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# The Exact Prime Counting Function

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***Abstract***
*We introduce the exact odd composites counting function and the exact prime counting function*

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## 1 Introduction

In this paper, we use the inclusion/exclusion method to define the exact odd composites counting function and deduct the exact prime counting function $\pi(x)$ up to a given integer $x$

Notations
$p_n$ : n-th prime number
$p_{n+1}$ : (n+1)-th prime number
$N$ : odd composite number
$G$ : gap of a sequence
$\lfloor ... \rfloor$ : Floor function

## 2 Distribution of primes

the prime number theorem describes the asymptotic distribution of the prime numbers among the positive integers. It formalizes the intuitive idea that primes become less common as they become larger by precisely quantifying the rate at which this occurs.

According to the Sieve of Rhafli [1], every odd composite $N$ with a variable $c \in Z^+$ and $p_n$ are all primes except 2 can be written as

$$p_n^2 + 2p_nc = N \tag{1}$$

When the same odd composite $N$ is generated by 2 different primes $p_n$ and $p_j$ we get

$$p_n^2 + 2p_nc_1 = p_j^2 + 2p_jc_2 = N; \qquad (c_1, c_2 \in Z^+; N \geq 9)$$

For multiple primes generating the same odd composite $N$ we have

$$p_n^2 + 2p_nc_1 = p_{n+1}^2 + 2p_{n+1}c_2 = p_{n+2}^2 + 2p_{n+2}c_3 = ... = p_h^2 + 2p_hc_n = N \tag{2}$$

# 3 The new prime counting function aspect

If we know the number of odd composites among the odd numbers, we can conclude by deduction the number of prime numbers included in a given interval The percentage of odd composite generated by the prime 3 is $(1/3 = 33,33..\%)$, similarly for the prime 5 $(1/5 = 20\%)$. Starting from $p_0 = 3$ The sum of the percentages for large number of primes is

$$\sum_{i=0}^{\infty} \frac{1}{p_i} \tag{3}$$

Leonard Euler proved[2] that this sum tends to $+\infty$, whereas the percentage off odd composite among the odd numbers (primes included) must tends to $\approx 99,99...\%$

According to the sieve of RHAFLI [1], the odd composites generated by equation 1 starts from $p_n^2$ when the integer $c = 0$, which means that the actual percentage of odd composites generated by a prime number is less than $\frac{1}{p}$

$$pc(p_n) < \frac{1}{p_n} \tag{4}$$

Where $pc(p_n)$ denotes the percentage of a prime $p_n$. For a large prime, this gap becomes bigger

The other important term to subtract from 3 is the redundant multiples. The equation 2 reveals that for given situations, the odd composite $N$ can be generated by different primes.

The new odd composites counting function is then based on subtracting the redundant odd composites from the gap of sequences to conclude the exact primes counting function

## 3.1 The gap of sequences

According to the equation 1, the series of multiples begin from $p_n^2$ when the integer $c = 0$.

Then the gap $G$ to eliminate, is the portion from 9 to $p_n^2$. We use a simple example to generate a general formula which bypasses the gap $G$

Given a prime number $p_n = 5$ and the interval $[25, 45]$, where the exact multiples $[25, 35, 45]$ generated by the prime 5 using sieve of Rhafli [1] equal to 3
From the figure 1 below, we see that the exact multiples of $p_n$ are $[25, 35, 45]$

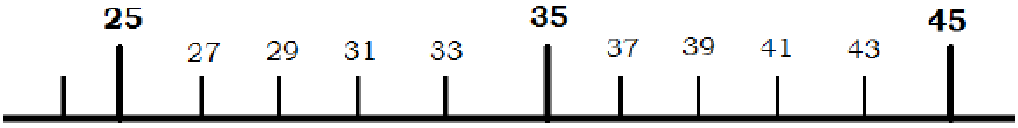


Figure 1

The number of odd numbers (primes includes) is

$$((45-25)+2)\frac{1}{2}=11=A$$

While the number of odd composites is

$$(\frac{A}{2}+2)\frac{1}{5}=3$$

We generalize the following theorem
Theorem 1 :
for any given prime $3 \leq p_n \leq N$, it generates exactly the number of multiples $n(M)$ as

$$n(M)=\left\lfloor \frac{N+2p_n-p_n^2}{2p_n} \right\rfloor$$

Thus in general

$$\sum n(M)=\sum_{p_n=3}^{p_n\leq\sqrt{N}}\left\lfloor \frac{N+2p_n-p_n^2}{2p_n} \right\rfloor \tag{5}$$

## 3.2 Redundant odd composites

The equation $p_n^2+2p_nc$ generates in some cases the same odd composites for different primes, ex: for primes $3,5,7$ the same multiple 105 is generated, our new prime counting function must bypasses these redundancies

As explained in page 4 of sieve of RHAFLI [1], the redundancy occur when 2 different prime numbers $p_n$ and $p_j$ satisfies the equation

$$p_n^2p_j+2p_np_jm=N, \qquad (m\in Z^+) \tag{6}$$

for instance, the primes 3 and 5 generate the same multiples at a rate of

$$45+30m=N, \qquad (m\in Z^+)$$

Note that, the equation 6 must calculate the redundancies only when they exist, for example, according to equation 1, the first odd composite generated by the prime 11 is $p_n^2=121$, whereas if we want to calculate its redundancies with prime 3, equation 6 ($p_n^2p_j+2p_np_jm=99+66m$) will start counting from 99 when $m=0$, to avoid having negative values during calculation, we'll consider divide the portion $(N-p_j)$ with the period $(2p_{np}j)$
Let $n(R)$ be the number of redundancies such that the primes $p_n<p_j$ and $N$

is an odd composite $p_j \leq \sqrt{N}$

$$n(R) = m = \left\lfloor \frac{N - p_j^2}{2p_n p_j} \right\rfloor$$

And in general

$$\sum n(R) = \sum m = \sum_{n=1}^{p_{n-1} \leq \sqrt{N}} \left( \sum_{j=1}^{p_n} \left\lfloor \frac{N - p_j^2}{2p_n p_{j+1}} \right\rfloor \right) \tag{7}$$

### 3.3 The odd composites counting function

Let $\pi'(x)$ be the number of odd composites up to $x$, we define the odd composites counting function as the (Gap of sequences - the odd composites redundancies),i.e,: $\pi'(x) = \sum n(M) - \sum n(R)$ as

$$\pi'(x) = \sum_{p_n=3}^{p_n \leq \sqrt{N}} \left\lfloor \frac{N + 2p_n - p_n^2}{2p_n} \right\rfloor - \sum_{n=1}^{p_{n-1} \leq \sqrt{N}} \left( \sum_{j=1}^{p_n} \left\lfloor \frac{N - p_j^2}{2p_n p_{j+1}} \right\rfloor \right)$$

## 4 The prime counting function

The new prime counting function is using inclusion/exclusion principle and deduct the exact primes up to $x$, as our odd composites counting function is defined in the interval of odd numbers in the interval $[9, x]$, we add the first 4 trivial primes and we conclude the exact prime counting function from $[0, x]$ as

$$\pi(x) = 4 + \left\lfloor \frac{x - 9}{2} + 1 \right\rfloor - \pi'(x)$$

And then as

$$\pi(x) = 4 + \left\lfloor \frac{x - 9}{2} + 1 \right\rfloor - \sum_{p_n=3}^{p_n \leq \sqrt{x}} \left\lfloor \frac{x + 2p_n - p_n^2}{2p_n} \right\rfloor + \sum_{n=1}^{p_{n-1} \leq \sqrt{x}} \left( \sum_{j=1}^{p_n} \left\lfloor \frac{x - p_j^2}{2p_n p_{j+1}} \right\rfloor \right) \tag{8}$$

## 5 Testing example

in order to clarify how the algorithm works, here is a simple example

Calculate the number of primes up to $N = 200$

### 5.1 Primes involved

The primes except 2 such as $p_n \leq \sqrt{(200)}$ are : [3,5,7,11,13]

### 5.2 Number of odd composites

According to equation 5
For $p_1 = 3$

$$\left\lfloor \frac{200+6-9}{6} \right\rfloor = 32$$

Similarly for primes $5, 7, 11, 13$ we get consecutively $18, 11, 4, 2$
Then the total number of odd composites generated by these primes are

$$\sum_{p=3}^{13} n(M) = 67$$

### 5.3 Redundant odd composites

According to equation 7, the number of redundancies between 2 primes is given by the following table

| Primes | 3 | 5 | 7 | 11 | 13 |
|---|---|---|---|---|---|
| 5 | 5 | | | | |
| 7 | 3 | 2 | | | |
| 11 | 1 | 1 | 0 | | |
| 13 | 1 | 1 | 0 | 0 | |

Then the total number of redundancies is

$$\sum n(R) = 14$$

We conclude then the number of primes up to $N = 200$ from equation 8

$$\pi(200) = 46$$

the actual number of primes up to 200 is 46, then the margin error in this example is 0

## References

[1] Marouane RHAFLI *Sublinear segmented prime sieve*. Volume 120, Number 2, 2019, Pages 147-159 ISSN: 0972-0871 DOI : 10.17654/MS120020147

[2] Euler, Leonhard (1737) *Variae observationes circa series infinitas*Commentarii Academiae Scientiarum Petropolitanae. 9: 160–188

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