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Solving The Assignment Problems Directly Without Any Iterations

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Abstract: *Assignment problems arise in different situation in our daily life where we have to assign* ***n*** *objects to* ***m****-other objects in such a way that can minimize the cost or maximize the profits. The assignment problems are a well studied topic in Operation Research especially in Optimization Problems. These problems find numerous application in production planning, telecommunication, economic …etc. It is done in such a way that the cost or time or distance involved in the process is to be minimized and the profits or returns are to be maximized. From here I can define two types of the assignments problems:*
1- *The Balanced assignment problems in which we are to assign number of resources (suppose equals N) to the same number of activities also equals N.*
2- *The Unbalanced Assignment problems is achieved when the number of resourses differs or not equal to the number of activities or tasks.*
in this research I propose a new method which gives the optimal solution directly without any iteration Contrary to previous methods such as Hungarian Method which begins with an initial solution and then begins to improve the solution in order to arrive with the Optimal solution after nearly two or three iterations . But my method is based on calculationg a new matrix from the given cost matrix, and using these two matrices to end up with a final matrix from which we can obtain the optimal assignments.This method can be applied for both balanced and unbalanced problems without using Dummy Cells as in Hungarian Method. And also it can be applied for both maximization and minimization problems. To show the efficiency of this method I'll consider some numerical examples for balanced and unbalanced assignment problems, and also for minimization and maximization assignment problems.
Keywords: *Assignment Problems, Balanced and Unbalanced assignment Problems, Constraints, Objective Function, Optimal Solution and Dummy Cell.*

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I. Introduction

The assignment problem is a standard topic discussed in operations research textbooks [8] and [10]. It is an important subject, put forward immediately after the transportation problem, is the assignment problem. This is particularly important in the theory of decision making. The assignment problem is one of the earliest application of linear integer programming problem. Different methods have been presented for assignment problem and various articles have been published on the subject. See [1], [7] and [9] and for the history of these methods.

The assignment problem is one of the fundamental combinatorial optimization problems in the branch of optimization and operations research in mathematics. The problem instance has a number of *agents* and a number of *tasks*. Any agent can be assigned to perform any task, It is required to perform all tasks by assigning exactly one agent to each task and exactly one task to each agent in such a way that the *total cost* of the assignment is minimized. Different methods have been presented for assignment problem and various articles have been published on the subject. Recently, I introduced a quick new method for solving the Balanced Assignment Problem which is a combination between the Hungarian method and Vogel method[1].

1. Assumptions of The Assignment Problem:

An Ssignment problem must satisfy the following assumptions:
1- The number of assignees and the number of assignments is the same and then it is called balanced assignment problem, otherwise it is unbalanced one.
2- Each assignee is to be assigned to one and only one task.
3- Each task is to be performed by exactly one assignee.
4- There is a cost Cij associated with each assignee.

II. Formulation Of The Problem

Suppose that an assignment problem has n machines and n jobs so as to minimize the total cost or time in such a way that each machine can be assigned to one and only one job [1],[3] and [6]. The cost matrix C_{ij} is given as:

$$\text{Resources}\overset{\textit{Activities}}{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}}$$

Let X_{ij} denotes the assignment of i[th] resource to j[th] activity such that:

$$X_{ij} = \begin{cases} 1; \textit{if} \ \text{ resource i is assigned to activity j} \\ 0; \text{ otherwise} \end{cases}$$

Thus, the mathematical formulation of the assignment problem is:

$$Minimize \ Z = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} X_{ij} \quad (or \ \ Maximize)$$

Subject to the constraints:

$$\sum_{i=1}^{n} X_{ij} = 1, \ for \ \text{all } i = 1, 2, ...n$$

$$\sum_{j=1}^{n} X_{ij} = 1, \ for \ \text{all j} = 1, 2, ...n$$

$$X_{ij} = 0 \ or \ 1$$

The first n constraints ensure that each origin is assigned to one and only one destination; the next n constraints ensure that each destination is assigned to one and only one origin

Theorem:
Consider the objective function of an assignment problem:

$$z = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij} x_{ij}$$

If a solution xij is optimal for the objective function z, then it is also optimal for the problem in which the objective function has been replaced by z' :

$$z' = \sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}\, x_{ij} \quad \textbf{(where } c'_{ij} \textbf{ is the expected cost matrix)}$$

Where $c'_{ij} = c_{ij} - \Delta_{ij}$,*where* Δ_{ij} is a constant for $i, j = 1....n$

Proof.

$$z' = \sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij} x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} (c_{ij} - \Delta_{ij}) x_{ij} =$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij} x_{ij} - \sum_{j=1}^{n}\sum_{i=1}^{n} \Delta_{ij} x_{ij} =$$

$$\text{thus} \quad z' = z - \sum_{j=1}^{n}\sum_{i=1}^{n} \Delta_{ij} x_{ij} = z - k.$$

The new objective function z' differs from the original z by a constant k. Therefore, the optimal values of xij are the same in both cases. The theorem enables us to transform the original assignment costs tableau into an equivalent one, by decreasing Δ_{ij} . So the objective function decreases by the value subtracted, but the solution to the assignment problem remains unchanged, since any solution must have exactly one xij = 1 in each row and column.

III. Algorithm Of The Problem

Let $X_1, X_2, \ldots, X_n$ denote the resources . And let A,B,.. ..denote the activities. Now perform the following steps:

1- Construct the cost matrix. Consider rows for resources and columns for activities.

2- We add an additional column in which we put the sum of cost in each row.

3- We add an additional row in which we put the sum of cost in each column.

4- We will have a one empty cell in which we put the total cost of all the matrix.

5- We costruct another matrix (The expected cost matrix), and we begin to calculate the expected cost for each real cell as follows: $E_{ij} = \frac{R_i C_j}{Total \cos t}$

6- After calculating all expected values and put the new expected cost matrix we subtract the original cost matrix from the calculated one and arrive with Δ_{ij} matrix

7- Now after subtraction we begin the assignment as follows:

A- If the problem is to be ***minimized*** we begin to assign from the ***most negative difference*** Δ_{ij} one after one untill we do all the assignments.

B- If the problem is to be ***maximized*** we begin to assign from the ***most positive difference*** Δ_{ij} one after one untill we do all the assignments.

2. Examples

Example 1. (***Minimization and balanced*** [2] ***assignment problem)***

Consider a minimization assignment problem with the following 3×3 cost matrix which represents 3 employees and 3 jobs.

$$\begin{bmatrix} 25 & 40 & 35 \\ 40 & 60 & 35 \\ 20 & 40 & 25 \end{bmatrix}$$

First: we add an additional row and column for the matrix as follows:

	Job1	**Job2**	**Job3**	**Row Sum**
Emp.1	**25**	**40**	**35**	**R_1= 100**
Emp.2	**40**	**60**	**35**	**R_2= 135**
Emp.3	**20**	**40**	**25**	**R_3= 85**
Column Sum	**C_1= 85**	**C_2= 140**	**C_3= 95**	***Total* cost=320**

Second: we calculate Eij for each cell as follows

$$E_{ij} = \frac{R_i C_j}{Total\ \text{cost}}$$

$$E_{11} = \frac{R_1 C_1}{Total} = \frac{100 \times 85}{320} = 26.56$$

$$E_{12} = \frac{R_1 C_2}{Total} = \frac{100 \times 140}{320} = 43.75$$

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$$E_{33} = \frac{R_3 C_3}{Total} = \frac{85 \times 95}{320} = 25.23$$

Third: Now we put all the calculated values in the expected cost matrix as given:

Expected cost matrix			
	Job1	**Job2**	**Job3**
Emp.1	26.56	43.75	29.68
Emp.2	35.86	59.06	40.078
Emp.3	22.578	37.188	25.23

Fourth:finding differences

Δ_{ij} **cost matrix**			
	Job1	**Job2**	**Job3**
Emp.1	-1.56	-3.75	5.32
Emp.2	4.14	0.94	-5.078
Emp.3	-2.578	2.812	-0.23

Most negative do assignment here

Means assign Emp.2 to Job3. Delete or mark this row and column.
Then from the remaining submatrix the most negative cell is (-3.75) which means assigning ***Emp1. To Job2***. The last one is to assign ***Emp.3 to Job 1***
Thus we left with the minimized optimal solution cost=35+40+20=95
Hint: This is the same solution see reference [2] and with the same assignments and the same cost.

Example 2.(Minimization and unbalanced assignment problem)
Consider the following assignment problem. Assign 4 jobs to 5 persons with minimum cost [4] . And here is the cost matrix:

$$\begin{array}{c} \qquad persons \\ Jobs \begin{bmatrix} 5 & 7 & 11 & 6 & 5 \\ 8 & 5 & 5 & 6 & 5 \\ 6 & 7 & 10 & 7 & 3 \\ 10 & 4 & 8 & 2 & 4 \end{bmatrix} \end{array}$$

Δ_{ij} Cost Matrix					
	A	B	C	D	E
1	-2.95	0.69	1.68	0.25	0.34
2	1.22	-0.38	-2.95	1.09	1.03
3	-1.71	0.88	0.95	1.41	-1.52
4	3.45	-1.19	0.33	-2.74	0.16

Most negative cells do assignments here

***Means assign* 1 to A and 2 to C, then mark or delete these rows and columns.**
The most negative cell becomes (-2.74) do assignment here, assign 4 to D.
Then the last assignment is 3 to E.
The total minimum optimal cost is = 5+5+2+3=15

Hint: This is the same solution see reference [4] but with different assignments and the same cost without using any Dummy cells
Note: *Thus this method can be used not only for balanced but also for unbalanced assignment problems***.**
Example 3. (Maximization and balanced assignment problem)
Consider the following assignment problem [5], where the total profit is to be maximized.
A company has 5 jobs to be done. The following matrix shows the return in terms of rupees on assigning i^{th} (i = 1, 2, 3, 4, 5) machine to the j^{th} job (j = A, B, C, D, E). Assign the five jobs to the five machines so as to maximize the total expected profit.

$$\begin{array}{c} \quad\quad\quad\quad Jobs \\ Machines \begin{bmatrix} 5 & 11 & 10 & 12 & 4 \\ 2 & 4 & 6 & 3 & 5 \\ 3 & 12 & 5 & 14 & 6 \\ 6 & 14 & 4 & 11 & 7 \\ 7 & 9 & 8 & 12 & 5 \end{bmatrix} \end{array}$$

Applying the previous algorithm we can obtain the Δ_{ij} profit matrix :

Δ_{ij} **profit matrix**					
	A	**B**	**C**	**D**	**E**
1	-0.2	0.4	**2.5**	0.2	-2.1
2	-0.5	-1.4	2.44	-2.6	**2.1**
3	-1.97	1.2	-2.13	**2.76**	0.2
4	0.8	**2.65**	-3.5	-0.8	0.9
5	**1.9**	-2.08	0.7	0.5	-0.98

Most positive cell, do assignment here

As the problem is to be maximized, so we begin to make the assignment from the most positive cell oppositely to the minimization problem.
We assign **3 to D**, mark that row and column.
Then the most positive cell which comes after is (2.65), **assign 4 to B**, mark this row and column.
The positive cell which comes after is (2.5), **assign 1 to C**, mark that row and column.
The other positive cell which follows is (2.1), **assign 2 to E**, mark that row and column.
Lastly we **assign 5 to A**.
***The optimal maximized profit is* = 10+5+14+14+7=50**
Hint: This is the same solution see reference [5] and with the same assignments and the same cost.

Example 3. (**Maximization and unbalanced assignment problem**)
Consider the following unbalanced assignment problem, where the total profit is to be maximized.

$$\begin{bmatrix} 3 & 6 & 2 & 6 \\ 7 & 1 & 4 & 4 \\ 3 & 8 & 5 & 8 \\ 6 & 4 & 3 & 7 \\ 5 & 2 & 4 & 3 \\ 5 & 7 & 6 & 4 \end{bmatrix}$$

Applying the previous algorithm we can obtain the Δ_{ij} profit matrix :

Δ_{ij} **Profit Matrix**				
	A	**B**	**C**	**D**
1	-1.36	1.79	-1.61	1.19
2	**2.894**	-2.96	0.61	-0.53
3	-3.16	**2.05**	-0.09	1.21
4	0.87	-0.95	-1.24	**1.34**
5	1.41	-1.46	1.03	-0.96
6	-0.64	1.55	**1.33**	-2.33

Most positive cell, do assignment here

Means assign 2 to A, mark that row and column.
The most positive cell which follows is (2.05) , assign 3 to B, mark the row and column.
The cell which follows is (1.34), assign 4 to D, mark that row and column.
Finally assign 6 to C.
The optimal maximized profit is = 7+8+7+6=28
Hint: I solve the problem without using any Dummy cells

IV. Conclusion

In this paper a new approach was introduced for solving assignment problem. This new method based on constructing an expected cost (or profit) matrix from the given real cost (or profit) matrix. And then subtracting these two matrices to obtain Δ_{ij} expected cost (or profit) matrix, from which we can do the assignments. Using the most negative cells in the minimization problems and the most positive cells in the maximization problems.

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