IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 3 Ser. II (May – June 2020), PP 10-12
www.iosrjournals.org

The special functions and the proof of the Riemann's hypothesis

Mohamed SGHIAR
Université de Bourgogne Dijon, Faculté des sciences Mirande, Département de mathématiques .
9 av alain savary 21078 , Dijon cedex, France

Abstract *: By studying the $\hat{S}$ function whose integer zeros are the prime numbers, and being inspired by the article [2], I give a new proof of the Riemann hypothesis.*

Résumé *: En étudiant la fonction $\hat{S}$ dont les zéros entiers sont les nombres premiers, et en m'inspirant de l'article [2], je donne une nouvelle preuve de l'hypothèse de Riemann.*

Date of Submission: 04-05-2020 Date of Acceptance: 18-05-2020

I. Introduction

The Riemann's hypothesis [2] conjectured that all nontrivial zeros of ζ are in the line $x=\frac{1}{2}$.

In this article, the study of the sghiar's function $\hat{S}$ which I introduced and whose integer zeros are the prime numbers inspired me to use the Gamma function Γ . And miraculously a proof similar to that used in [2] allowed me to give a short and elegant proof of the Riemann Hypothesis.

In order not to recall everything, I suppose known - among others - the functions zeta ζ , Gamma Γ : $z \rightarrow \int_0^{+\infty} t^{z-1} e^{-t} dt$ and their properties (See [3] and [4]).

II. The Proof Of The Riemann Hypothesis

Theorem 1[The Riemann hypothesis] : All non-trivial zeros of ζ are in the line $x=\frac{1}{2}$.

Lemma 1 : $0<\Re(z)<1 \Rightarrow \left| \int_0^{+\infty} \frac{t^{z-1}}{e^t-1} dt \right| \neq 0$

Proof :

It suffices to prove that $\Re\left(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1} dt\right) \neq 0$ or $\Im\left(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1} dt\right) \neq 0$

Let $z=x+iy$, by change of variable, and by setting $t^{x-1}=e^u$, we deduce :

$$-\Re\left(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1} dt\right)=\int_{-\infty}^{+\infty} \frac{e^u}{e^{e^{\frac{u}{x-1}}}-1} \cos\left(y\frac{u}{x-1}\right) \frac{1}{x-1} e^{\frac{u}{x-1}} du$$

Note :

As $\frac{e^u}{e^{e^{\frac{u}{x-1}}}-1} \cos\left(y\frac{u}{x-1}\right) \frac{1}{x-1} e^{\frac{u}{x-1}}=0$ for $u_k=(2k+1)\frac{\pi}{2}\frac{x-1}{y}$, $k \in \mathbb{Z}$ and oscillates increasing in amplitude because $g(u)=\frac{e^u}{e^{e^{\frac{u}{x-1}}}-1} \frac{1}{x-1} e^{\frac{u}{x-1}}$ is decreasing with u, we deduce that :

$\int_{u=(2k+1)\frac{\pi}{2}\frac{x-1}{y}}^{u=(2(k+2)+1)\frac{\pi}{2}\frac{x-1}{y}} \frac{e^u}{e^{e^{\frac{u}{x-1}}}-1} \cos\left(y\frac{u}{x-1}\right) \frac{1}{x-1} e^{\frac{u}{x-1}} du$ is different from 0 and its sign does not depend on $k \in 2\mathbb{Z}$ (we have the same result if $k \in 2\mathbb{Z}+1$):

Because :

$$\int_{u=(2k+1)\frac{\pi}{2}\frac{x-1}{y}}^{u=(2(k+2)+1)\frac{\pi}{2}\frac{x-1}{y}} \frac{e^u}{e^{e^{\frac{u}{x-1}}}-1}\cos(y\frac{u}{x-1})\frac{1}{x-1}e^{\frac{u}{x-1}}du = \int_{u_k}^{u_{k+2}} g(u)\cos(y\frac{u}{x-1})du$$

$$= \int_{u_k}^{u_{k+1}} g(t)\cos(y\frac{t}{x-1})dt + \int_{u_{k+1}}^{u_{k+2}} g(u)\cos(y\frac{u}{x-1})du \;.$$

$$= \int_{u_{k+1}}^{u_{k+2}} \cos(y\frac{u}{x-1})(g(u)-g(u-\tau))du$$

where $\tau = \frac{\pi}{\frac{y}{x-1}}$ (it is found by changing the variable $u=t+\tau$). and so the integral $\int_{u=(2k+1)\frac{\pi}{2}\frac{x-1}{y}}^{u=(2(k+2)+1)\frac{\pi}{2}\frac{x-1}{y}} \frac{e^u}{e^{e^{\frac{u}{x-1}}}-1}\cos(y\frac{u}{x-1})\frac{1}{x-1}e^{\frac{u}{x-1}}du$ is different from 0 and its sign does not depend on $k\in 2\mathbb{Z}$) (we have the same result if $k\in 2\mathbb{Z}+1$).

By using the note above :

Let $f(u)=\frac{e^u}{e^{e^{\frac{u}{x-1}}}-1}\cos(y\frac{u}{x-1})\frac{1}{x-1}e^{\frac{u}{x-1}}$, and $u_k=(2k+1)\frac{\pi}{2}\frac{x-1}{y},\; k\in\mathbb{Z}$.

we have $-\Re(\int_0^{+\infty}\frac{t^{z-1}}{e^t-1}dt)=\lim_{u_k\to+\infty}\int_{-\infty}^{u_k} f(u)du$

If $\int_{-\infty}^{u_l} f(u)du\geq 0$:

So :

- Either $f'(u_l)\geq 0$ (f increasing in the vicinity of u_l)

In this case : $-\Re(\int_0^{+\infty}\frac{t^{z-1}}{e^t-1}dt)=\int_{-\infty}^{u_l} f(u)du+\int_{u_l}^{u_{l+1}} f(u)du+\sum_{k\in 2\mathbb{N}}\int_{u_{k+l+1}}^{u_{(k+2)+l+1}} f(u)du>0$

- Or either $f'(u_l)\leq 0$ (f decreasing in the vicinity of u_l)

In this case : $-\Re(\int_0^{+\infty}\frac{t^{z-1}}{e^t-1}dt)=\int_{-\infty}^{u_l} f(u)du+\sum_{k\in 2\mathbb{N}}\int_{u_{k+l}}^{u_{(k+2)+l}} f(u)du>0$

Similarly:

If $\int_{-\infty}^{u_l} f(u)du\leq 0$:

So :

- Either $f'(u_l)\geq 0$,

In this case : $-\Re(\int_0^{+\infty}\frac{t^{z-1}}{e^t-1}dt)=\int_{-\infty}^{u_l} f(u)du+\sum_{k\in 2\mathbb{N}}\int_{u_{k+l}}^{u_{(k+2)+l}} f(u)du<0$

- Or either $f'(u_l)\leq 0$,

In this case : $-\Re(\int_0^{+\infty}\frac{t^{z-1}}{e^t-1}dt)=\int_{-\infty}^{u_l} f(u)du+\int_{u_l}^{u_{l+1}} f(u)du+\sum_{k\in 2\mathbb{N}}\int_{u_{k+l+1}}^{u_{(k+2)+l+1}} f(u)du<0$

Proof of the theorem

We know ([3,4]) that : $\zeta(z)\Gamma(z)=\int_0^{+\infty}\frac{t^{z-1}}{e^t-1}dt$

As $\Gamma(z+1)=z\Gamma(z)$, then $\zeta(z)(z-1)\Gamma(z-1)=\int_0^{+\infty}\frac{t^{z-1}}{e^t-1}dt$

But the gamma function also cheks the Legendre duplication formula [3] :

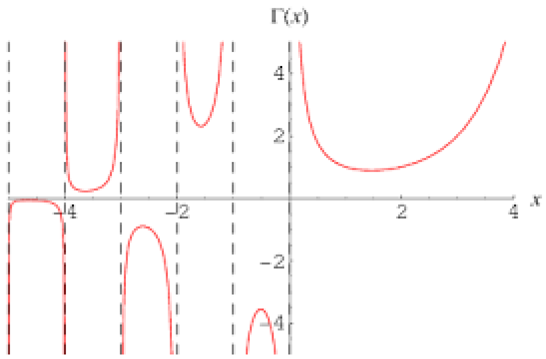
$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\pi^{\frac{1}{2}}\Gamma(2z)$. so $\Gamma(z-1)\Gamma\left(z-\frac{1}{2}\right)=2^{3-2z}\pi^{\frac{1}{2}}\Gamma(2z-2)$.

And we deduce : $\zeta(z)(z-1)2^{3-2z}\pi^{\frac{1}{2}}\Gamma(2z-2)=\Gamma\left(z-\frac{1}{2}\right)\int_0^{+\infty}\frac{t^{z-1}}{e^t-1}dt$

If $\zeta(s)=0$ with s a non trivial zero of ζ , then, by symmetry of the zeros about the critical line $\Re(z)=\frac{1}{2}$, we can assume that $s=\frac{1}{2}-\alpha+i\beta$ with $0\leq\alpha<\frac{1}{2}$ (because it is known that any non-trivial zero belongs to the critical strip : $\{s\in\mathbb{C}:0<\Re(s)<1\}$

But from the Euler's reflection formula: $\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin(\pi z)},\forall z\notin\mathbb{Z}$, we have $\Gamma\left(s-\frac{1}{2}\right)\neq 0$, so by tending z towards s and by using the **lemma 1**, we will have: $|\Gamma(2s-2)|=|\Gamma(-1-2\alpha+i2\beta)|=+\infty$, and consequently we deduce that: : $|\Gamma(-1-2\alpha)|=+\infty$

The study of *Gamma* -**See Figure** [gamma function] - Shows that the only possible case is $-1-2\alpha=-1$, so $\alpha=0$.



Theorem [The sghiar's function and the prime numbers] :

Let $\hat{S}(z)=\zeta\left(-\frac{\Gamma(z)+1}{z/2}\right)$. If $z\in\mathbb{N}^*$, then $\hat{S}(z)=0 \Leftrightarrow$ z is a prime number

Proof : It follows from Wilson's theorem [1] - which assures that p is a prime number if and only if $(p-1)!\equiv -1$ mod(*p)* , and the fact that the trivial zeros of ζ are $-2\mathbb{N}^*$.

III. Conclusion

The Gamma function Γ and the Mertens function *M* are closely linked to the Riemann zeta function ζ . What is curious is that by the same techniques the Mertens function allowed the proof of the Riemann hypothesis in [2], and the gamma function allowed also in this article a simple, short and elegant proof of the Riemann hypothesis.

References

[1]. Roshdi Rashed, Entre arithmétique et algèbre : Recherches sur l'histoire des mathématiques arabes, journal Paris, 1984,
[2]. M. Sghiar. The Mertens function and the proof of the Riemann's hypothesis, International Journal of Engineering and Advanced Technology (IJEAT), ISNN:2249-8958, Volume- 7 Issue-2, December 2017
[3]. https://en.wikipedia.org/wiki/Gamma_function.
[4]. https://en.wikipedia.org/wiki/Riemann_zeta_function.