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An analysis of flow of Temperature by heat equation on using Double Laplace transforms

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Abstract: *In Integral transform, The Laplace transform has wide application to study of differential equation. We study some partial differential equations and analyze flow of temperature in to a rod by using Double Laplace Transform and getting a length independent temperature relation.*
Kew words*: Laplace Transform, Circuit, current, temperature, time, length etc.*

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I. Introduction

Integral Transforms are used to analyze functions. For example, Fourier transforms are used to compare functions to sines and cosines, while Laplace Transforms can be used to compare functions to the exponential function.

The main feature in using Laplace Transforms for differential equations is the fact that transforming the derivative results in an expression with no derivatives. Therefore, the Laplace transform used on linear differential equations with constant coefficients changes a differential equation into an algebraic equation.

This includes understanding what the improper integral represents. Existence of the Laplace Transform: f (t) should be piecewise continuous (no vertical asymptotes), and should be of exponential order.

The Laplace- Bi Lateral Laplace transform is used to find the Laplace - Mellin integral transform in the range [0, 0] to [∞, ∞], properties like Linear property, Scaling property, Power property, theorems like inversion theorem, convolution theorem, Parseval's theorem, Shifting theorem. By using this integral transform we obtain the results of derivative w. r. t. x, and we find the generalized result of derivative of the function f(x, y) w. r. t. x.. Using these derivatives we solve the Laplace equation in Cartesian form, one dimensional wave and heat flow equations.

The two-sided Laplace transform or Bi-Lateral Laplace transform is an integral transform equivalent to probability's moment generating function. Two-sided Laplace transforms are closely related to the Fourier transform, the Mellin transform, and the ordinary or one-sided Laplace transform. If $f(t)$ is a real or complex valued function of the real variable t defined for all real numbers, then the two-sided Laplace transform is defined by the integral

$$\mathcal{B}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt.$$

The integral is most commonly understood as an improper integral, which converges if and only if each of the integrals

$$\int_0^{\infty} e^{-st} f(t)\, dt, \quad \int_{-\infty}^{0} e^{-st} f(t)\, dt$$

exists. There seems to be no generally accepted notation for the two-sided transform; the $\mathcal{B}$ used here recalls "bilateral". The two-sided transform used by some authors is

$$\mathcal{T}\{f(t)\} = s\mathcal{B}\{f\} = sF(s) = s\int_{-\infty}^{\infty} e^{-st} f(t)\, dt.$$

The argument t can be any variable, and Laplace transforms are used to study how differential operators transform the function.

Bilateral transforms don't respect causality. They make sense when applied over generic functions but when working with functions of time (signals) unilateral transforms are preferred. so there is need to integrate and analysis of differential equations with Bi-Lateral transform and is to obtain the solution of the certain partial differential equations by using convolutions and Bi-Lateral Laplace transform. Differential equations have most important role in all field like biology physics and mathematics, there are so many applications are available to represent this, mycelia growth, water pollution in bio and velocity, acceleration in physics etc. So there is more scope generating for work on Partial differential equation.

In science and engineering applications, the argument *t* often represents time (in seconds), and the function *f*(*t*) often represents a signal or waveform that varies with time. In these cases, the signals are transformed by filters that work as mathematical operator, but with a restriction. They have to be causal, which means that the output in a given time *t* cannot depend on an output which is a higher value of *t*. In population ecology, the argument *t* often represents spatial displacement in a dispersal kernel.

When working with functions of time, *f*(*t*) is called the **time domain** representation of the signal, while *F*(*s*) is called the **s-domain** (or *Laplace domain*) representation. The inverse transformation represents a *synthesis* of the signal as the sum of its frequency components taken over all frequencies, whereas the forward transformation represents the *analysis* of the signal into its frequency components.

The Bi-lateral Laplace transforms we use convolution operations together with Bi- lateral Laplace transform to find the particular solution of certain Partial differential equations.

II. Double Laplace Transform:

Analytical Theory of Probability contained some basic results of the Laplace transform which is one of the oldest and most commonly used linear integral transforms available in the mathematical literature. This has effectively been used in finding the solutions of linear differential, difference and integral equations. On the other hand, Joseph Fourier's (1768–1830) monumental treatise on La Theories [4], [5].

The double Laplace transform of a function *f (x, y)* of two variables *x* and *y* defined in the first quadrant of the x–y plane is defined by the double integral in the form

$$L_2[f(x,y);s_1,s_2]=\int_0^\infty\int_0^\infty e^{-s_1x-s_2y}f(x,y)dxdy=u(s_1,s_2) \qquad \textbf{(2.1)}$$

Where

$$u(s_1,s_2)=L_2[f(x,y)]=L[L\{f(x,y):x\to s_1\}:y\to q]=L\{u(s_1,y):y\to q$$

Provided the integral exists, where we fallow Debnath and Bhatta [28] to denote Laplace transform $\overline{f(s)}$ = L{f(x)}= $\int_0^\infty e^{-sx}\, f(x)dx$, Re(s)>0 (2.2)

The inverse double Laplace Transform $L_2^{-1}\{\, u(s_1,s_2)\}$= f(x, y) is defined by the complex double integral

$L_2^{-1}\{\, u(s_1,s_2)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s_1x}ds_1\,\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{s_2x}u(s_1,s_2)ds_2$ (2.3) Where $u(s_1,s_2)$ must be an analytic function for all s_1 and s_2 in the region defined by the inequalities Re(s_1)>c and Re(s_2)>d, where c and d are arbitrary suitable constant.

2.1 Linear Property of double Laplace and Inverse Laplace Transform:

1. Laplace transform

$$L_2\{a_1f_1(x,y)+a_2f_2(x,y)\}=\int_0^\infty\int_0^\infty e^{-s_1x-s_2y}\{a_1f_1(x,y)+a_2f_2(x,y)\}dxdy$$
$$=a_1\int_0^\infty\int_0^\infty e^{-s_1x-y}f_1(x,y)dxdy+a_2\int_0^\infty\int_0^\infty e^{-s_1x-s_2y}f_2(x,y)\}dxdy$$
$=a_1L_2\{f_1(x,y)\}+a_2L_2\{f_2(x,y)\}$, where a_1 and a_2 are constant.

2. Inverse Laplace Transform:

$$L_2^{-1}\{a_1u_1(s_1,s_2)+a_2u_2(s_1,s_2)\}=a_1\, L_2^{-1}\{u_1(s_1,s_2)\}+a_2L_2^{-1}\{u_2(s_1,s_2)\}$$

2.2 First translating shifting Property:

If $L_2[f(x,y)]=u(s_1,s_2)$

Then show that

$$L_2\left[e^{(ax+by)}f(x,y)\right]=u(s_1-a,s_2-b)$$

Proof:
We know that

$$L_2[f(x,y);s_1,s_2] = \int_0^\infty \int_0^\infty e^{-s_1x-s_2y} f(x,y)dxdy = u(s_1,s_2) \quad \ldots(2.4)$$

Then we have

$$L_2\left[e^{(ax+by)}f(x,y);s_1,s_2\right] = \int_0^\infty \int_0^\infty e^{-s_1x-s_2y}e^{(ax+by)}f(x,y)dxdy$$

$$= \int_0^\infty \int_0^\infty e^{-(s_1-a)x-(s_2-b)y} f(x,y)dxdy$$

by using (3.5.1), we can say

$$L_2\left[e^{(ax+by)}f(x,y)\right] = u(s_1-a,s_2-b)$$

2.3 CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS FOR SIGNAL SYSTEM:
Spatially distributed systems are mathematically described by PDEs. The most common types of PDEs are called *elliptic*, *parabolic* and *hyperbolic* equations. [6][9]In what follows us roughly clear these terms on examples of second-order PDEs. Let $a>0$, $b>0$, $c>0$ be constants, u be a solution, f right-hand side, ∇ denotes Del and $\Delta = \nabla^2$.
The elliptic equation has the form

$$-c\Delta u - b\nabla u - au = f.$$

Elliptic equations do not contain the time variable. They describe stationary states whose control is not subject of this paper.
The parabolic equation has the form

$$d\frac{\partial u}{\partial t} - c\Delta u - b\nabla u - au = f \quad \ldots (2.3.1)$$

Where d >0 Parabolic equations contain first derivation with respect to time.
Heat conduction, diffusion, chemical reactions and others irreversible processes are described by a parabolic PDE. If boundary conditions are non-negative and f is non-negative, then a solution u is non-negative too.

The hyperbolic equation has the form

$$d_1\frac{\partial^2 u}{\partial t^2} + d_2\frac{\partial u}{\partial t} - c\Delta u - b\nabla u - au = f,$$ Where $d_1>0$ and $d_2>0$.

Many of the equations of mechanics, including waves, oscillations and deformations, are hyperbolic.

2.4 Partial Differential Equation for heat conduction:
A typical representative of parabolic PDE is *heat conduction*. A model of heat conduction in a rod equipped with an array of temperature sensors and heaters is schematically sketched in Fig. 2.1. It is described by well-known heat equation

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \quad \ldots (2.4.1)$$ with initial condition u(0, t)= 1, $u_x(0, t)\} = 0$ and u(x,0) = 1, for k = 1

Where u denotes temperature (^{0}C), f the input heat ($^0\text{C s}^{-1}$), t and x denote time (s) and a spatial coordinate (m), respectively, and $\kappa = \frac{\chi}{\rho c_p}$, is a constant ($\text{m}^2\ \text{s}^{-1}$), where κ is the thermal conductivity ($\text{W m}^{-1}\text{k}^{-1}$), ρ is the density(kgm^{-3}) and $c_p$is the heat capacity per unit mass ($\text{J K}^{-1}\ \text{kg}^{-1}$).
Since when we input heat in a rod then temperature is directly proportional to input heat. So $u(x,t) \propto f(x,t)$, then we can say $u(x,t) = \lambda f(x,t)$, where λ is heat resistive coefficient.

Then $f(x,t) = \frac{1}{\lambda}u(x,t)$ … (2.4.2) with initial if u(0, t)=1, then f(0, t) = $\frac{1}{\lambda}$, if $u(x,0) = 1$, then $f(x,0) = \frac{1}{\lambda}$ and $f_x(0, t) = 0$.

Taking Laplace both side in equation 2.4.2

$$L_2\{u_t(x,t)\} = kL_2\{u_{xx}(x,t)\} + L_2\{f(x,t)\}$$

$$S_2\breve{\breve{u}}(s_1,s_2) - L\{u(x,0)\}$$

$$= k\left[s_1^2\breve{\breve{u}}(s_1,s_2) - s_1L\{u(0,t) - \ L\{u_x(0,t)\}\right] + L_2\left\{\frac{1}{\lambda}\ \mathrm{u(x,t)}\right\} \quad \ldots (2.4.3)$$

$$s_2\breve{\breve{u}}(s_1,s_2) - \frac{1}{s_1} = k\left[s_1^2\breve{\breve{u}}(s_1,s_2) - s_1\frac{1}{s_2}\right] + \left\{\frac{1}{\lambda}u(s_1,s_2)\right\}$$

$$\{s_2 - ks_1^2\}\breve{\breve{u}}(s_1,s_2) = k\frac{s_1}{s_2} + \frac{1}{s_1} + \left\{\frac{1}{\lambda}u(s_1,s_2)\right\}$$

$$\{s_2 - ks_1^2 - \frac{1}{\lambda}\}\breve{\breve{u}}(s_1,s_2) = k\frac{s_1}{s_2} + \frac{1}{s_1}$$

$$\breve{\breve{u}}(s_1,s_2) = k\frac{s_1}{s_2\{s_2 - ks_1^2 - \frac{1}{\lambda}\}} + \frac{1}{s_1\{s_2 - ks_1^2 - \frac{1}{\lambda}\}}$$

$$\breve{\breve{u}}(s_1,s_2) = k\frac{s_1}{s_2\{s_2 - ks_1^2 - \frac{1}{\lambda}\}} + \frac{1}{s_1\{s_2 - ks_1^2 - \frac{1}{\lambda}\}}$$

If k = 1 then above equation can be written as

$$\breve{\breve{u}}(s_1,s_2) = \frac{s_1}{s_2\{s_2 - s_1^2 - \frac{1}{\lambda}\}} + \frac{1}{s_1\{s_2 - s_1^2 - \frac{1}{\lambda}\}}$$

$$\breve{\breve{u}}(s_1,s_2) = -\frac{s_1}{s_2\left\{s_1^2 - s_2 + \frac{1}{\lambda}\right\}} - \frac{1}{s_1\{s_1^2 - s_2 + \frac{1}{\lambda}\}}$$, by using partial fraction this equation can be written as

$$\breve{\breve{u}}(s_1,s_2) = \frac{s_1}{(s_1^2 - \frac{1}{\lambda})}\left[\frac{1}{\{s_2 - (s_1^2 - \frac{1}{\lambda})\}} - \frac{1}{s_2}\right] + \frac{1}{s_1\{s_2 - (s_1^2 - \frac{1}{\lambda})\}}$$

Taking inverse Laplace with respect to s_2

$$\breve{u}(s_1,t) = \frac{s_1}{(s_1^2 - \frac{1}{\lambda})}\left[e^{\left(s_1^2 - \frac{1}{\lambda}\right)t} - 1\right] + \frac{1}{s_1}e^{\left(s_1^2 - \frac{1}{\lambda}\right)t}$$

$$\breve{u}(s_1,t) = \frac{s_1}{(s_1^2 - \frac{1}{\lambda})}.e^{\left(s_1^2 - \frac{1}{\lambda}\right)t} - \frac{s_1}{(s_1^2 - \frac{1}{\lambda})} + \frac{1}{s_1}e^{\left(s_1^2 - \frac{1}{\lambda}\right)t}$$

$$since\ e^{\left(s_1^2 - \frac{1}{\lambda}\right)t} = 1 + \left(s_1^2 - \frac{1}{\lambda}\right)t + \frac{\{\left(s_1^2 - \frac{1}{\lambda}\right)t\}^2}{2!} + \frac{\{\left(s_1^2 - \frac{1}{\lambda}\right)t\}^3}{3!} + \cdots$$

Then above equation can be written as:

$$\breve{u}(s_1,t) = \frac{s_1}{\left(s_1^2 - \frac{1}{\lambda}\right)}\left[1 + \frac{\left(s_1^2 - \frac{1}{\lambda}\right)t}{1!} + \frac{\left\{\left(s_1^2 - \frac{1}{\lambda}\right)t\right\}^2}{2!} + \frac{\left\{\left(s_1^2 - \frac{1}{\lambda}\right)t\right\}^3}{3!} + \cdots\right] - \frac{s_1}{\left(s_1^2 - \frac{1}{\lambda}\right)}$$

$$+\frac{1}{s_1}\left[1+\left(s_1^2-\frac{1}{\lambda}\right)t+\frac{\{\left(s_1^2-\frac{1}{\lambda}\right)t\}^2}{2!}+\frac{\{\left(s_1^2-\frac{1}{\lambda}\right)t\}^3}{3!}+\cdots\right]$$

$$\breve{u}(s_1,t)=\left[\frac{s_1}{\left(s_1^2-\frac{1}{\lambda}\right)}+\frac{s_1 t}{1!}+\frac{\left(s_1^2-\frac{1}{\lambda}\right).s_1.t^2}{2!}+\frac{\left\{\left(s_1^2-\frac{1}{\lambda}\right)\right\}^2 s_1.t^3}{3!}+\cdots\right]$$

$$-\frac{s_1}{\left(s_1^2-\frac{1}{\lambda}\right)}+\left[\frac{1}{s_1}+\frac{(s_1-\frac{1}{\lambda s_1})t}{1!}+\frac{\left(s_1^3-\frac{2}{\lambda}s_1+\frac{1}{\lambda^2 s_1}\right)t^2}{2!}+\cdots.\right]$$

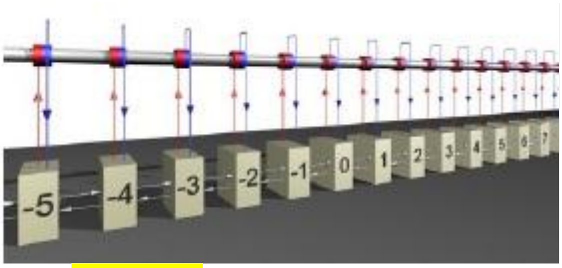
Now taking inverse Laplace along S_1

$$u(x,t)=\left[\cosh\sqrt{\frac{1}{\lambda}}x+0+0+\cdots\right]-\cosh\sqrt{\frac{1}{\lambda}}x+1-\frac{1}{\lambda}t+\frac{\left(\frac{1}{\lambda}t\right)^2}{2!}-\frac{\left(\frac{1}{\lambda}t\right)^3}{3!}+\cdots$$

Therefore

$$u(x,t)=e^{-\frac{1}{\lambda}t} \qquad (2.4.4)$$

Here we see that u(x, t) is free from x so we can say the decay of temperature depend only on time not the length of rod. That is when the thermal conductivity is taken as unity (k = 1), that is on unit thermal conductivity the input heat in the rod decrease its temperature with time and u(x, t) does not depend distance from one end. It is also shows that it is bounded as $t \to \infty$, the temperature $u(x,t) \to 0$.



(Fig.2.1)

III. Results and Discussion:

Laplace transformations and inverse Laplace Transformations. We have given numerous illustrative examples on applications of these results in two dimensions. The heat conduction in a rod equipped with an array of temperature sensors and heaters, from equation 2.4.4 it is clearly found that temperature of rod by heat conduction is depend only on time because $u(x,t)=e^{-\frac{1}{\lambda}t}$ **(Dayal's Formula)** is length independent function.

We believe that these results will further enhance the use of 2 dimensional Laplace Transformation and help further development of more theoretical results. Several initial boundary value problems (IBVPs) characterized by Non-Homogenous linear partial differential equations (PDEs) are explicitly. Even though multi-dimensional Laplace transformation have been studied extensively since the early 1920s, or so, still a table of three on N-dimensional Laplace transforms is not available. To fill this gap much work is left to be done. To this end, we have developed several new results on 2-dimensional Laplace transformations as well as inverse Laplace transformation and many more are still under our investigation. A successful completion of this task will be a significant endeavor, which will be extremely beneficial to the further research in Applied Mathematics, Engineering and Physical Sciences. Specially, by the use of multi-dimensional Laplace transformations a PDE and its associated boundary conditions can be transformed into an algebraic equation in n independent variables, this algebraic equation can be solved to obtain the desired solution.

IV. Conclusion:

After seeing above temperature relation in equation 2.4.4 to rod we should familiar about few applications that if we use this sensor into car, this may be maintain temperature of car and also during outer flow of temperature it could be help to remove fog during driving the car. So we can develop a sensor which is length independent that is small in size easily can fit in to car. Therefore after removing the car the driving is very safe and smooth.

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