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On Discrete Game Models for Treating the Spread of a Disease

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Abstract
In this paper, we provide discrete game models controlled by a finite control set for the treatment of a widely spread disease. In this regard, we build available strategies for curing defected people with possible number of treatments, each of which emerges from a discrete model which is controlled by a finite set. More precisely, we assume that the exact spreading time for the disease is not known, namely there are some certain number of infected people at the initial time and the aim is to build a discrete controlled system which would be our treatment model limited initially in such a way that those patients receive their treatments accordingly. Hence, we consider a two-player game as one player determines the total number of infected people which is increasing day by day and the other tries to build its corresponding number of treatments in which one patient receives only one treatment at a given period. In this context, we observe that the game is similar to pursuer-evasion discrete games proposed in [1]. Moreover, the paper connects to a reachability problem in robotics and the idea could be implemented on the model proposed in the recent paper [2] related to COVID-19.
Keywords: *Kronecker's theorem, two-player games, discrete control model, infectious disease, finite control set.*

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I. Introduction

More recently, there has been a great attempt for modelling discrete environments, most of which are emerged from continuous dynamical processes for real life problems in diff erent fields of science and technology. When those models are controlled by some certain sets, interesting results appear, especially analysing diff erent behaviour of candidates each of which emerges from the discrete system under consideration, provide useful knowledge about the problem itself and motivates for designing two-player games or roughly speaking, pursuit-evasion type games in this context. In these games, one can see challenges and interesting results by building game theoretical thinking for gaining winning positions over rival forces in battleground where decision making analysis and strategies are involved.

For pursuit-evasion games, we provide to the book of Isaacs [10] for an introduction of modelling and solving those games with both discrete and continuous environments. Moreover, we refer to the paper [11] which examines two-player zero-sum diff erential game known as the target guarding problem through which second-order dynamics and acceleration control are examined. On the other hand we provide the paper [12] for a new approach for developing an idea of using matrix norms in the context of two-player zero sum matrix game. Beside, we refer to the papers [13] for stochastic diff erential games in mathematical economy. Moreover, recent papers [14–16] exam-ine some numerical results of stochastic diff erential models applied in mathematical finance.

Recently, there has been a growing tendency toward building eff ective and con-trol based environments for preventing the spread of an infectious disease, so called COVID-19. For that reason, new mathematical models have been designed to describe and analyse the dynamics of the widely spread diseases. Here, we refer to the recent papers [2, 3, 6–9] for modelling and analysing COVID-19 and its evolution process in diff erent fields of the subject under certain treatment measures.

In this work, we construct discrete game models with controls ranging from a finite set to build a treatment policy for patients who are suff ering from a widely spread disease like COVID-19. In this regard, we build certain strategies for delivering pos-sible number of treatments to infected people. Naturally, we suppose that the exact spreading time of the disease is unknown and the goal is to design a proper discrete controlled model which we call it as the treatment model in which initial candidates are zero, i.e., there is no treatment at the first moment when the disease outbreaks. in a way that those patients receive their treatments accordingly. Hence, we consider a two-player game as one player determines the total number of infected people which is increasing day after day and the other tries to build its corresponding number of treatments in which one patient receives only one treatment at a given period. Hence, we mark the first player as evader and second player as pursuer. In [1], two-player game with that sort is investigated where the position of each player is formulated by controls taken from finite sets and the game finishes whenever the capturing appears within a small neighbourhood.

The paper is organized as follows. In section 2, we introduce some basic concepts and notions, together with theorems and lemma related to the research. In this part, we provide Kronecker's density theorem [1, 5] and provide one application of it for later purposes. In section 3, we propose a game model for the treatment of an infectious disease and analyse the model in different platforms with a discrete fashion. In this section, we provide two treatment models for the pursuer for chasing its opponent, which is nothing but the total number of infected patients at a given period of time. In section 4, we provide an example for how to deal with the simple spread model of the disease introduced in section 3. Finally, the paper concludes with some perspectives and future works in section 5.

II. Preliminaries

In this section, we give definitions, theorems and a lemma which would be used for later purposes. Now, we define the notion of transcendental and algebraic number.

Definition 2.1. An algebraic number is a real number which could be seen as a root of a non-zero polynomial in one variable with rational coefficients. If a number is not algebraic then it is called a transcendental number.

Hence, there is no polynomial with rational coefficients such that a transcendental number be a root of it. Now, we define the concept of rationally independent numbers

Definition 2.2. ξ and η is called rationally independent numbers if and only if ξ/η is an irrational number. Generally, $\xi_1, \xi_2, \cdots, \xi_m$ are called rationally independent if and only if the one cannot be written as a linear combination of the others with rational coefficients.

Let $\{\cdot\}$ and $\lfloor \cdot \rfloor$ be the fractional part and integer part of a real number respectively.
Here, we provide Kronecker's Theorem (see [1, 4, 5]).

Theorem 2.3. *If ξ is irrational, then the set of points* $(\{n\xi\} : n \in \mathrm{N})$ *is dense in the open interval* $(0, 1)$.

As a consequence of Theorem 2.3, we consider the following useful result [4, 5].

Lemma 2.4. *Let ξ and η be rationally independent positive real numbers. Then, the set*

$$K := \{ j\eta + i\xi \mid i, j \in \mathrm{Z}\},$$

is dense in R.

The proof of Lemma 2.4 is straightforward (see e.g., [5]). Let $\gamma \in (0, 1)$ be a tran-scendental number. For any $\sigma > 0$, we consider the following sets

$$I_\sigma = \sigma(i + \gamma j)/(1 + \gamma)\lfloor\lambda\rfloor, \qquad i, j \in \{0, \pm 1, ..., \pm\lfloor\lambda\rfloor\},$$

where $\lambda > 1$ and it is rationally independent with γ. Finally, we conclude this part with the following result obtained in [4].

Theorem 2.5. *The set*

$$S = \{ u_0 + u_1\lambda + ... + u_n\lambda^n : u_i \in I_\sigma \}$$

is dense in R *for $\lambda > 1.62$*

Remark 1. The importance of the set I_σ is that each elements in the set is less than or equal to σ which could be any positive real number. Hence, coefficients of each polynomial in S could be picked so small or large depending on the problem under consideration.

III. A Game Model for the Treatment of an Infectious Disease

In this part, we provide a two-player discrete game model. At each periods, one player counts the total number of patients suff ering from the disease which is increasing as the days go by, but the other player tries to reach the total number of sick people by delivering necessary treatments in which one treatment goes for one patient. Here, the game is over whenever all infected people nearly receive their medical treatments. More precisely, the game becomes a pursuit-evasion type in which the first player could be seen as evader, and the second as pursuer. Therefore, the game finishes whenever the evader is caught by the pursuer within a small neighbourhood. In this game, we concentrate on the following discrete model of the form which determines the move of each players

$$x_{n+1} = \lambda x_n + u, \qquad x_0 = x^0, \quad |u| \leq S, \tag{2}$$

where $\lambda \geq 1$ which could be seen as growth rate constant and u is taken from a finite control set I_u in which each elements bounded by a positive real number S and $x^0 \in \mathrm{R}$, an initial status of the player. Here, the position of the player at the s^{th} period, which represents number of infected/treated patients at s^{th} day or moment, could be seen as

$$x_s = u_0 + u_1\lambda + \dots + x^0\lambda^s.$$

where $\{u_0, u_1, \dots, u_{s-1}\}$ is a finite control sequence with $u_i \in I_u$.

Let an arbitrary $\epsilon > 0$ is given. The first player (evader) with positions $\{X_i\}_{i\in K}$, is caught by the second player (pursuer) with positions $\{Y_i\}_{i\in K}$ after s period means that

$$|X_s - Y_s| < \epsilon,$$

which ends the game. The problem of that sort and playability assumptions are de-signed in [1]. For that reason, we assume that the control set of the pursuer ranges more than the one with the evader (see e.g., [1]). This is necessary to provide an advantageous environment for the pursuer to catch its opponent after finitely many steps. Of course, those routes are characterized by the control sequence which each of its candidate ranges from constructed finite sets. Hence, designing a proper control set plays a vital role in this process. In general, we notice here that diff erent positions of the player at any periods starting from from the origin, belong to the set

$$X_m(\lambda) := \{\beta_0 + \beta_1\lambda + \dots + \beta_k\lambda^k : |\beta_i| \leq m,\ k \in \mathrm{N}\}$$

Hence, the topological structure of the set $X_m(\lambda)$ is important to understand the behaviour of each player or interactions in between. There has been recent progress on the topology of the set $X_m(\lambda)$ in which each candidate is a polynomial with its coefficients are of integers taken from a discrete set [4]. Those results would guide us to get into more complicated infection spread models and to build a suitable schedule for the treatment process.

3.1. A Treatment Model for Simple Spread of a Disease

In this section, we consider the following simple spread of a disease model

$$x_{n+1} = x_n + u_n \quad \text{with} \quad x(0) = x_0 \quad \text{and} \quad u_n \in I_u. \tag{3}$$

which is the discrete model (2) for the case $\lambda = 1$. Hence, for given any daily re-ports at n^{th} period could be seen as the sequence of positions $\{x_0, x_1, x_2, \cdots, x_n\}$ which determines the control set I_u, and vice versa, namely given a control sequence $\{u_0, u_1, \cdots, u_{n-1}\}$ determines the total number of infected people at n^{th} period.

Now, we provide the following first main result which simply assert that for any path that the evader follows would be eventually caught by the pursuer within a small neighbourhood.

Theorem 3.1. *Assume that there are x_0 number of infected patients initially and the spread model of the disease satisfies the discrete system (3) with control set I_u. Then, there always exists a treatment policy satisfying (3) with finite control set I_v curing the total infected patients approximately even in the absence of the treatment in the first days of infection.*

Proof. Let p and q be rationally independent real numbers. Let $\{x_0, x_1, x_2, \cdots\}$ and $\{y_0, y_1, y_2, \cdots\}$ be the path that emerged from (3) for first and second player re-spectively. Moreover, let $\epsilon > 0$ be given and fixed. Assume that $k \geq 1$ be the first period time in which the cure or treatment for the disease is firstly applied, i.e.,

$y_0 = y_1 = \cdots = y_k = 0$, but $y_{k+1} > 0$.

From Lemma 2.4, one can have integers ℓ, η such that

$$| x_k - (p\eta + q\ell) | < \epsilon.$$

Here, without loss of generality assume that η and ℓ are positive integers and define $N = \eta + \ell$. Let the control sequence for the first player be $\{u_0, u_1, \cdots, u_k, \cdots\}$.

Now design the second player's control sequence as $\{v_0, v_1, \cdots, v_k, \cdots\}$ in the following sense

$$v_i = u_{k+i} + p, \text{ for } i = 0, \cdots, \eta - 1, \qquad \text{and } v_j = u_{k+j} + q, \text{ for } j = \eta, \cdots, N - 1$$

As a result, we have

$$|x_{k+N} - y_{k+N}| < \epsilon. \tag{5}$$

Hence, the total treatment would nearly save the total infected patients at $(k + N)^{th}$ period □

Remark 2. It is seen for Theorem 3.1 that the choice of rationally independent numbers p, q are optional which shows that there exist infinitely many such policy for the pursuer to win the game. Moreover, the design of the control set for the pursuer could be flexible which motivates to build different control sets for the pursuer.

3.2. A Treatment Model for Widely Spread of a Disease

In this part, we examine the following discrete model which could be seen as a widely spread infection model (see e.g., [1])

$$x_{n+1} = \lambda x_n + u_n \quad \text{with} \quad x(0) = x_0 \quad \text{and} \quad u_n \in I_u, \tag{6}$$

As it is considered for the case $\lambda = 1$ in the previous section, we examine the same problem, i.e., whether it is possible to design a finite control set I_v under which the second player treatment policy nearly cures the total number of infected patients. The difference is the growth rate in (6) could be much more greater than the model (3) which makes the infection widely spread out.

Now, we are ready to provide the main result of this section

Theorem 3.2. *Suppose that there are x_0 number of infected people in the first place and the total number of sick patients at each period satisfies the model (6) with control set I_u. Then, there always exists a treatment policy satisfying (6) with finite control set I_v curing the total infected patients approximately even in the absence of the treatment in the first days of infection.*

Proof. Let $\epsilon > 0$ be given and fixed. Moreover, we assume that $\{x_0, x_1, x_2, \cdots\}$ and $\{y_0, y_1, y_2, \cdots\}$ are the positions that emerged from (6) for first and second player respectively. Suppose that $k \geq 1$ be the period time such that $y_0 = y_1 = \cdots = y_{k-1} = 0$, but $y_k > 0$.

Let σ be arbitrary parameter which could be taken so big or small depending on the purpose and design the set I_σ as it is discussed in section 2. From Theorem 2.5,
one could find a positive integer $m > k$ such that the following satisfies

$$|x_k\lambda^m - (\mu_0\lambda^m + \mu_1\lambda^{m-1} + ... + \mu_{m-1}\lambda + \mu_m)| < \epsilon, \tag{7}$$

where $\mu_i \in I_\sigma$ for $i = 0, 1, ..., m$. player be $\{u_0, u_1, \cdots, u_k, \cdots\}$. Now we build the $\{v_0, v_1, \cdots, v_k, \cdots\}$ as the following way

Let the control sequence for the first build the second player's control sequence

$$v_0 = \mu_0, \quad \text{and} \quad v_i = u_{k+i-1} + \mu_i, \quad \text{for} \quad i = 1, \cdots, m.$$

As a result, we have

$$|x_{k+m} - y_{k+m}| < \epsilon. \tag{8}$$

Hence, the total treatment would save nearly the whole infected patients at $(k + m)^{th}$ period.

□

Remark 3. Since the control set for the pursuer depends on the parameter σ, which could be any real number other than zero, together with Theorem 2.5, one can build diff erent control sets each of which changes the chasing strategies of the pursuer. More precisely, there are many diff erent treatment models for infected people to deliver all possible treatments to each patient.

IV. An Example

Here, we provide a simple example for the construction (2) for both players when $\lambda = 1$.

Since, the disease outbreaks suddenly, it is assumed that there is no medical treatment initially.

Now, consider the following two tables each of which represents a path followed by each player in daily life. In the first table, the system (2) is controlled by the spread of a disease, but in the second, we see that available treatment with a suitable control set exists and cures all the patients after some periods.

Number of Infected Patients (Evader)			
Periods (k)	Patients (x_k)	Control (u)	Bound (S_e)
Initial	5	4	4
First	9	5	5
Second	14	3	5
Third	17	6	6
Forth	23	5	6
Fifth	28	5	6

Number of Treatment Delivered (Pursuer)			
Periods (k)	Treatments (y_k)	Control (v)	Bound (S_p)
Initial	0	5	5
First	5	6	6
Second	11	6	6
Third	17	6	6
Forth	23	5	6
Fifth	28	5	6

Here, all data is taken as integers and the pursuer catches the evader at the third period and follows the same path of its opponent for the rest of time periods. Hence, we see that the range of control set determines the future of the game in favor of the pursuer

V. Conclusion

In this paper, we establish two-player game models for treating the spread of a disease. The role of the first player in this game is to collect all infected patients whereas the other one tries to provide necessary treatments in such a way that every patient makes use of the medication as one patient receives only one treatment at some periods. Here, we concentrated on the discrete models which of course totally derived from continuous environments and this motivates us to focus on some continuous models as spread of COVID-19 [2, 3] and any other infectious diseases. Hopefully, some discretization techniques applied for those continuous models could provide us discrete models like (2) (see e.g., [1, 10]) and hence this creates a possible playing environment for designing corresponding two-player games controlled by a finite set.

On the other hand, the work has a connection to the reachability problem of a convex polyhedra rolling in R. Roughly speaking, the problem is a discrete model of the form

$$x_{n+1} = \lambda x_n + \eta, \;\; x_0 = 0, \tag{9}$$

where $\lambda > 1$ and η is ranging from finite control set. In literature, the problem is related to robotics and we refer to the papers [4, 17, 18] for analysing reachability property of the system under finite control set. Our motivation for connecting controllability and two-player games in this regard, comes from the fact that each discrete points of the system can be seen as a polynomial in λ with bounded coefficients emerged from the finite set which could be seen as a position of the player at some periods As a result, reachability problems and two-player game models may meet at the same point in the context of discrete control models.

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