W^*)\{\beta X - a_1 - \mu_2 - [p_2Y/(m+W)] - [p_2Z/(m+W)]\}$$
$$+\alpha_2(Y-Y^*)(Y-Y^*)\{[q\,p_1X/(m+X)] + [q\,p_2W/(m+W)] + [b_2T/Y] - \alpha Z - \mu_3\}$$
$$+\alpha_3(Z-Z^*)(Z-Z^*)\{[q\,p_2W/(m+W)] + [q\,p_3X/(m+W)] + \alpha Y - a_2 - \mu_4\}$$
$$+\alpha_4(T-T^*)(T-T^*)\{[a_1W/T] + [a_2Z/T] - \mu_1 - b_1 - b_2\}$$

By rearranging, it could be obtained

$$dL/dt = -(X-X^*)^2\{-r(1-[(X+W)/k]) - [b_1T/X] + \beta W + [p_1Y/(m+X)] + [p_3Z/(m+X)]\}$$
$$-\alpha_1(I-I^*)^2\{-\beta X + a_1 + \mu_2 + [p_2Y/(m+W)] + [p_2Z/(m+W)]\}$$
$$-\alpha_2(Y-Y^*)^2\{-[q\,p_1X/(m+X)] - [q\,p_2W/(m+W)] - [b_2T/Y] + \alpha Z + \mu_3\}$$
$$-\alpha_3(Z-Z^*)^2\{-[q\,p_2W/(m+W)] - [q\,p_3X/(m+W)] - \alpha Y + a_2 + \mu_4\}$$
$$-\alpha_4(T-T^*)^2\{-[a_1W/T] - [a_2Z/T] + \mu_1 + b_1 + b_2\}$$

Thus it is possible to set $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$ such that $dL/dt \le 0$ and endemic equilibrium point is globally stable. It is to be noted that the parameters $k, m,\ q$ play a vital role in controlling the stability aspects of the system.

1.3.3 Reproduction number or Threshold number R_0

If $R_0 < 1$ then each infected individual produces on average less than one new infected individual so it is expected that the disease would die out. On the other hand if $R_0 > 1$ then each individual produces more than one new infected individual so it is expected that the disease would continue spreading in the population.

Theorem 9 Infected Prey Threshold The Reproduction number for infected prey at Disease free equilibrium point is given by $R_0 = m\beta\bar{X}/[(a_1 + \mu_2)m + p_2\bar{Y}]$.

Proof: Consider infected prey equation (6) $dW/dt = \beta XW - a_1W - e_2W - [p_2WY/(m+W)] - [p_2WZ/(m+W)] = \{\beta X - \{a_1 + \mu_2 + [p_2Y/(m+W)] + p_2Z/m + W\}\}W$. Now Let $F = \beta X$ and $V = a_1 + \mu_2 + [p_2Y/(m+W)] + [p_2Z/m] + W$. Evaluate F and V at Disease equilibrium point $(\bar{X}\ \ 0\ \ \bar{Y}\ \ 0\ \ 0)$, then $F = \beta\bar{X}$ and $V = a_1 + \mu_2 + p_2\bar{Y}/m$. It is known that $R_0 = FV^{-1}$ and hence $R_0 = m\beta\bar{X}/[(a_1 + \mu_2)m + p_2\bar{Y}]$

Theorem 10 Infected Predator Threshold The Reproduction number for infected predators at disease free equilibrium point takes the form as $R_0 = [qp_3\bar{X} + m\alpha Y]/[m(a_2 + \mu_4)]$.

Proof: Consider the infected predator model equation (8)

$$dZ/dt = [q\,p_2WZ/(m+W)] + [q\,p_3XZ/(m+W)] + \alpha YZ - a_2Z - \mu_4Z$$
$$= \{[q\,p_2W/(m+W)] + [q\,p_3X/(m+W)] + \alpha Y - (a_2 + \mu_4)\}Z$$

Let $F = [q\,p_2W/(m+W)] + [q\,p_3X/(m+W)] + \alpha Y$ and $V = a_2 + \mu_4$

Evaluate F and V at disease free equilibrium point $(\bar{X}\ \ 0\ \ \bar{Y}\ \ 0\ \ 0)$, then $F = q\,p_3\bar{X}/m + \alpha\bar{Y}$ and $V = a_2 + \mu_4$. It is known that $R_0 = FV^{-1}$ and hence $R_0 = [qp_3\bar{X} + m\alpha\bar{Y}]/[m(a_2 + \mu_4)]$

IV. Numerical simulations

In this section, Numerical simulation of model equations (5) - (9) is carried out using the software DEDiscover version: 2.6.4. Model equation and parameters were arranged for DEDiscover software in this way for simulation purpose:

```
dX/dt=r*X*(1-(X+W)/k)+b_1*T-beta*X*W-P_1*X*Y/(m+X)-P_3*X*Z/(m+X) // Susceptible prey
dW/dt=beta*X*W-a_1*W-u_1*W-P_2*W*Y/(m+W)-P_2*W*Z/(m+W) // Infected prey                (12)
dY/dt=b_2*T+q*P_1*X*Y/(m+X)+q*P_2*W*Y/(m+W)-u_3*Y+alpha*Y*Z // Susceptible predator
dZ/dt=q*P_2*W*Z/(m+W)+q*P_3*X*Z/(m+X)-a_2*Z-u_4*Z-alpha*Y*Z // Infected predator
dT/dt=a_1*W+a_2*Z-b_1*T-b_2*T-u_1*T //  Treatment
```

Table 1 Initial Conditions for the given Variables

Name	Current Initial Conditions	Source
$X[t_o]$	50.0000	Assumed
$W[t_o]$	90.0000	Assumed
$T[t_o]$	40.0000	Assumed
$Y[t_o]$	5.0000	Assumed
$Z[t_o]$	65.0000	Assumed

Table 4 Model Parameters and Their Descriptions and Values

Parameters	Parameter value	Description of parameters	Source
r	11.2000	Intrinsic growth rate of susceptible prey	Mukhopadhyay and Bhattacharyya(2009)
k	30.0000	Carrying capacity of susceptible prey	Mukhopadhyay and Bhattacharyya(2009)
b_1	0.0010	Recovery rate of infected prey	Estimated
β	1.2000	Convolution rate of susceptible prey to be infected prey	Mukhopadhyay and Bhattacharyya(2009)
p_1	0.4000	Predation coefficient of susceptible prey due to susceptible predator	Mukhopadhyay and Bhattacharyya(2009)
p_3	0.2000	Predation coefficient of susceptible prey due to infected predator.	Estimated
m	0.5000	Half-saturation constant	Mukhopadhyay and Bhattacharyya(2009)
a_1	0.0100	Treatment rate of infected prey	Estimated
μ_2	0.4000	Death rate of infected prey	Mukhopadhyay and Bhattacharyya(2009)
p_2	0.6000	Predation coefficients of infected prey due to *predator*	Mukhopadhyay and Bhattacharyya(2009)
b_2	0.0020	Recovery rate of infected predator	Estimated

q	0.2500	Efficiency of predation.	Mukhopadhyay and Bhattacharyya(2009)
α	1.3000	Convolution rate of susceptible predator to be infected predator	Assumed.
μ_3	0.0800	Death rate of susceptible predator	Mukhopadhyay and Bhattacharyya(2009)
a_2	0.0300	Treatment rate of infected predator	Estimated
μ_4	0.0100	Death rate of infected predator	Estimated
μ_1	0.0500	Death rate of both infected prey and infected predator	Estimated

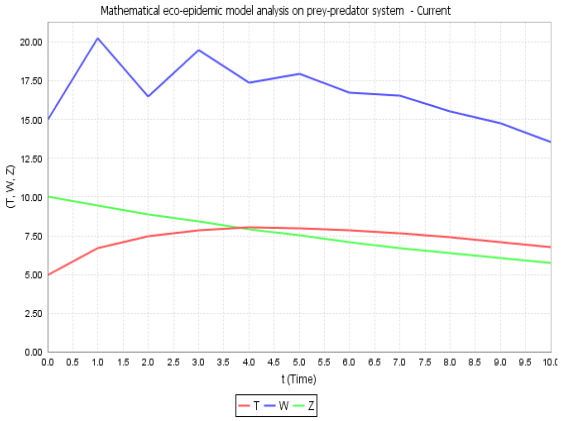

Figure 1a Infected prey-predator with increasing treatment

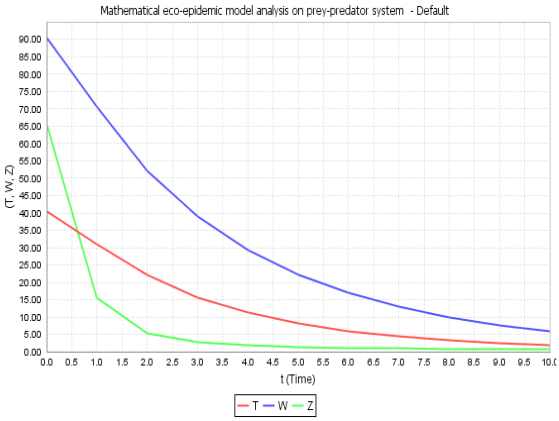

Figure 1b Infected prey-predator with less treatment

It can be seen from figure 1a, as the number of treatment increase on the infected prey – predator, there will be a decrease in the infected population slowly. Meaning disease dies out and infected pry-predator recover from disease. and from figure 1b it has been observed that if we neglect treatment to the infected population, then the population decrease very soon, which means the infected population will get more disease that leads to more death on both infected prey and predator. In any of the cases the population of infected prey and predator decreases .Therefore infected prey-predator population always decrease independent of treatment.

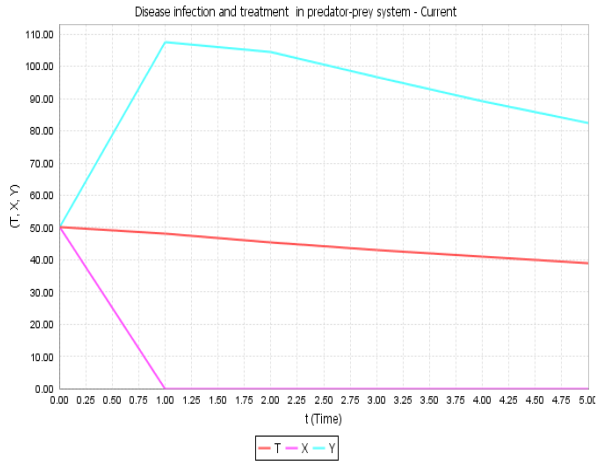

Figure 2a Susceptible prey-predator with initial 50

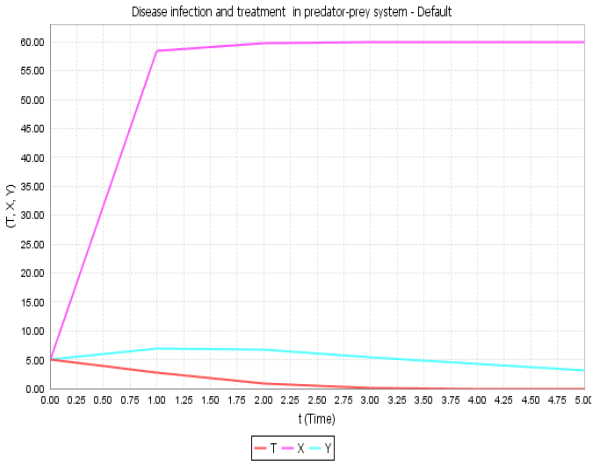

Figure 2b Susceptible prey-predator with all initial 5

In figure 2a applying treatment Continuously, then susceptible predators will be strong, increase in number, and got the capacity to predate the susceptible prey easily that leads to the loss of many susceptible preys. From figure 2b, the decrease in level of treatment differently, there is no big influence on susceptible prey and predator, especially on susceptible prey will be free from predation by infected predator that improves the prey to grow well logistically. This ensures susceptible prey is the only population that can reproduce.

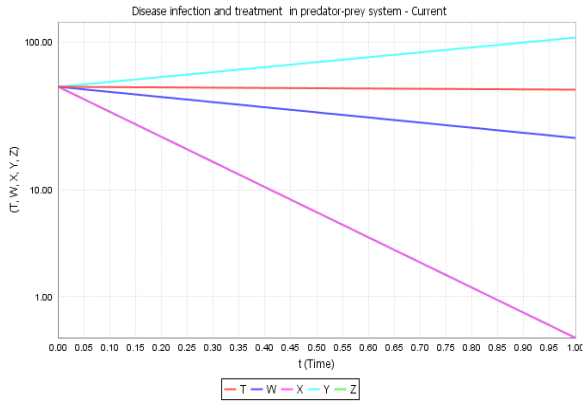

Figure 3a prey-predator with initial data 90

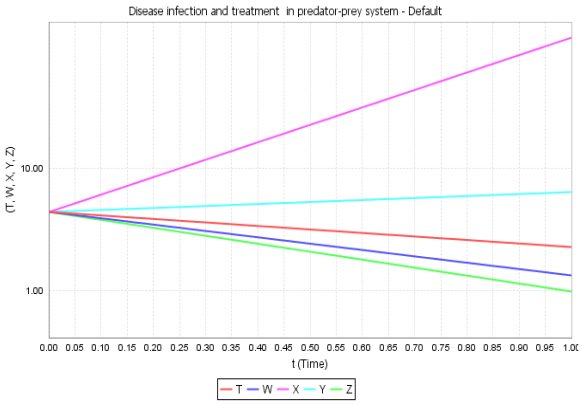

Figure3b.prey-predator with initial data 5

From figure 3a As long as the infected prey and predator populations are treated constantly, the infected prey dies out from disease which is blue in color or the susceptible predator increase in population due to more recovered predators goes in to susceptible predator class labeled turquoise in color. From Figure 3b,it is clear that increase the susceptible prey population X, leads for susceptible predator Y got more food sufficiently, again this improves healthy prey-predator system which turns out disease dies out from infected prey-predator i.e. decrease in W and Z.

V. Conclusions

The Boundedness, positivity, and existence of solutions of the system are shown to hold implying that the system is biologically well behaved. Disease free equilibrium points were obtained and their stability analysis has been done.

It is observed that giving treatment to infected prey and infected predator into the system save the population from extinction. Increasing the rate of treatment in the infected prey and predator leads to increase susceptible prey and predator density populations but decreasing the infected prey and predator population that results the infected prey and predator recover from disease or disease dies out.

A numerical simulation of the model was carried out and it was observed that the increase of the number of infected prey tends to lower the whole population. It can be concluded that the disease can be eradicated in a population through treatment.

VI. Recommendations

One can extend this work by assuming the predator grows logistically or by adding parameters like natural death rate on the prey or by including other variables like vaccination, immigration, migration on prey-predator populations. And these things can be considered as limitation of this paper.

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