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Comparison Results and Estimates of Amplitude for Oscillatory Solutions of Some Quasilinear Equations

TADIE

ABSTRACT. In this work we investigate some qualitative properties of solutions of problems of the type

$$\nabla\cdot\{A(x)|\nabla u|^{\alpha-1}\nabla u\}+C(x)|u|^{\alpha-1}u+F(x,u,\nabla u)=0\quad x\in\mathbb{R}^n$$

where $\alpha\geq 1$ and $n\geq 3$;the functions $A\in C^1(\mathbb{R}^n,\ \mathbb{R}^+)$, $C\in C(\mathbb{R}^n,\ \mathbb{R}^+)$ and $F\in C(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$ are strictly positive functions.

This work investigates via some comparison results the estimates of the amplitudes of bounded and non-trivial (strongly) oscillatory solutions for such problems.

Namely if w is such a solution, its nodal set say, denotes any bounded $D:=D(w)=\{x\in\mathbb{R}^n\mid w(x)\neq 0\ \textit{inside } D\ \textit{and } w|_{\partial D}=0\}$. The main result obtained here is as follows: when e.g. $D(u)$ lies in $\Omega_R^T:=\{x\in\mathbb{R}^n\mid R<|x|<T\}$, as $R\nearrow\infty$,

$$|u^+|_{D(u^+)}:=\max_{x\in D(u^+)}\ u^+(x)=O\Big(\ \max_{\Omega_R^T}\ \Big\{\frac{A(x)}{C(x)}\Big\}^{1/(1+\alpha)}\Big).$$

I. Introduction

This work will be focussed on bounded solutions in $\mathbb{R}^n$, $n\geq 3$ of problems of the types

$$(P)\begin{cases}(i)\quad \nabla\cdot\Big\{A(x)\Phi(\nabla u)\Big\}+C(x)\phi(u)=0\quad\text{(half-linear equations)},\\ (ii)\quad \nabla\cdot\Big\{A(x)\Phi(\nabla v)\Big\}+C(x)\phi(v)+F(x,v,\nabla v)=0\\ (iii)\quad \nabla\cdot\Big\{A(x)\Phi(\nabla w)\Big\}+C(x)\phi(w)+\nabla K(x)\cdot\Phi(\nabla w)=0\\ \text{where } K\in C^1(\mathbb{R}^n)\text{ and } A,\ C,\ F\in C(\mathbb{R}^n)\text{ are positive real valued-functions.}\end{cases}$$

Here for some $\alpha\geq 1$, $t>0$, $\zeta\in\mathbb{R}^n$, so are defined

$\phi(t):=\phi_\alpha(t)=|t|^{\alpha-1}t$; $\Phi(\zeta):=\Phi_\alpha(\zeta)=|\zeta|^{\alpha-1}\zeta$ with the properties that $t\phi(t)=|t|^{\alpha+1}$; $\zeta\cdot\Phi(\zeta)=|\zeta|^{\alpha+1}$; $\phi(t)\Phi(\zeta)=\Phi(t\zeta)$ and $t\phi'(t)=\alpha\phi(t)$.

In the equations, $F(x,.,.)$ denotes a perturbation term and $\nabla K(x)\cdot\Phi(\nabla w)$ a damped term.

Definition 1.1. *Let $h\in C(E)$ where E denotes $\mathbb{R}$ or $\mathbb{R}^n$. h will be said to be*

(i) Oscillatory (weakly) in E if h has a zero in any $\Omega_T:=\{x\in E\,;\,|x|>T\}$;

(ii) Strongly oscillatory if it has a nodal set in any Ω_T, where a nodal set is any non trivial connected and bounded component of the support $supp(h)$ of h. A nodal set of say h will be denoted as $D(h)$.

For a $T > 0$, $D_T(h)$ will denote a nodal set for h lying inside Ω_T.

(iii) A differential equation will be said to be oscillatory if any of its non trivial and bounded solution is oscillatory.

Here, a function u is called a solution for (P) if for any bounded domain $D \subset E$, $u, \nabla u \in C^1(\overline{D})$, $u \in C^2(D)$ and satisfies the equations (i) or (ii).

(iv) Therefore a function w will be said not to be oscillatory if either there are $\mu, R > 0$ such that $|w| > \mu$ in Ω_R or $\liminf_{t\nearrow\infty} |w(t)| = 0$.

With functions y and u defined respectively in $\mathbb{R}$ and $\mathbb{R}^n$, we will be dealing with differential operators of the types

$$\begin{cases} P(y) := \Big\{a(t)\phi(y')\Big\}' + c(t)\phi(y) + f(t,y,y'), \quad t \in \mathbb{R} \quad \text{and} \\ Q(u) := \quad \nabla\cdot\Big\{A(x)\Phi(\nabla u)\Big\} + C(x)\phi(u) + F(x,u,\nabla u), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

The functions $f(t,y,y')$ and $F(x,u,\nabla u)$ are the perturbations terms added to the respective half-linear equations. But if they have the form $g'(t)\phi(y')$ (or $\nabla G(x)\cdot\Phi(\nabla u)$ for any continuously differentiable g (*or* G) they are called damping terms of the half-linear equations. General hypotheses will be on the coefficients of the half-linear parts of the equations mainly

(H)

A solution of the problems $P(y) = 0$ or $Q(u) = 0$ in say E will be a non-trivial and bounded function y (u) which satisfies (weakly) the respective equations with y, $a(t)\phi(y')$, u and $A(x)\Phi(\nabla u)$) continuously differentiable in E.

H1) In the equations (1.1) the numerical functions (or any functions representing them in equations) a and A are strictly positive and continuously differentiable in their respective domains ; C and c are continuous in their respective arguments and strictly positive .

H2) On the perturbation terms, for the leading terms in y or u in (1.1) for small values of these unknown functions to remain $c(t)\phi(y)$ and $C(x)\phi(u)$

$$\lim_{|y|\searrow 0} \frac{|f(t,y,y')|}{|\phi(y)|} = 0 \quad \textit{and} \quad \lim_{|u|\searrow 0} \frac{|F(x,u,\nabla u)|}{|\phi(u)|} = 0 \tag{1.2}$$

are required.

H3) Because our main interest is on the estimates for decaying oscillatory solutions i.e. u such that $\lim_{|x|\to\infty}\Big[\max_{x\in D(u^+)} u^+(x)\Big] = 0$, the following extra assumptions will be required for the coefficients a and c of the half-linear parts of the equations: for large $|x|$

$$\left|\frac{a(x)}{c(x)}\right| \simeq \chi(|x|) \text{ where } \chi \in C(\mathbb{R}^+,\ \mathbb{R}^+) \text{ is a decreasing function.} \tag{1.3}$$

Note that the goal of the hypothesis H2) is to avoid the solutions to have compact support and to ensure some application of maximum principle (see [4], [2]).

In the sequel, the following notations will be used for any continuous function

II. Comparison Results

$$\begin{cases} h \in C(\mathbb{R}^n, \mathbb{R}) \; ; \; x \in \mathbb{R}^n \text{ and } r := |x|, \\ a) \quad h^+(x) := \max\{0, \; h(x) \} \; ; \; h^-(x) := \min\{0, \; h(x) \}; \\ b) \quad H^+(r) := \overline{h^+}(r) := \max_{|x|=r} h(x) \text{ and } C^-(r) := \overline{h^-}(r) := \min_{|x|=r} h(x). \end{cases} \quad (1.4)$$

2. Some Comparison principle for half-linear equation in $\mathbb{R}^n$

If we take $\alpha \geq 1$ then $t \mapsto \phi(t) := |t|^{\alpha-1}t$ or $\zeta \mapsto \Phi(\zeta) := |\zeta|^{\alpha-1}\zeta$ are monotonic increasing in the sence that $\forall t, s \in \mathbb{R}$ $(\zeta_1, \; \zeta_2 \in \mathbb{R}^n)$ $\Big(\phi(t) - \phi(s)\Big)(t-s)$ and $\Big(\Phi(\zeta_1) - \Phi(\zeta_2)\Big)(\zeta_1 - \zeta_2)$ are strictly positive whenever $|t-s|$ (respectively $|\zeta_1 - \zeta_2|$) is non zero.
Let $\Omega \subset \mathbb{R}^n$ be a connected and bounded regular domain.

Theorem 2.1. *(Comparison principle)*
Let $E \subset \mathbb{R}^n$ be a connected and regular domain , $c_1, \; c_2 \in C(E)$ be non-negative and $a \in C^1(\overline{E}, \; (0, \; \infty) \;)$. Let $\Omega \subset E$ be a bounded subdomain. If two distinct and nonegative $u, v \in C^2(\Omega)$ satisfy in Ω

$$\begin{cases} (i)a \quad \nabla \cdot \Big\{ a(x)\Phi(\nabla u) \Big\} + c_1(x)\phi(u) = 0 \quad and \\ (i)b \quad \nabla \cdot \Big\{ a(x)\Phi(\nabla v) \Big\} + c_2(x)\phi(v) = 0; \\ (ii) \quad (a) \;\; (u-v)|_{\partial\Omega} \geq 0 \; ; \quad (b) \quad \exists x_1 \in \Omega \, ; \; (u-v)(x_1) > 0. \end{cases} \quad (2.1)$$

Then $(u-v) \geq 0$ in Ω, provided that $c_2 \geq c_1$ in Ω. In addition if Ω is connected then $u > v$ there. If a numerical function $F \in C(\Omega, \mathbb{R}, \mathbb{R}^n)$ is non-negative, then the results of the theorem still hold with (i)b replaced by

$$\nabla \cdot \Big\{ a(x)\Phi(\nabla v) \Big\} + c_2(x)\phi(v) + F(x, v, \nabla v) = 0.$$

Proof. In fact assume that $\Omega^- := \{x \in \Omega \mid u(x) < v(x) \}$ has a positive measure. Let $\tau > 0$ be such that $w := w_\tau = u - v + \tau > 0$ in some non-neglected $D_\tau \subset \Omega^-$ and $w|_{\partial D_\tau} = 0$. Such τ exists because of (ii)(b) above.
As $w \in C^1(\Omega^-)$ and non-negative ,

$$\begin{aligned} &\int_{D_\tau} w(x)\Big[\nabla \cdot \Big(a(x)[\, \Phi(\nabla u) - \Phi(\nabla v) \,] \Big) \Big] dx = \\ &- \int_{D_\tau} a(x)\Big(\Phi(\nabla u) - \Phi(\nabla v) \Big) \cdot \nabla w \, dx = \\ &\int_{D_\tau} w(x)\Big(c_2(x)\phi(v) - c_1(x)\phi(u) \Big) dx > 0 \end{aligned} \quad (2.2)$$

as $v > u$ there.
Because of the strict monotonicity of $t \to t\phi(t)$, the equation above is absurd as
$-\int_{D_\tau} a(x)\Big(\Phi(\nabla u) - \Phi(\nabla v) \Big) \cdot \nabla w dx =$
$-\int_{D_\tau} a(x)(\nabla u - \nabla v) \cdot [\Phi(\nabla u) - \Phi(\nabla v)] dx \leq 0$ contradicting (2.2);
the assumption is false and $u - v \geq 0$ in Ω. So,

$$\nabla\cdot\Big\{a(x)\Phi(\nabla u)\Big\}\geq\nabla\cdot\Big\{a(x)\Phi(\nabla v)\Big\}$$ and $u\geq v$ in Ω implying that $u\neq v$ therein (see e.g. [5] , Theorem 2.2).

The last part of the result is a mere verification. □

Corollary 2.2. *Let* $a,\ c_1,\ c_2\ \in C(\mathbb{R}^n)$ *be strictly positive functions with* $c_1\leq c_2$ *. Let* $u_i\in C^2(\mathbb{R}^n)$ *be respectively two oscillatory solutions (with overlapping nodal sets) for*

$$\nabla\cdot\Big[a(x)\Phi(\nabla u_i)\Big]+c_i(x)\phi(u_i)=0\quad x\in\mathbb{R}^n;\quad i=1,2.$$

Assume that there is a connected domain $\Omega\subset D(u_1^+)\bigcap D(u_2^+)$ *which contains their local maxima. Then*

$$\max_{\Omega}(\ u_2^+(x)\ )\leq\max_{\Omega}(\ u_1^+(x)\ ).$$

Proof. This is due to the fact that $\Omega^-:=\{x\in\Omega\mid(u_1^+-u_2^+)(x)<0\}$ would have a zero maesure by the Theorem 2.1 . □

III. Some Half-Linear Operators And Identities

3.1. Half-linear equations. (Some identities). Given some positive functions $a,\ C\ \in C^1(\mathbb{R}^n)$ we consider the problem

$$\nabla\cdot\Big[a\Phi(\nabla u)\Big]+\alpha c\phi(u)=0;\qquad x\in\mathbb{R}^n \tag{3.1}$$

where for some $m>0$ and $\alpha\geq 1$

$$\text{the functions } a,\ C>m \text{ and } \phi:=\phi_\alpha. \tag{3.2}$$

Note that the multiplier parameter α is added to the coefficient c in (3.1) just for easing the obtension of the identities. When the departing equation has no such α to c , it would be enough to replace c by $\dfrac{c}{\alpha}$ in the formulas later on.

Any non trivial and bounded solution of (3.1) is strongly oscillatory and easy calculations show that

(i) $\nabla\cdot\Big(a\Phi(\nabla u)\Big)=\nabla a\cdot\Phi(\nabla u)+a\alpha|\nabla u|^{\alpha-1}\Delta u;$

(ii) $\nabla\cdot\Big[a\Phi(\nabla u)\Big]\nabla u=|\nabla u|^{\alpha+1}\nabla a+\dfrac{\alpha}{\alpha+1}a\nabla\Big[|\nabla u|^{\alpha+1}\Big]=$
$\nabla\Big\{a|\nabla u|^{\alpha+1}\Big\}-\dfrac{a}{\alpha+1}\nabla\Big(|\nabla u|^{\alpha+1}\Big)$ and

(iii) $\alpha c\phi(u)\nabla u=\dfrac{\alpha}{\alpha+1}c\nabla\Big[|u|^{\alpha+1}\Big]=\dfrac{\alpha}{\alpha+1}\Big\{\nabla\Big(c|u|^{\alpha+1}\Big)-|u|^{\alpha+1}\nabla c\Big\}.$

So, from $\nabla u\Big\{\nabla\cdot\Big(a\Phi(\nabla u)\Big)+\alpha c\phi(u)\Big\}=0$

We get

$$\begin{cases} \nabla \cdot \Big\{ (\alpha+1)a(x)|\nabla u|^{\alpha+1} + \alpha c(x)|u|^{\alpha+1} \Big\} = \\ \alpha|u|^{\alpha+1}\nabla c(x) + a(x)\nabla\Big(|\nabla u|^{\alpha+1}\Big) \\ \text{or} \quad \nabla \cdot \Big\{ \alpha a(x)|\nabla u|^{\alpha+1} + \alpha c(x)|u|^{\alpha+1} \Big\} = \\ \alpha|u|^{\alpha+1}\nabla c(x) - |\nabla u|^{\alpha+1}\nabla a(x). \end{cases} \tag{3.3}$$

If the coefficients $a, c \in C^1(\mathbb{R}^n)$ and a is strictly positive any non-trivial oscillatory solution u of (3.1) satisfies

$$\begin{cases} (i) \quad \nabla \cdot \Big[\Phi(\nabla u)\Big] + A(x)\cdot\Phi(\nabla u) + \alpha C(x)\phi(u) = 0 \quad x \in \Omega \quad ; \; u|_{\partial\Omega} = 0 \\ \text{where } \; A(x) := \dfrac{\nabla a(x)}{a(x)} \quad \text{and} \;\; C(x) := \dfrac{c(x)}{a(x)} \quad \text{from which} \\ (ii) \quad \nabla \cdot \Big\{ |\nabla u(x)|^{\alpha+1} + C(x)|u(x)|^{\alpha+1} \Big\} = \\ \qquad |u(x)|^{\alpha+1}\nabla C(x) - \dfrac{\alpha+1}{\alpha} A(x)|\nabla u|^{\alpha+1} \end{cases} \tag{3.4}$$

after processing as before.

3.2. Half-linear equation with constant coefficient of the principal part. Consider

$$\begin{cases} (i) \quad \nabla \cdot \Big\{ A\Phi(\nabla u) \Big\} + c(x)\phi(u) = 0 \quad \text{in} \quad \mathbb{R}^n \\ (ii) \quad \text{where } \; A > 0 \; \text{ is a constant and } \; c \in C(\mathbb{R}^n, \; (0, \; \infty) \;). \end{cases} \tag{3.5}$$

It is clear that with $C^1(x) := \dfrac{c(x)}{A}$ the equation (3.5)(i) is equivalent to

$$\nabla \cdot \Big\{ \Phi(\nabla u) \Big\} + C^1(x)\phi(u) = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{3.6}$$

It is known that any non-trivial and bounded solution for (3.6) is strongly oscillatory.

Also if the coefficient C^1 depends only on $r := \{\sum_{i=1}^n x_i^2\}^{1/2}$ the equation would be axially symmetric (see e.g. [11]) and would have the form

$$\begin{cases} (i) \quad \Big\{ r^{n-1}\phi(U') \Big\}' + r^{n-1}C^1(r)\phi(U) = 0; \quad r \geq 0 \\ (ii) \quad \text{or} \quad \Big\{ \phi(U') \Big\}' + \dfrac{n-1}{r}\phi(U') + C^1(r)\phi(U) = 0 \\ \text{where } \; C^1(r) := C^1(|x|) \;\; . \end{cases} \tag{3.7}$$

3.3. One-dimensional cases. For (3.6), we have with $c(t) > 0 \ \forall t \geq 0$

$$\Big\{\phi(u')\Big\}' + \alpha c(t)\phi(u) = 0, \quad t > 0; \tag{3.8}$$

and it is known that any non-trivial and bounded solution of that equation is strongly oscillatory.

We also know (e.g. from [10]) that if c is an increasing and unbounded function, for any nodal set $D(u^+)$ of (3.8), we have the following estimates for large $T_1 > 0$:

$$D(u^+) := [T_1,\, T_2] \Longrightarrow$$
$$\begin{cases} (i) \quad \max\limits_{[T_1,\, T_2]} \Big[u^+\Big] = Const.\Big\{\dfrac{1}{c(T_1)}\Big\}^{1/(\alpha+1)} \\ (ii) \quad \text{and} \quad |T_2 - T_1| = Const.\Big\{\dfrac{1}{c(T_1)}\Big\}^{1/(\alpha+1)}. \end{cases} \tag{3.9}$$

3.4. Some Picone-type identities and some applications. (some recalls)

Because of those identities and formulae will be refered to from now on, it is necessary to recall them before hand.

For ease in writing, the operators $P_i(.)$ will denote the $P(.)$ in (1.1) in which the coefficients a, c and the function f carry the index i. Similarly is defined $Q_i(.)$.

Let y_1 and y_2 be respectively used in $P_i(y_i) = 0$, $t > 0$; $i = 1, 2$. Then wherever y_2 is non zero, a version of Picone's identity reads

$$\begin{aligned} &\Big\{y_1 a_1(t)\phi(y_1') - y_1\phi(\frac{y_1}{y_2})a_2(t)\phi(y_2')\Big\}' = \\ &= a_2(t)\zeta_\alpha(y_1, y_2) + \Big[a_1(t) - a_2(t)\Big]|y_1'|^{\alpha+1} + \Big[c_2(t) - c_1(t)\Big]|y_1|^{\alpha+1} \\ &+ |y_1|^{\alpha+1}\Big[\frac{f_2(t, y_2, y_2')}{\phi(y_2)} - \frac{f_1(t, y_1, y_1')}{\phi(y_1)}\Big] + |y_1|^{\alpha+1}\Big[\frac{P_1(y_1)}{\phi(y_1)} - \frac{P_2(y_2)}{\phi(y_2)}\Big]. \end{aligned} \tag{3.10}$$

where, $\forall\gamma > 0$, the two-form function ζ_γ is defined $\forall u, v \in C^1(\mathbb{R},\, \mathbb{R})$ by

$$(\mathbf{Z1}): \qquad \zeta_\gamma(u, v) \begin{cases} = |u'|^{\gamma+1} - (\gamma+1)u'\phi_\gamma(\frac{u}{v}v') + \gamma v'\frac{u}{v}\phi_\gamma(\frac{u}{v}v') \\ = |u'|^{\gamma+1} - (\gamma+1)u'\phi_\gamma(\frac{u}{v}v') + \gamma|\frac{u}{v}v'|^{\gamma+1} \end{cases}$$

is strictly positive for non null $u \neq v$ and null only if $u = \lambda v$ for some $\lambda \in \mathbb{R}$. Similarly, if u_1 and u_2 are respectively used in Qu_i, $i = 1, 2$, then wherever u_2 is non zero, a version of Picone's identity reads

$$\nabla\cdot\Big\{u_1A_1(x)\Phi(\nabla u_1)-u_1\phi(\frac{u_1}{u_2})A_2(x)\Phi(\nabla u_2)\Big\}=A_2(x)Z_\alpha(u_1,u_2)$$
$$+\Big(A_1(x)-A_2(x)\Big)|\nabla u_1|^{\alpha+1}+\Big(C_2(x)-C_1(x)\Big)|u_1|^{\alpha+1}$$
$$+|u_1|^{\alpha+1}\Big[\frac{F_2(x,u_2,\nabla u_2)}{\phi(u_2)}-\frac{F_1(x,u_1,\nabla u_1)}{\phi(u_1)}\Big]+ \tag{3.11}$$
$$|u_1|^{\alpha+1}\Big[\frac{Q_1(u_1)}{\phi(u_1)}-\frac{Q_2(u_2)}{\phi(u_2)}\Big]\quad \text{where}\quad \forall\gamma>0,\quad \forall u,v\in C^1(\mathbb{R}^n)$$

(*Z2*) : $\quad Z_\gamma(u,v):=|\nabla u|^{\gamma+1}-(\gamma+1)\Phi_\gamma(\frac{u}{v}\nabla v)\cdot\nabla u+\gamma|\frac{u}{v}\nabla v|^{\gamma+1}$

$$=|\nabla u|^{\gamma+1}-(\gamma+1)|\frac{u}{v}\nabla v|^{\gamma-1}\ \frac{u}{v}\nabla v\cdot\nabla u+\gamma|\frac{u}{v}\nabla v|^{\gamma+1}.$$

We recall that $\forall\gamma>0$ the two-form $Z_\gamma(u,v)\ge 0$ and is null only if $\exists k\in\mathbb{R};\ u=kv$. (see e.g. [1], [6, 7]).

We note that Z_γ is associated to the two-form Ψ_γ defined on $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ by

(Ψ) : $\quad \Psi_\gamma(X,Y):=|X|^{\gamma+1}-(\gamma+1)|Y|^{\gamma-1}Y\cdot X+\gamma|Y|^{\gamma+1}$

which is positive for non-null X and Y and null only if one of them is null or if $X=\mu Y$ for some $\mu\in\mathbb{R}$.

Lemma 3.1. *Consider for some strictly positive functions* $a,\ c_1,\ c_2$ *the strongly and bounded oscillatory solutions* u *and* v *with some overlapping nodal sets* $D(u^+)$ *and* $D(v^+)$ *of*

$$\begin{cases}\Big\{a(t)\phi(u')\Big\}'+c_1(t)\phi(u)=0=\Big\{a(t)\phi(v')\Big\}'+c_2(t)\phi(v);\quad t>0\\ \textit{where}\ \ \forall t\ge 0\qquad c_1(t)\le c_2(t).\end{cases}$$

If through some translation of variable say, $w(t):=v(t+\xi),\quad \xi\in\mathbb{R}$ *the new function satisfies for some* $s\in D(u^+)\quad w'(s)=u'(s)=0,$ *then*

$$D(w^+)\subset D(u^+)\qquad \textit{and}\qquad u^+(s)\ge w^+(s)=\max_{D(v^+)}\ v(t) \tag{3.12}$$

and $\qquad diam(v^+)=diam(w^+)\le diam(u^+).$

The same conclusions hold even when the equation in v *reads*

$\Big\{a(t)\phi(v')\Big\}'+c_2(t)\phi(v)+F(t,v,v')=0$ *with positive* $F(.,.,.)$.

Proof. Let s be the root of u' in $D(u^+)$. A Picone formula for the solutions is

$$\Big\{a(t)\Big[u\phi(u')-u\phi(\ \frac{u}{v}\ )\phi(v')\Big]\Big\}'=a(t)\zeta_\alpha(u,v)+[c_2(t)-c_1(t)]|u|^{\alpha+1}$$

which is strictly positive in $D(u^+)$. Therefore v has to have a zero inside $D(u^+)$. By a translation like $t\to t+\xi$ for a suitable choice of $\xi\in\mathbb{R}$ such that $V(t-\xi):=v(t)$ satisfies $V'(s)=u'(s)=0$, the Picone formula above in terms of V is

$$\Big\{a(t)\Big[u\phi(u')-u\phi(\ \frac{u}{V}\ )\phi(V')\Big]\Big\}'=a(t)\zeta_\alpha(u,V)+[c_2(t)-c_1(t)]|u|^{\alpha+1}$$

which remains also positive in $D(u^+)$. Let $[t_0,\ t_1] := D(u^+)$. The integration over $(t_0,\ s)$ in one hand and over $(s,\ t_1)$ in other hand of the equation give
$0 = \int_{t_0}^{s}\{a(t)\zeta_\alpha(u,V) + [c_2(t) - c_1(t)]|u|^{\alpha+1}\}dt$ and
$\int_{s}^{t_1}\{a(t)\zeta_\alpha(u,V) + [c_2(t) - c_1(t)]|u|^{\alpha+1}\}dt = 0$
each of which is absurd as each of the integrand is strictly positive. Therefore V has a zero inside each of the subintervals, leading to $D(V^+) \subset D(u^+)$ (see [12, 10]). By Corollary 2.2 , $\max_{[t_0,\ t_1]} V \le \max_{[t_0,\ t_1]} u$.
When F is introduced, the Pinone formula becomes
$$\left\{a(t)\left[u\phi(u')-u\phi(\ \frac{u}{V}\ )\phi(V')\right]\right\}' = a(t)\zeta_\alpha(u,V)+\left([c_2(t)-c_1(t)]+\frac{F(t,V,V')}{\phi(V)}\right)|u|^{\alpha+1}$$
which brings no extra difficulties.

□

Now let $\Omega \subset \mathbb{R}$ be an intervall and $m > 0$; $a \in C^1(\Omega,\ (m,\ \infty)))$ be increasing and $c \in C(\Omega,\ (m,\ \infty))$ be given.

Lemma 3.2. *Let* $F \in C(\ \mathbb{R}^3,\ [0,\ \infty)\ )$ *be a positive function and*

$b \in C^1(\mathbb{R})$ *with* $b' \neq 0$ *fore large* $t > 0$.

If u *is a strongly oscillatory solution and* v *a non-trivial and bounded solution in* Ω *of*

$$\begin{cases} (i) & \left\{a(t)\phi(u')\right\}' + c(t)\phi(u) = 0 \qquad \textit{and} \\ (ii) & \left\{a(t)\phi(v')\right\}' + b'(t)\phi(v') + c(t)\phi(v) + F(t,v,v') = 0, \end{cases} \tag{3.13}$$

then v *is also strongly oscillatory. Moreover in any* Ω_T *with large* $T > 0$ *any* $D(u^+)$ *overlaps with a* $D(v^+)$. *Let* s *denote the singularity of* u^+ *(i.e* $s \in D(u^+)$ *and* $u'(s) = 0$ *). There is* $\xi \in \mathbb{R}$ *such that the new oscillatory function* $V(t) := v(t+\xi)$ *satisfies* $V'(s) = u'(s) = 0$ *leading to*

$$D(V^+) \subset D(u^+) \quad \textit{and} \quad \max_{D(u^+)} V^+ \le \max_{D(u^+)} u^+. \tag{3.14}$$

Proof. a) That v is oscillatory follows from [8], Theorem 3.4 .
For the self containing concern, we give the sketch of the proof mainly because of the presence of the extra term F.
The Picone-type formula we chose for (3.13) is:

$$\begin{cases} \left\{a(t)u\phi(u') - u\phi(\ \frac{u}{v}\ )a(t)\phi(v') - u\phi(\ \frac{u}{v}\ )b(t)\phi(v')\right\}' \\ = a(t)\zeta_\alpha(u,v) + u\phi(\ \frac{u}{v}\ )F(t,v,v') - b(t)\left[u\phi(\ \frac{u}{v}v'\ )\right]'. \end{cases} \tag{3.15}$$

If in (3.13) $b(t)$ is replaced by $b(t)+\mu$, any $\mu \in \mathbb{R}$, we get (3.15) with $b(t)+\mu$ replacing $b(t)$ i.e.

$$\begin{cases} \left\{a(t)u\phi(u') - u\phi(\ \frac{u}{v}\ )a(t)\phi(v') - u\phi(\ \frac{u}{v}\ )a(t)(b(t)+\mu)\phi(v')\right\}' \\ = a(t)\zeta_\alpha(u,v) + u\phi(\ \frac{u}{v}\ )F(t,v,v') - (b(t)+\mu)\left[u\phi(\ \frac{u}{v}v'\ )\right]'. \end{cases} \tag{3.16}$$

If we assume that $v > 0$ in any $D := D(u^+)$ the integration over D of the above equation leads to

$$\forall\mu \in \mathbb{R}, \quad 0 = \int_D \left[a(t)\zeta_\alpha(u,v) + u\phi(\frac{u}{v})F(t,v,v') \right] dt \\ - \int_D \left(b(t) + \mu) \left[u\phi(\frac{u}{v}v') \right]' \right) dt \tag{3.17}$$

and that can hold only if each of the integrand is zero! Obviously the first integrand is not zero hence v has a zero in any $D(u^+)$ and is then also strongly oscillatory.

Let $D(u^+) := [t_0,\ s] \bigcup [s,\ t_1] = D_1 \bigcup D_2$ where $t_0 > T$, $b' \neq 0$ in Ω_T and $u'(s) = 0$. We chose $\xi \in \mathbb{R}$ such that the associate function to v, $V(t) := v(t+\xi)$ satisfies $V'(s) = u'(s) = 0$. Using D_1 or D_2 in the equations (3.16) and (3.17) we similarly conclude that V has one zero in each of them and $diam(V^+) := diam(v^+) \leq diam(u^+)$.

Consequently in both cases

$$D(V^+) \subset D(u^+) \ \text{ and } \ \max_{D(u^+)} V^+ \leq \max_{D(u^+)} u^+$$

by the Theorem 2.1 . In fact in one of $[t_0,\ s]$ or $[s,\ t_1]$, $b'(t)\phi(v') + F(t,v,v') > 0$ and the Theorem applies over that sub-interval.

The proof is completed . □

IV. Main Results

Let $m > 0$ and for some $T > 0$ $\quad \Omega := \Omega_T = (T,\ \infty)$. For some

$$\begin{cases} \text{increasing } a \in C^1(\Omega,\ (m,\ \infty));\ c, c_1 \in C(\Omega,\ (m,\ \infty)) \\ \text{with } \ c \leq c_1 \ \text{ and } \ b \in C^1(\Omega,\ \mathbb{R}) \ \text{ with } \ b' \neq 0 \ \text{ in } \ \Omega_T \end{cases} \tag{4.1}$$

the non-trivial and bounded solutions $u,\ v,\ w$ respectively of

$$\begin{cases} \left\{ a(t)\phi(u') \right\}' + c(t)\phi(u) = 0; \quad \left\{ a(t)\phi(v') \right\}' + c_1(t)\phi(v) = 0 \\ \text{and} \quad \left\{ a(t)\phi(w') \right\}' + c_1(t)\phi(w) + b'(t)\phi(w') = 0 \end{cases} \tag{4.2}$$

are strongly oscillatory (see e.g. [8]). Let a fixed nodal set $D(u^+) \subset \Omega_T$ of the solution u in (4.2) be set and $s \in D(u^+)$ be its singularity i.e. $u'(s) = 0$. Then there are $\xi, \theta \in \mathbb{R}$ such that the oscillatory oscillatory functions

$V(t) := v(t+\xi)$ and $W(t) := w(t+\theta)$ satisfy $W'(s) = V'(s) = u'(s) = 0$. Consequently

Theorem 4.1. *Consider a nodal set $D(u^+) \subset \Omega_T$ for a large $T > 0$ where u denotes a strongly oscillatory solution in (4.2) and $s \in D(u^+)$ its singularity. With $\delta(A) := diam(A)$ for any $A \subset \Omega_T$ we have the following estimates for the oscillatory functions described above:*

$$(i) \quad \delta(D(w^+)) \leq \delta(D(v^+)) \leq \delta(D(u^+)) \quad as\ W^+(s) \leq V^+(s) \leq u^+(s) \quad or$$
$$(ii) \quad \max_{t\in D(w^+)} w^+(t) \leq \max_{t\in D(v^+)} v^+(t) \leq \max_{t\in D(u^+)} u^+(t) \quad and \tag{4.3}$$
$$\max_{t\in D(u^+)} u^+(t) = Const. \max_{t\in D(u^+)} \left(\frac{a(t)}{c(t)}\right)^{1/(1+\alpha)}.$$

Proof. The translation function conserves the distance and the last lemmae explain the relations between the diameters. Lemma 3.1 applies for the comparison of the estimates of u and V; Lemma 3.2 for those of v and W.

To complete the proof, the equation $\left\{a(t)\phi(u')\right\}' + c(t)\phi(u) = 0$ being equivalent to

$$\left(\phi(u')\right)' + \frac{a'(t)}{a(t)}\phi(u') + \frac{c(t)}{a(t)}\phi(u) = 0,$$

from the relations above (Lemmas 3.1 and 3.2) and (3.9)

$$\max_{t\in D(u^+)} u^+ = Const. \max_{t\in D(u^+)} \left(\frac{a(t)}{c(t)}\right)^{1/(1+\alpha)}. \tag{4.4}$$

□

When the coefficients of the equation are radially symmetric (i.e. depends only on $r := |x|$) and satisfy H1), the radially symmetric version of the equation

$$\begin{cases} (i) \quad \nabla\cdot\Big(A(x)\Phi(\nabla u)\Big) + C(x)\phi(u) = 0, \quad x \in \mathbb{R}^n \\ \text{would read} \\ (ii) \quad \left\{r^{n-1}A(r)\phi(U')\right\}' + r^{n-1}C(r)\phi(U) = 0, \quad r > 0 \end{cases} \tag{4.5}$$

the coefficients in (4.5)(ii) being strictly positive with increasing A; that equation is equivalent to

$$\begin{cases} \Big[\phi(U')\Big]' + K(r)'\phi(U') + \dfrac{C(r)}{A(r)}\phi(U) = 0, \quad r > 0 \\ \text{where} \quad K'(r) := \{\ \log[r^{n-1}A(r)]\ \}'\ . \end{cases} \tag{4.6}$$

So, from the Theorem 4.1 and the estimates (3.9) we have the following

Theorem 4.2. *If the coefficients A and C are radially symmetric, strictly positive and A increasing, any bounded and non-trivial solution U of (4.5) is radially symmetric and satisfies (4.6). It satisfies in any Ω_T for large $T > 0$ and $D(U^+) := [R_1,\ R_2]$*

$$|R_1 - R_2| = R_2 - R_1 = const.\left[\frac{A(R_2)}{C(R_1)}\right]^{1/(\alpha+1)} = \max_{r\in D(U^+)} U^+(r). \tag{4.7}$$

Remark 4.3. *In one-dimensinal case the notion of the diameter of a $D(u)$ is without any confusion the length of the interval $[t_1,\ t_2] := D(u)$. But in multi-dimensional case, the situation is quite different. Thus we will estimate only the local maxima of u^+ for the moment.*

Define for any $w \in C(\mathbb{R}^n,\ \mathbb{R})$ and $\Omega_s^t := \{x \in \Omega\ ;\ s \le |x| \le t\}$

$$W_{st}^+(t) := \max_{x\in\Omega_s^t} w(x) \quad \text{and} \quad W_{st}^-(s) := \min_{x\in\Omega_s^t} w(x). \tag{4.8}$$

Given the strictly positive functions $a,\ c \in C(\mathbb{R}^n)$, to the equation

$$\begin{cases} (i) \quad \nabla\cdot\Big[a(x)\Phi(\nabla u)\Big] + c(x)\phi(u) = 0, \quad x\in\mathbb{R}^n \\ \text{ we associate} \\ (ii) \quad \nabla\cdot\Big[A_{ST}^+(T)\Phi(\nabla U)\Big] + C^-(r)\phi(U) = 0, \quad r>0 \\ \quad \text{when we assume that } D(u^+)\subset\Omega_R^T. \end{cases} \tag{4.9}$$

In fact the Picone-type formula, where $A = A(T) := A_{RT}^+(T)$ and $C(r) := C^-(r) = \min_{|x|=r} c(x)$ reads

$$\nabla\cdot\Big[AU\phi(\nabla U) - a(x)U\phi(\frac{U}{u})\Phi(\nabla u)\Big] =$$
$$a(x)Z_\alpha(U,u) + (A-a)|\nabla U|^{\alpha+1} + (c - C(r))|U|^{\alpha+1} > 0.$$

So, whenever (4.9)(ii) is oscillatory so is (4.9)(i) as u would have a zero in any nodal set $D(U^+)$ (see e.g. [1, 8]).

Any non-trivial and bounded Solution U of

$$\nabla\cdot\Big[A\Phi(\nabla U)\Big] + C(r)\phi(U) = 0, \quad r>0$$

is oscillatory and radially symetric and as in (3.7), the equation reads

$$\Big\{r^{n-1}A\phi(U')\Big\}' + r^{n-1}C(r)\phi(U) = 0, \quad r>0.$$

Theorem 4.4. *Consider for some strictly positive $a,\ c\in C(\mathbb{R}^n)$ the problem*

$$\nabla\cdot\Big\{a(x)\Phi(\nabla u)\Big\} + c(x)\phi(u) = 0, \quad x\in\mathbb{R}^n, \quad n\ge 3. \tag{4.10}$$

The equation is oscillatory .

1) If $a(x)$ is constant or is radially symmetric and differentiable (i.e. $a(x)\equiv a(|x|)$), then

$$\max_{x\in D(u^+)} u^+(x) = O\bigg(\Big[\frac{a(r)}{C(r)}\Big]^{1/(1+\alpha)}\bigg) \quad \textit{for large } |x| = r. \tag{4.11}$$

2) In general, when we take $D(u^+) := D_{RT}(u^+)\subset\Omega_R^T$ for a large $R>0$, the same conclusion holds and (4.11) becomes

$$\max_{x\in D(u^+)} u^+(x) = O\bigg(\Big[\frac{A^+(T)}{C(r)}\Big]^{1/(1+\alpha)}\bigg) \quad \textit{for large } r = |x| \in (R\ ,\ T). \tag{4.12}$$

Proof. 1) From before, we can assume that under the hypotheses, (4.8)(i) and (4.8)(ii) fulfill the conditions (2.1)(i)a and (i)b of the Theorem 2.1. In addition u^+ has a zero inside any $D(U^+)$. Therefore Theorem 2.1 applies here; $U^+ \geq u^+$ whenever through a translation of U, the singularity x_0 of u is such that $|x_0|$ is close enough to r_0 where $\nabla u(x_0) = 0$ and $U'(r_0) = 0$. All this show that

$$\max_{x \in D(u^+)} u^+(x) \leq \max_{x \in D(U^+)} U^+(x)$$

whenever $|x_0|$ is very close to r_0 through a suitable translation of U. Let V be an oscillatory solution of

$$\begin{cases} (i) \quad \Big[r^{n-1}A(r)\phi(V')\Big]' + r^{n-1}C(r)\phi(V) = 0, \quad r > 0 \\ \text{or with } K(r) := \log\{r^{n-1}A(r)\} , \\ (ii) \quad \Big(\phi(V')\Big)' + K'(r)\phi(V') + \dfrac{C(r)}{A(r)}\phi(V) = 0, \quad r > 0 \end{cases} \tag{4.13}$$

where $A(r) = a(r)$ if a is radially symmetric or constant. From Theorem 4.1 , if $\dfrac{C(r)}{A(r)}$ is unboubded above and continuous, then as $r \nearrow \infty$

$$\max_{r \in D(V^+)} V^+(r) = const. \max_{r \in D(V^+)} \Big(A(r)/C(r)\Big)^{1/(1+\alpha)}.$$

2) is a mere application of 1) as $A^+(T)$ is constant.

□

Dedicated to my late mother and aunts:
Meguem Homsi Justine *(1926-January 1, 2019);*
Moyum Victorine *(1938- 2018) and*
Mafoko Veronique *(+ Oct. 2018)*

"Dû de la bonté, de la bienveillance et gratitude dont vous avez fait preuve durant votre vie, que nos ancêtres vous accueillent et vous guident dans la paix et la sérénité perpétuelles."

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