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A simple proof of the Fermat theorem

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Abstract
The paper gives the simple way of proving Fermat's last theorem. The proof is based on the study of the properties of natural numbers, an analysis of the constraints on the proposed solutions, and uses some general theorems on the roots of algebraic equations.
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I. Introduction

Great (last) Fermat's theorem was formulated over 300 years ago. In view of the significance of the problem in many areas of mathematic, large, but unsuccessful efforts have been made to prove it. Finally, a proof of the theorem was given in [5], based on the connection of the theorem with the properties of modular elliptic curves. The proof is too complicated, so attempts were made to find a simple proof of the theorem. In particular, in [1 – 3], alternative approaches to the proof of this theorem were proposed. The theorem of P. Fermat, as is known, asserts that equation

$$x^p + y^p = z^p \tag{1}$$

has no positive integer solutions for $p > 2$. The purpose of this paper is to give a simple proof of Fermat's theorem within the framework of the elementary number theory. Define a few terms that we will use. The quantities x, y, z and their values are called bases of degree, or briefly bases. The bases from the first ten we call elementary bases. The set (combination) of three bases (x, y, z) we call a triplet or, accordingly, an elementary triplet. Thus, equation (1) depends on three bases and an exponent. For definiteness, let $x < y$, i.e. we assume that x is the base taking the least value. We put $y – x = u$, $z – x = v$, then equation (1) will depend on one base x, the exponent p, and two parameters u and v. It is easy to see that the parameters u and v do not change, if all three bases are increased by an arbitrary number a. Hereinafter (see sections III, IV, V) we use this technique, because it allows us to simplify the analysis of equation (1) and apply the Descartes' rule of signs to determine the position of the roots of the equation.

II. Restrictions on possible solutions of the equation and admissible transformations

Consider restrictions on the possible solutions of equation (1). Analysis of the restrictions allows us to establish the conditions that the bases and exponent must satisfy, so that they can be solutions of equation (1). Let us formulate the first restriction. Put for definiteness that $x < y$, i.e. x always means the smallest number on the left. Since the numbers x, y, z are all different, we have the following inequality:

$$x^p + y^p < (x + y)^p . \tag{2}$$

If a is a positive integer, then *a fortiori*

$$x^p + y^p < (x + y + a)^p . \tag{3}$$

From inequalities (2), (3) and the form of equation (1), *the first restriction* for numbers as possible solutions of equation (1) follows

$$\max(x, y) < z < (x + y) . \tag{4}$$

The second restriction is associated with the obvious requirement that the number $x^p + y^p$ ends in the same digit as the number z^p . In particular, it follows that the left and right sides of equation (1) must be of the same parity. Let us formulate *the third restriction*. If the following relation holds

$$x^2 + y^2 \le z^2 , \tag{5}$$

then x, y, z are not solutions of the basic equation (1). In this case, strict inequalities hold

$$x^3 + y^3 < z^3 ,$$

$x^4 + y^4 < z^4$,...

$x^p + y^p < z^p$. (6)

Indeed, multiplying (5) by *z* and using the left-hand side of inequality (4), we have $z^3 \geq zx^2 + zy^2 > x^3 + y^3$. Multiplying this inequality by *z* and using (4), we obtain $z^4 > z(x^3 + y^3) > x^4 + y^4$ etc. *The fourth restriction* is the equality of the exponents of all components in equation (1). The restrictions formulated are necessary conditions for numbers to be solutions of the basic equation (1). They are quite strong and allow us to select the proposed solutions for equation (1). *The second and fourth restrictions we call basic*, since their fulfillment is an unconditional requirement. The first and third restrictions are auxiliary and their implementation can be ensured through transformations (see below). Now consider the permissible transformations that keep safe the restrictions. As the starting point, we take the elementary bases, i.e. natural numbers from the first ten, that satisfy the basic restrictions. Such transformations include:

1. Multiplication of all bases of degree in (1) by a positive integer *q* = 2, 3, 4, etc. Since the starting point is elementary bases, division is excluded.
2. An increase of one, two or all three bases by a number *a* = 10*l* that is a multiple of 10, where *l* = 1, 2, 3, etc. Since the starting point is elementary bases, the first increase is 10.
3. If two triplets of numbers x, y, z and x', y', z' satisfy the second restriction, then the triplet of numbers (bases) { $x'' = x^p + x'^p$, $y'' = y^p + y'^p$, $z'' = z^p + z'^p$ }satisfies this restriction.

The first transformation is useful for obtaining from the known solution all solutions of the same class. For example, it can be used to obtain solutions of equation (1) for *p* = 2. This transformation does not change the "status" of the triplet, i.e. if the triplet is not a solution of (1), then after this transformation it will not be a solution of (1). Therefore, in proving the theorem without loss of generality, it is sufficient to consider only *prime* triplets, namely, those in which the bases do not have a common divisor different from 1. The second transformation is used to provide the first restriction (4) if it does not hold for elementary bases, but the second restriction holds. The third transformation can be applied only if some solutions are known, for example, when solving equation (1) for *p* = 2. Thus, the main "generator" of allowed combinations of numbers (triplets) is the second transformation; it allows us to go over all the numbers that are admissible by restrictions. Let's take a detailed look at the second restriction, for which we analyze the degrees of "elementary" numbers (bases) from 1 to 10, starting with degree 3. We have replaced 10 by 0, so as not to violate the uniformity of the representation (see below). The results are given in table 1. It follows from the data of table 1 that the repetition period of the last digit for bases 2, 3, 7 and 8 is 4, for bases 4 and 9 the period is 2, for bases 5, 6, 0 and 1, the period is 1. Now we consider combinations of powers of different bases, taking into account the basic restrictions. The analysis is performed in the following order. First, we consider combinations of numbers with period 4, i.e., powers of 2 are combined consistently with the degrees of the numbers 3, 4,..., 0, 1, then the number 3 is combined with the remaining, the number 7 – with the remaining, the number 8 – with the remaining. After that, the bases with period 2 are combined, i.e. number 4 – with the numbers 5, 6, 9, 0 and 1, then the number 9 – with 5, 6, 0 and 1. Lastly, the numbers with period 1 are combined, i.e. number 5 with the numbers 6, 0 and 1, the number 6 – with 0 and 1, the number 0 – with 1. Some of the combinations can be immediately excluded due to violation of the second restriction. In addition, we excluded some combinations that are not used in the preliminary analysis and proof of the lemmas (see below). For example, we have excluded combinations indifferent to the exponent of the form $0^p + 0^p = 0^p$, $1^p + 0^p = 1^p$, $2^p + 0^p = 2^p$ etc., $5^p + 5^p = 0^p$, as well as trivial combinations of the form $1^{5+4k} + 1^{5+4k} = 2^{5+4k}$, $2^{5+4k} + 2^{5+4k} = 4^{5+4k}$, $3^{5+4k} + 3^{5+4k} = 6^{5+4k}$, $4^{5+4k} + 4^{5+4k} = 8^{5+4k}$, $6^{5+4k} + 6^{5+4k} = 2^{5+4k}$, $7^{5+4k} + 7^{5+4k} = 4^{5+4k}$, $8^{5+4k} + 8^{5+4k} = 6^{5+4k}$, $9^{5+4k} + 9^{5+4k} = 8^{5+4k}$ and some others. Selected results are given in table 2. The data of tables 1, 2 are used to select the permissible combinations of bases (triplets) that satisfy the formulated restrictions.

III. Proof of auxiliary statements

We prove Fermat's theorem for the initial triplets (*x*, *y*, *z*) in which *x* takes values from 1 to 10 (Lemma 1), and for additional triplets in which *x* takes values from 11 to 14. We use these classes of triplets as particular case in the general proof of the theorem. Lemma 1. *Triplet (x, y, z), where x is an elementary base (x < y < z), y and z are arbitrary positive integers, cannot be a solution of equation (1) for p > 2.* The validity of the lemma is established by direct verification. The number of admissible triplets turns out to be finite and if we take into account the restrictions formulated above, then the verification procedure ends rather quickly. We must consistently consider triplets (*x*, *y*, *z*), for which *x* is an elementary base, and $z - y = 1$, $z - y = 2$,

$z - y = 3$ and so on. We call the triplet (*x*, *y*, *z*) boundary triplet if it satisfies equation (1) for *p* = 2, i.e. it is a solution of the quadratic Fermat equation. If equation $x^2 + y^2 = z^2$ is satisfied for the triplet (*x*, *y*, *z*), then the verification procedure ends. From (5) it is easy to obtain a relation that allows us to determine the number of admissible triplets. We have

$$x^2 > 2(x+u)m + m^2 \quad (5a)$$

or

$$u < (x^2 - m^2)/2m - x, \quad (5b)$$

where *x* – elementary base, $u = y - x$, $m = z - y$; *u* = 1, 2, 3, etc.; *m* = 1, 2, 3, etc. Relation (5b) determines the upper limit of the variation of *u*. The equality of the left and right sides in (5b) corresponds to the boundary triplet. If *x* is an odd number, then only such triplets are allowed, in which *y* and *z* have different parity, which follows from the basic restrictions. Therefore, the difference $m = z - y$ can take only odd values, i.e. 1, 3, 5 and so on. If *x* is an even number, then only such triplets are allowed, in which *y* and *z* have the same parity, which follows from the basic restrictions. So, *m* can take only even values, i.e. 2, 4, 6 and so on. Now consider the procedure for obtaining admissible triplets. It is clear that triplets of the form (1, *y*, *z*) and (2, *y*, *z*) cannot be solutions of equation (1), since conditions (5) are certainly satisfied for them. Moreover, for triplets of the form (1, *y*, *z*) the relation $1 + y \le z$ holds. For example, 1 + 2 = 3, 1 + 12 < 15, etc. If *y* and *z* are both even, the triplets of the form (2, *y*, *z*) after dividing all the components by 2 reduce to triplets of the form (1, *y*, *z*), for example, (2, 4, 6) reduces to (1, 2, 3) etc. If *y* and *z* are both odd, then for the triplets (2, *y*, *z*) the relation $2 + y \le z$ holds, for example, 2 + 3 = 5, 2 + 15 < 19, etc. For triplets of the form (1, *y*, *z*) and (2, *y*, *z*), there are no boundary triplets. Let's start with the triplet (3, 4, 5). Since it is boundary triplet, it satisfies relation (5). Therefore, triplets of the form (3, *y*, *z*), where *y* > 4, *z* > 5, cannot be solutions of equation (1), so the verification procedure ends. Consider the following triplets of the form (4, *y*, *z*). For *x* = 4 and *m* = 2, we obtain from (5b) the inequality *u* < 0, so there are no such triplets. Note that although triplets of the form (1, *y*, *z*), (2, *y*, *z*), (3, *y*, *z*), (4, *y*, *z*) cannot be solutions of equation (1), but from them it is possible to obtain admissible triplets by successively increasing their bases simultaneously by *a* = 10*l*, where *l* = 1, 2, 3, etc. (see below). Consider the following triplets of the form (5, *y*, *z*) in more detail to show how restrictions are used. For *x* = 5 and *m* = 1, we obtain from (5b) the inequality *u* < 7, i.e. there are 6 such triplets. The value *u* = 7 corresponds to the boundary triplet (5, 12, 13). We write these triplets explicitly (5, 6, 7), (5, 7, 8), (5, 8, 9), (5, 9, 10), (5, 10, 11), (5, 11, 12). The triplets (5, 9, 10) and (5, 10, 11) can be excluded, since they do not satisfy the second (basic) restriction. Let *m* = 3. It follows from (5b) that *u* < 0, so there are no such triplets. In the future, when considering other bases, we will write down only the final result. Consider the following triplets of the form (6, *y*, *z*). For *x* = 6 and *m* = 2, we obtain from (5b) the inequality *u* < 2, so there is only one such triplet (6, 7, 9). The triplet (6, 8, 10) = 2(3, 4, 5) is the boundary triplet. Consider the following triplets of the form (7, *y*, *z*). For *x* = 7 and *m* = 1, we obtain from (5b) the inequality *u* < 17, so there are 16 such triplets. The triplet (7, 24, 25) is the boundary triplet. For *x* = 7 and *m* = 3, we obtain from (5b) the inequality *u* < 0, so there are no such triplets. In fact, the number of permissible triplets is much smaller if we take into account the basic restrictions (see below). Consider the triplets of the form (9, *y*, *z*). For *x* = 9 and *m* = 1, we obtain from (5b) the inequality *u* < 31, so there are 30 such triplets. The triplet (9, 40, 41) is the boundary triplet. The number of triplets is rather large, but the analysis is simplified, because only triplets can be considered, in which *y* changes within the interval 10 ... 19, since the last digits are repeated in every ten. For the remaining triplets, the degrees allowed according to the basic restrictions will be repeated (see below). After exclusion, there remain twelve triplets. For *x* = 9 and *m* = 3, we obtain from (5b) the inequality *u* < 3, so there are 2 such triplets. One triplet is excluded due to the basic restrictions, so there is only one permissible triplet. The boundary triplet is (9, 12, 15) = 3(3, 4, 5). For *x* = 9 and *m* = 5, we obtain from (5b) the inequality *u* < 0, so there are no such triplets. Consider the triplets of the form (8, *y*, *z*). For *x* = 8 and *m* = 2, we obtain from (5b) the inequality *u* < 7, so there are 6 such triplets. Two triplets (8, 11, 13) and (8, 13, 15) can be excluded because of the basic restrictions. It should also be taken into account that if all bases in the triplet are even, then such triplet can be excluded, since after dividing all the components by 2 we obtain triplet considered earlier. For example, (8, 10, 12) reduces to (4, 5, 6), (8, 12, 14) to (4, 6, 7), etc. So there is only one permissible triplet (see below). The triplet (8, 15, 17) is the boundary triplet. For *x* = 8 and *m* = 4, we obtain from (5b) the inequality *u* < 0, so there are no such triplets. Consider the triplets of the form (10, *y*, *z*). For *x* = 10 and *m* = 2, we obtain from (5b) the inequality *u* < 14, so there are 13 such triplets. In view of the foregoing, after exclusion, there remain four triplets. The boundary triplet is (10, 24, 26) = 2(5, 12, 13). For *x* = 10 and *m* = 4, we obtain from (5b) the inequality *u* < 84/8 − 10, so there are no such triplets. The final result is given in table 3. It is easy to verify by direct verification that the admissible triplets from table 3 are not solutions of equation (1). For example, for triplet (5, 6, 7) we have $5^2 + 6^2 > 7^2$, but $5^3 + 6^3 < 7^3$, therefore, all other powers of these bases will give the same result (zero is passed), which follows from relations (4), (6). For triplet (5, 7, 8) we have $5^2 + 7^2 > 8^2$, but $5^3 + 7^3 < 8^3$ (zero is passed). For triplet (5, 8, 9) we have $5^2 + 8^2 > 9^2$, but $5^3 + 8^3 < 9^3$ (zero is passed). For triplet (5, 11, 12) we have $5^2 + 11^2 > 12^2$, but $5^3 + 11^3 < 12^3$ (zero is passed). For triplet (6, 7, 9) we obtain $6^2 + 7^2 > 9^2$, but $6^3 + 7^3 < 9^3$ (zero is passed). For triplet (7, 9, 10) we obtain $7^2 + 9^2 > 10^2$, $7^3 + 9^3 > 10^3$, but $7^4 + 9^4 < 10^4$ (zero is passed). For triplet (7, 10, 11) we have $7^2 + 10^2 > 11^2$, $7^3 + 10^3 > 11^3$, but $7^4 + 10^4 < 11^4$ (zero is passed). For triplet (7, 20, 21) we have $7^2 + 20^2 > 21^2$, but $7^3 + 20^3 < 21^3$ (zero is passed). For triplet (8, 9, 11) we have $8^2 + 9^2 > 11^2$, but $8^3 + 9^3 < 11^3$ (zero is passed). For triplet (9, 10, 11) we obtain $9^2 + 10^2 > 11^2$, $9^3 + 10^3 > 11^3$, $9^4 + 10^4 > 11^4$, but $9^5 + 10^5 < 11^5$ (zero is passed). For triplet (9, 37, 38) we obtain $9^2 + 37^2 > 38^2$, but $9^3 + 37^3 < 38^3$ (zero is passed). For triplet (10, 11, 13) we have $10^2 + 11^2 > 13^2$, $10^3 + 11^3 > 13^3$, but $10^4 + 11^4 < 13^4$ (zero is passed). For triplet (10, 21, 23) we have $10^2 + 21^2 > 23^2$, but $10^3 + 21^3 < 23^3$. Similarly, the check is performed for the remaining triplets. Lemma 1 is proved. Hereinafter the triplets from table 3 will be called *initial triplets*. Triplets of the form (1, *y*, *z*), (2, *y*, *z*), (3, *y*, *z*) and (4, *y*, *z*) we have excluded, so as they do not satisfy the restrictions. However, if all the bases of these triplets are increased by 10, then we obtain permissible triplets that satisfy all the restrictions. Hereinafter these triplets will be called *additional triplets*. Prove Lemma 2. *Additional triplets cannot be solutions of equation (1) for p > 2.* The analysis of additional triplets was performed in the same way as for the initial triplets. We write them explicitly using the following notation: (p; x, y, z; p_{th}; u, v, m), where p is the allowed exponent, p_{th} is the threshold exponent, at which the difference between the left and the right sides of equation (1) changes sign from plus to minus; $u = y - x$, $v = z - x$, $m = z - y = v - u$. The number of permissible triplets of the form (11, *y*, *z*) is 51, of which 48 are triplets with m = 1 and 3 triplets with m = 3. For m = 1 we have triplets (5+4*k*; 11, 12, 13; 6; 1, 2, 1), (5+4*k*; 11, 13, 14; 5; 2, 3, 1), (3+2*k*; 11, 14, 15; 5; 3, 4, 1), (3+*k*; 11, 15, 16; 4; 4, 5, 1), (5+4*k*; 11, 16, 17; 4; 5, 6, 1), (5+4*k*; 11, 17, 18; 4; 6, 7, 1), (5+4*k*; 11, 18, 19; 4; 7, 8, 1), (3+2*k*; 11, 19, 20; 4; 8, 9, 1), (3+*k*; 11, 20, 21; 4; 9, 10, 1), (5+4*k*; 11, 21, 22; 3; 10, 11, 1), (5+4*k*; 11, 22, 23; 3; 11, 12, 1), (5+4*k*; 11, 23, 24; 3; 12, 13, 1), (3+2*k*; 11, 24, 25; 3; 13, 14, 1), (3+*k*; 11, 25, 26; 3; 14, 15, 1), (5+4*k*; 11, 26, 27; 3; 15, 16, 1), (5+4*k*; 11, 27, 28; 3; 16, 17, 1), (5+4*k*; 11, 28, 29; 3; 17, 18, 1), (3+2*k*; 11, 29, 30; 3; 18, 19, 1), (3+*k*; 11, 30, 31; 3; 19, 20, 1), (5+4*k*; 11, 31, 32; 3; 20, 21, 1), (5+4*k*; 11, 32, 33; 3; 21, 22, 1), (5+4*k*; 11, 33, 34; 3; 22, 23, 1), (3+2*k*; 11, 34, 35; 3; 23, 24, 1), (3+*k*; 11, 35, 36; 3; 24, 25, 1), (5+4*k*; 11, 36, 37; 3; 25, 26, 1), (5+4*k*; 11, 37, 38; 3; 26, 27, 1), (5+4*k*; 11, 38, 39; 3; 27, 28, 1), (3+2*k*; 11, 39, 40; 3; 28, 29, 1), (3+*k*; 11, 40, 41; 3; 29, 30, 1), (5+4*k*; 11, 41, 42; 3; 30, 31, 1), (5+4*k*; 11, 42, 43; 3; 31, 32, 1), (5+4*k*; 11, 43, 44; 3; 32, 33, 1), (3+2*k*; 11, 44, 45; 3; 33, 34, 1), (3+*k*; 11, 45, 46; 3; 34, 35, 1), (5+4*k*; 11, 46, 47; 3; 35, 36, 1), (5+4*k*; 11, 47, 48; 3; 36, 37, 1), (5+4*k*; 11, 48, 49; 3; 37, 38, 1), (3+2*k*; 11, 49, 50; 3; 38, 39, 1), (3+*k*; 11, 50, 51; 3; 39, 40, 1), (5+4*k*; 11, 51, 52; 3; 40, 41, 1), (5+4*k*; 11, 52, 53; 3; 41, 42, 1), (5+4*k*; 11, 53, 54; 3; 42, 43, 1), (3+2*k*; 11, 54, 55; 3; 43, 44, 1), (3+*k*; 11, 55, 56; 3; 44, 45, 1), (5+4*k*; 11, 56, 57; 3; 45, 46, 1), (5+4*k*; 11, 57, 58; 3; 46, 47, 1), (5+4*k*; 11, 58, 59; 3; 47, 48, 1), (3+2*k*; 11, 59, 60; 3; 48, 49, 1). The triplet (11, 60, 61) is the boundary triplet. For m = 3 we have triplets (6+4*k*: 11, 12, 15; 3; 1, 4, 3), (4+4*k*: 11, 15, 18; 3; 4, 7, 3), (6+4*k*: 11, 17, 20; 3; 6, 9, 3). The number of permissible triplets of the form (12, *y*, *z*) is 12, of which 11 are triplets with m = 2 and 1 triplet with m = 4. For m = 2 we have triplets (3+2*k*; 12, 13, 15; 4; 1, 3, 2), (3+*k*; 12, 15, 17; 4; 3, 5, 2), (5+4*k*; 12, 17, 19; 3; 5, 7, 2), (5+4*k*; 12, 19, 21; 3; 7, 9, 2), (5+4*k*; 12, 21, 23; 3; 9, 11, 2), (3+2*k*; 12, 23, 25; 3; 11, 13, 2), (3+*k*; 12, 25, 27; 3; 13, 15, 2), (5+4*k*; 12, 27, 29; 3; 15, 17, 2), (5+4*k*; 12, 29, 31; 3; 17, 19, 2), (5+4*k*; 12, 31, 33; 3; 19, 21, 2), (3+2*k*; 12, 33, 35; 3; 21, 23, 2). The triplet (12, 35, 37) is the boundary triplet. For m = 4 we have triplet (4+4*k*; 12, 15, 19; 3; 3, 7, 4). The triplet (12, 16, 20) = 4(3, 4, 5) is the boundary triplet. The number of permissible triplets of the form (13, *y*, *z*) is 69, of which 56 are triplets with m = 1 and 13 triplets with m = 3. For m = 1 we have triplets (6+4*k*; 13, 14, 15; 7; 1, 2, 1), (4+4*k*; 13, 15, 16; 6; 2, 3, 1), (3+4*k*; 13, 16, 17; 6; 3, 4, 1), (3+4*k*; 13, 18, 19; 5; 5, 6, 1), (6+4*k*; 13, 19, 20; 4; 6, 7, 1), (4+4*k*; 13, 20, 21; 4; 7, 8, 1), (3+4*k*; 13, 21, 22; 4; 8, 9, 1), (3+4*k*; 13, 23, 24; 4; 10, 11, 1), (6+4*k*; 13, 24, 25; 4; 11, 12, 1), (4+4*k*; 13, 25, 26; 4; 12, 13, 1), (3+4*k*; 13, 26, 27; 4; 13, 14, 1), (3+4*k*; 13, 28, 29; 3; 15, 16, 1), (6+4*k*; 13, 29, 30; 3; 16, 17, 1), (4+4*k*; 13, 30, 31; 3; 17, 18, 1), (3+4*k*; 13, 31, 32; 3; 18, 19, 1), (3+4*k*; 13, 33, 34; 3; 20, 21, 1), (6+4*k*; 13, 34, 35; 3; 21, 22, 1), (4+4*k*; 13, 35, 36; 3; 22, 23, 1), (3+4*k*; 13, 36, 37; 3; 23, 24, 1), (3+4*k*; 13, 38, 39; 3; 25, 26, 1), (6+4*k*; 13, 39, 40; 3; 26, 27 1), (4+4*k*; 13, 40, 41; 3; 27, 28, 1), (3+4*k*; 13, 41, 42; 3; 28, 29, 1), (3+4*k*; 13, 43, 44; 3; 30, 31, 1), (6+4*k*; 13, 44, 45; 3; 31, 32, 1), (4+4*k*; 13, 45, 46; 3; 32, 33, 1), (3+4*k*; 13, 46, 47; 3; 33, 34, 1), (3+4*k*; 13, 48, 49; 3; 35, 36, 1), (6+4*k*; 13, 49, 50; 3; 36, 37, 1), (4+4*k*; 13, 50, 51; 3; 37, 38, 1), (3+4*k*; 13, 51, 52; 3; 38, 39, 1), (3+4*k*; 13, 53, 54; 3; 40, 41, 1), (6+4*k*; 13, 54, 55; 3; 41, 42, 1), (4+4*k*; 13, 55, 56; 3; 42, 43, 1), (3+4*k*; 13, 56, 57; 3; 43, 44, 1), (3+4*k*; 13, 58, 59; 3; 45, 46, 1), (6+4*k*; 13, 59, 60; 3; 46, 47, 1), (4+4*k*; 13, 60, 61; 3; 47, 48, 1), (3+4*k*; 13, 61, 62; 3; 48, 49, 1), (3+4*k*; 13, 63, 64; 3; 50, 51, 1), (6+4*k*; 13, 64, 65; 3; 51, 52, 1), (4+4*k*; 13, 65, 66; 3; 52, 53, 1), (3+4*k*; 13, 66, 67; 3; 53, 54, 1), (3+4*k*; 13, 68, 69; 3; 55, 56, 1), (6+4*k*; 13, 69, 70; 3; 56, 57, 1), (4+4*k*; 13, 70, 71; 3; 57, 58, 1), (3+4*k*; 13, 71, 72; 3; 58, 59, 1), (3+4*k*; 13, 73, 74; 3; 60, 61, 1). (6+4*k*; 13, 74, 75; 3; 61, 62, 1), (4+4*k*; 13, 75, 76; 3; 62, 63, 1), (3+4*k*; 13, 76, 77; 3; 63, 64, 1), (3+4*k*; 13, 78, 79; 3; 65, 66, 1), (6+4*k*; 13, 79, 80; 3; 66, 67, 1), (4+4*k*; 13, 80, 81; 3; 67, 68, 1), (3+4*k*; 13, 81, 82; 3; 68, 69, 1), (3+4*k*; 13, 83, 84; 3; 70, 71, 1). The triplet (13, 84, 85) is the boundary triplet. For m = 3 we have triplets (5+4*k*; 13, 14, 17; 4; 1, 4, 3), (3+*k*; 13, 15, 18; 3; 2, 5, 3), (5+4*k*; 13, 16, 19; 3; 3, 6, 3), (3+2*k*; 13, 17, 20; 3; 4, 7, 3), (5+4*k*; 13, 18, 21; 3; 5, 8, 3), (5+4*k*; 13, 19, 22; 3; 6, 9, 3), (3+*k*; 13, 20, 23; 3; 7, 10, 3), (5+4*k*; 13, 21, 24; 3; 8, 11, 3), (3+2*k*; 13, 22, 25; 3; 9, 12, 3), (5+4*k*; 13, 23, 26; 3; 10, 13, 3), (5+4k; 13, 24, 27; 3; 11, 14, 3), (3+k; 13, 25, 28; 3; 12, 15, 3), (5+4k; 13, 26, 29; 3; 13, 16, 3). The number of permissible triplets of the form (14, y, z) is 10, of which 7 are triplets with m = 2 and 3 triplets with m = 4. For m = 2 we have triplets (4+4k; 14, 15, 17; 5; 1, 3, 2), (6+4k; 14, 23, 25; 3; 9, 11, 2), (4+4k; 14, 25, 27; 3; 11, 13, 2), (6+4k; 14, 33, 35; 3; 19, 21, 2), (4+4k; 14, 35, 37; 3; 21, 23, 2), (6+4k; 14, 43, 45; 3; 29, 31, 2), (4+4k; 14, 45, 47; 3; 31, 33, 2). The triplet (14, 48, 50) = 2(7, 24, 25) is the boundary triplet. For m = 4 we have triplets (3+4k; 14, 15, 19; 3; 1, 5, 4), (5+4k; 14, 17, 21; 3; 3, 7, 4), (3+4k; 14, 21, 25; 3; 7, 11, 4). Direct verification shows that the additional triples are not solutions of equation (1). So Lemma 2 is proved. We summarize the results of the analysis of the initial and additional triplets. For the initial triplets from table 3, the permissible exponent can take the values $p = 3 + 4k$, $4 + 2k$, $4 + 4k$ or $6 + 4k$; the smallest permissible exponent takes the values $p_{\min}$ = 3, 4 or 6; the repetition period of permissible ends takes the values b = 2 or 4; threshold exponent p_{th} = 3, 4 or 5. For additional triplets, we have $p = 3 + k$, $3 + 2k$, $3 + 4k$, $4 + 4k$, $5 + 4k$ or $6 + 4k$; $p_{\min}$ = 3, 4, 5 or 6; b = 1, 2 or 4; p_{th} = 3, 4, 5, 6 or 7. For most initial and additional triplets, $p_{\min} \geq p_{\text{th}}$. The exceptions are initial triplets (9, 10, 11), (9, 12, 13) and additional triplets (11, 12, 13), (11, 14, 15), (11, 19, 20), (11, 20, 21), (12, 13, 15), (12, 15, 17), (13, 14, 15), (13, 15, 16), (13, 16, 17), (13, 18, 19), (13, 21, 22), (13, 23, 24), (13, 26, 27), (14, 15, 17), for which the value of the exponent p_0 closest to $p_{\min}$ should be taken, taking into account the period, namely $p_0 = p + b$, where b is the period (b = 1, 2 or 4). In this case the inequality $p_0 \geq p_{\text{th}}$ will be satisfied (see above). For example, for a triplet (9, 10, 11) $p_{\min}$ = 4, p_{th} = 5, so we should take the exponent p_0 = 4 + 2 = 6, where 2 is period. For a triplet (9, 12, 13) $p_{\min}$ = 3, and p_{th} = 4, therefore it is necessary to take p_0 = 3 + 4 = 7, where 4 is period (see table 3). The exponent for other such triplets is selected in the same way. All admissible triplets can be obtained from the initial triplets of table 3 and additional triplets, successively increasing all bases of these triplets by $a = 10l$. Simple reasoning confirms this statement. Indeed. If the bases x and y do not change or only one of them y ($y > x$) increases by a, and simultaneously z increases by a, then either we obtain the already taken into account initial or additional triplets or condition (5) is satisfied for the obtained triplets, i.e. the third restriction is violated, since we go beyond the boundary triplet, and such triplets are excluded (see also (7), Section IV). If z does not change, and y ($y > x$) increases by a, then condition (4) is not satisfied, i.e. the first restriction is violated and such triplets are excluded. Therefore, so that the restrictions are not violated, all bases must increase by the same value a. In triplets obtained from the same initial or additional triplet, the bases have the same ends, the same permissible exponent p, and the same values of the parameters u and v. Therefore, the set of admissible triplets is divided into groups, in each of which the triplets have the same values of p, u and v.

IV. Study of the function $F(p; x, y, z) = x^p + y^p - z^p$

We study the properties of the function $F(p; x, y, z) = x^p + y^p - z^p$. On the set of natural numbers, it takes discrete values and the roots of this function are solutions of equation (1). From the previous analysis (see the proof of Lemmas 1 and 2), we have Corollary 1: *For initial and additional triplets $F > 0$ for $p < p_{th}$ and $F < 0$ for $p \geq p_{th}$. For triplets, which are exceptions, $F > 0$ for $p < p_0$ and $F < 0$ for $p \geq p_0$.* Therefore, the function takes negative values for all admissible exponent $p \geq p_{\min}$ (or $p \geq p_0$). The equality of the function to zero does not occur at natural values of x, y, z. To continue the analysis, we write a formal representation for the function $F(p; x + a, y + a, z + a)$. We have for arbitrary fixed p

$$F(p; x+a, y+a, z+a) \equiv (x+a)^p + (y+a)^p - (z+a)^p = (x^p + y^p - z^p) + C_p^1(x^{p-1} + y^{p-1} - z^{p-1})a + C_p^2(x^{p-2} + y^{p-2} - z^{p-2})a^2 + \ldots + C_p^{p-2}(x^2 + y^2 - z^2)a^{p-2} + , C_p^{p-1}(x + y - z)a^{p-1} + a^p \quad (6)$$

where $a = 10l$, l = 1, 2, 3, etc. Obviously, when a = 0, $F(p; x + a, y + a, z + a) = F(p; x, y, z)$. If only one base on the left side of equation (1) is increased by $a = 10l$, then (6) takes the form

$$(x+a)^p + y^p - (z+a)^p = (x^p + y^p - z^p) + C_p^1(x^{p-1} - z^{p-1})a + C_p^2(x^{p-2} - z^{p-2})a^2 + \ldots + C_p^{p-2}(x^2 - z^2)a^{p-2} + C_p^{p-1}(x - z)a^{p-1} , \quad (7)$$

where $y = x + u$, $z = x + v$. If $p \geq p_{\text{th}}$ (or $p \geq p_0$), all terms in the right-hand side of (7) are less than zero, therefore equality is impossible, and such bases can be ignored, since they cannot be solutions of equation (1). *A fortiori* this conclusion is valid if both bases x and y do not change. We represent the functions $F(p; x + a, y + a, z + a)$ and $F(p; x , y, z)$ in canonical form in decreasing powers of one base x, treating x as a variable, and u, v as parameters. Since u and v do not change with an increase in all bases on a, such a representation is convenient for analysis and allows us to apply the Descartes' rule of signs. We write the equation for finding the roots of the function $F(p; x + a, y + a, z + a)$

$$F(p;x+a,y+a,z+a)\equiv(x+a)^p+(y+a)^p-(z+a)^p=F(p;x+a,u,v)=x^p+$$
$$x^{p-1}[C_p^1a+C_p^1(u-v)]+x^{p-2}[C_p^2a^2+C_p^1C_{p-1}^1a(u-v)+C_p^2(u^2-v^2)]+$$
$$x^{p-3}[C_p^3a^3+C_p^1C_{p-1}^2a^2(u-v)+C_p^2C_{p-2}^1a(u^2-v^2)+C_p^3(u^3-v^3)]+...+$$
$$x[C_p^{p-1}a^{p-1}+C_p^1C_{p-1}^{p-2}a^{p-2}(u-v)+C_p^2C_{p-2}^{p-3}a^{p-3}(u^2-v^2)+$$
$$C_p^3C_{p-3}^{p-4}a^{p-4}(u^3-v^3)+...+C_p^{p-1}(u^{p-1}-v^{p-1})]+[a^p+C_p^1a^{p-1}(u-v)+$$
$$C_p^2a^{p-2}(u^2-v^2)+...+C_p^{p-1}a(u^{p-1}-v^{p-1})+(u^p-v^p)]$$

(6a)

If in (6a) we put $a = 0$, then we obtain the equation for finding the roots of the initial function $F(p;\, x\,,\, y,\, z)$. We have

$$F(p;x,y,z)\equiv x^p+y^p-z^p=F(p;x,u,v)=x^p+C_p^1x^{p-1}(u-v)+C_p^2x^{p-2}(u^2-v^2)+...$$
$$+C_p^{p-1}x(u^{p-1}-v^{p-1})+(u^p-v^p)=0$$

(6b)

Equations (6a) and (6b) are equivalent to equation (1) for different values of bases. Find out what happens if we simultaneously increase all the bases of the triplet by the value of $a = 10l$. This increase must be simultaneous and by a multiple of 10 so that the restrictions stated above are not violated. With a simultaneous increase in the bases of the triplet, if the function F was positive, i.e. $F > 0$, then its sign does not change; if it was negative, i.e. $F < 0$, its sign changes to the opposite, i.e. will be $F > 0$. Let us explain this with an example. Take the triplet (5, 6, 7). From table 3 it follows that $5^4 + 6^4 - 7^4 < 0$ (here $p = 4$ is the smallest allowed exponent, $p_{\text{th}} = 3$ is the threshold exponent). If we increase all bases by 10, we get $15^4 + 16^4 - 17^4 > 0$, and the transition to the negative region occurs only when $p_{\text{th}} = 8$ (we have $5^8 + 6^8 - 7^8 < 0$. If only one or two bases are increased by 10, then the restrictions formulated above are violated. In our example, we have triplets (5, 6, 17), (5, 16, 17), (6, 15, 17), (15, 16, 7). It is easy to verify that these triplets do not satisfy the first or the third restriction, so they can be ignored. A similar result is observed for other triplets. The larger a, the greater must be the exponent p_{th}, at which F changes the sign (becomes negative). So, for arbitrary admissible fixed p, increasing the bases of initial and additional triplets by $a = 10l$, we can observe a change in the sign of the function $F(p, x, y, z)$ from minus to plus due to the fact that, as follows from (6), the quantity a^p will predominate over the rest of members. On the other hand, for arbitrary fixed base (or, which is the same, for arbitrary $a = 10l$), increasing the exponent p, we can observe a change in the sign of the function $F(p, x, y, z)$ from plus to minus due to the fact that, as follows from (6), the quantity z^p will predominate over the rest of members (as $z > \max(x, y)$). Below we show that the equality $F(p, x, y, z)$ to zero on the set of natural numbers is impossible for any fixed $a = 10l$ and for any fixed $p > 2$. We now consider the equation $F(p;\, x\,,\, y,\, z) = 0$, where the function $F(p;\, x\,,\, y,\, z)$ is given by expression (6b), in the field of real numbers; x is a variable that changes continuously. The characteristic (essential) parameters of this equation are p, u and v. In other words, in order to maintain succession (continuity) with the Fermat equation, it is necessary to consider different equations (6b) for different admissible values of p, u and v. The coefficients of equation (6b) have one change of sign; therefore, according to the Descartes' rule of signs, this equation has one positive real root. In our case, this conclusion does not depend on the parity of the number p, since the first and the last coefficients of the equation have different signs, namely, $a_0 = 1 > 0$, $a_n = (u^p - v^p) < 0$. From the previous analysis (see Lemmas 1, 2) it follows that for the initial and additional triplets, this root is not a natural number. It also cannot be a rational number, since otherwise it follows from the theory that it would be an integer [4], which is not the case. From (6) it follows that if $p < p_{\text{th}}$, then all terms in (6) are positive and the increase in a does not change the sign of the function. If $p = p_{\text{th}}$, then only the first term is negative, and the rest are positive. If $p = p_{\text{th}} + 1$, then the first two terms are negative. If $p = p_{\text{th}} + 2$, then the first three members are negative, and so on. The sign of the function depends on the value of a, which, in turn, depends on the value of u for a given p and m. So, when $p \geq p_{\text{th}}$, for all initial and additional triplets $F(p;\, x\,,\, u,\, v) < 0$, and $F(p;\, x + a\,,\, u,\, v) > 0$. Therefore, according to the well-known Weierstrass theorem the root of equation (6b) is located between x and $x + a$. It cannot be a natural or rational number, since in the interval from x to $x + a$ there are no admissible values of the bases or that the same there are no admissible values of x, u and v. We give the illustrative example. The triplet (5, 6, 7) from table 3 corresponds to three numbers $p = 4 + 4k$, $u = 1$, $v = 2$. Let $p = p_{\min} = 4$, then this triplet is described by the function $F(4;\, x\,,\, 1,\, 2)$. Calculations by (6b) show that $F(4;\, 5\,,\, 1,\, 2) < 0$ and for any admissible p, it remains negative, since $p = 4 + 4k > p_{\text{th}} = 3$. Increase the base x by 10, then the calculations by (6a) give that $F(4;\, 15,\, 1,\, 2) > 0$.

V. Determining the position of real positive root of the function *F*(*p*; *x* , *y*, *z*)

The position of root can be determined more accurately using the Descartes' rule of signs, which can be useful for large a. We apply this rule to equation (6b), which is equivalent to (1), but depends on one variable x (u and v are parameters). The Descartes' rule allows us to determine the number of roots of equation (6b), exceeding a certain number a. In our case, $a = 10l$ (l = 1, 2, 3, etc.). Consider equation (6b), in which the change of variables is made $x \rightarrow x + a$. Equation (6b) transforms into (6a). In this case, as is known, all the roots of the initial equation (6b) are reduced by the same value a. It turns out that for all triplets (in particular, initial and additional), only two cases are possible, all the coefficients of equation (6a) are positive or they have one change of sign (from plus to minus). According to the Descartes' rule, in the first case all the roots of the initial equation (6b) are less than a, and in the second case there is one root larger than a. From the analysis of equation (6a) it follows that it is sufficient to determine the sign of the free term in (6a). In the first case, the sign is positive, and in the second, the sign is negative. We carried out calculations of the position of the roots for all permissible values of p, u and v by equation (6a) and additionally checked the results on the table of powers of numbers. Since the values of u and v remain constant when the bases simultaneously increase by a, the results have a general applicability. The sign of the free term depends on p and m, and for identical p and m on the value of u. We prove Lemma 3. *For initial and additional triplets, the real positive root of equation (6b) is located between* $x + a - 10$ *and* $x + a$. The validity of Lemma 3 is established by direct verification. We give the results of calculation for the initial and additional triplets. For initial triplets, value of a varies from 10 to 30. For all triplets of the form (5, y, z) with $m = 1$, $p_{min} = 4 > p_{th} = 3$ we obtain that $a = 10$. Moreover, it is easy to verify using (6a) that for triplets (5, 6, 7), (5, 7, 8) and (5, 8, 9) the root of (6b) is less than 10 and is located between 5 and 10. For triplet (5, 11, 12) the root is greater than 10 and is located between 10 and 15. For triplet (6, 7, 9) with $m = 2$, $p_{min} = 3 = p_{th}$ we get that $a = 10$ and the root of (6b) is less than 10 and is located between 6 and 10. For triplet (7, 9, 10) with $m = 1$, $p_{min} = 6 > p_{th} = 4$ we get that $a = 10$ and the root is located between 10 and 17. For triplet (7, 10, 11) with $m = 1$, $p_{min} = 4 = p_{th}$ we have $a = 10$ and the root is located between 7 and 10. For triplet (7, 14, 15) with $m = 1$, $p_{min} = 6 > p_{th} = 3$ we have $a = 20$ and the root is located between 17 and 27. For triplet (7, 15, 16) with $m = 1$, $p_{min} = 4 = p_{th}$ we have $a = 10$ and the root is located between 10 and 17. For triplet (7, 19, 20) with $m = 1$, $p_{min} = 6 > p_{th} = 3$ we have $a = 30$ and the root is located between 27 and 37. For triplet (7, 20, 21) with $m = 1$, $p_{min} = 4 > p_{th} = 3$ we have $a = 20$ and the root is located between 17 and 27. For triplet (8, 9, 11) with $m = 2$, $p_{min} = 3 = p_{th}$ we have $a = 10$ and the root is located between 8 and 10. For triplet (9, 10, 11) with $m = 1$, $p_0 = 6 > p_{th} = 5$ we have $a = 10$ and the root is located between 9 and 19. For triplet (9, 12, 13) with $m = 1$, $p_0 = 7 > p_{th} = 4$ we have $a = 20$ and the root is located between 19 and 29. For triplet (9, 15, 16) with $m = 1$, $p_{min} = 4 = p_{th}$ we have $a = 10$ and the root is located between 9 and 19. For triplet (9, 20, 21) with $m = 1$, $p_{min} = 4 > p_{th} = 3$ we have $a = 10$ and the root is located between 9 and 19. For triplets (9, 25, 26), (9, 30, 31), (9, 35, 36), with $m = 1$, $p_{min} = 4 > p_{th} = 3$ we have $a = 20$ and the root is located between 19 and 29. For triplets (9, 17, 18), (9, 22, 23), (9, 27, 28), (9, 32, 33), (9, 37, 38) with $m = 1$, $p_{min} = 3 = p_{th}$ we have $a = 10$ and the root is located between 9 and 19. For triplet (9, 10, 13) with $m = 3$, $p_{min} = 4 > p_{th} = 3$ we have $a = 10$ and the root is located between 9 and 19. For triplet (10, 11, 13) with $m = 2$, $p_{min} = 4 = p_{th}$ we have $a = 10$ and the root is located between 10 and 20. For triplets (10, 17, 19), (10, 19, 21), (10, 21, 23), with $m = 2$, $p_{min} = 4 > p_{th} = 3$ we have $a = 20$ and the root is located between 20 and 30. The regularity (behavior) is obvious. If the values of p and m increase, then a increases; with the same p and m, if u is increased by 10, then a, as a rule, increases by 10. So, the root of equation (6b) for initial triplets is between x and 10, if $p = 4$ and ($u = 1$, $v = 2$), ($u = 2$, $v = 3$), ($u = 3$, $v = 4$) or $p = 3$ and ($u = 1$, $v = 3$). The root of equation (6b) is located between 10 and ($x + 10$) if $p = 6$ and ($u = 2$, $v = 3$) or $p = 4$ and ($u = 8$, $v = 9$). For the remaining initial triplets, the root of equation (6b) is located between $x + a - 10$ and $x + a$. Thus, Lemma 3 is valid for initial triplets. For additional triplets calculations are carried out similarly. Below are the results of calculations. The dependence of a on the parameters p, m and u for additional triplets is the same as for the initial ones. For additional triplets, a varies from 10 to 80. For triplets of the form (11, y, z) for $m = 1$, the following results are obtained. If $p = 5+4k$, then $a = 10$, when $p_{th} = 6$ and $u = 1$; $a = 10$, when $p_{th} = 5$ and $u = 2$; $a = 10$, when $p_{th} = 4$ and u = 5, 6, 7; $a = 20$, when $p_{th} = 3$ and u = 10, 11, 12, 15, 16, 17; $a = 30$, when $p_{th} = 3$ and u = 20, 21, 22, 25, 26, 27; $a = 40$, when $p_{th} = 3$ and u = 30, 31, 32, 35, 36, 37; $a = 50$, when $p_{th} = 3$ and u = 40, 41, 42, 45, 46, 47. If $p = 3+2k$, then $a = 10$, when $p_{th} = 5$ and $u = 3$; $a = 10$, when $p_{th} = 4$ and $u = 8$; $a = 10$, when $p_{th} = 3$ and u = 13, 18, 23, 28, 33; $a = 20$, when $p_{th} = 3$ and u = 38, 43, 48. If $p = 3+k$, then $a = 10$, when $p_{th} = 4$ and u = 4, 9; $a = 10$, when $p_{th} = 3$ and u = 14, 19, 24, 29, 34; $a = 20$, when $p_{th} = 3$ and u = 39, 44. For $m = 3$, the following results are obtained. If $p = 6+4k$, then $a = 20$, when $p_{th} = 3$ and $u = 3$; $a = 40$, when $p_{th} = 3$ and $u = 6$. If $p = 4+4k$, then $a = 20$, when $p_{th} = 3$ and $u = 4$. For triplets of the form (12, y, z) for $m = 2$, the following results are obtained. If $p = 3+2k$, then $a = 10$, when $p_{th} = 4$ and $u = 1$; $a = 10$, when $p_{th} = 3$ and $u = 11$; $a = 20$, when $p_{th} = 3$ and $u = 21$. If $p = 3+k$, then $a = 10$, when $p_{th} = 4$ and $u = 3$; $a = 10$, when $p_{th} = 3$ and $u = 13$. If $p = 5+4k$, then $a = 20$, when $p_{th} = 3$ and u = 5, 7, 9; $a = 30$, when $p_{th} = 3$ and u = 15, 17, 19. For $m = 4$ we have the following results. If $p = 4+4k$, then $a = 20$, when $p_{th} = 3$ and u = 3. For triplets of the form (13, y, z) for $m = 1$, the following results are obtained. If $p = 6+4k$, then $a = 10$, when

p_{th} = 7 and u = 1; a = 10, when p_{th} = 4 and u = 6; a = 20, when p_{th} = 4 and u = 11; a = 30, when p_{th} = 3 and u = 16; a = 40, when p_{th} = 3 and u = 21, 26; a = 50, when p_{th} = 3 and u = 31, 36; a = 60, when p_{th} = 3 and u = 41, 46; a = 70, when p_{th} = 3 and u = 51, 56; a = 80, when p_{th} = 3 and u = 61, 66. If p = 4+4k, then a = 10, when p_{th} = 6 and u = 2; a = 10, when p_{th} = 4 and u = 7, 12; a = 10, when p_{th} = 3 and u = 17; a = 20, when p_{th} = 3 and u = 22, 27, 32; a = 30, when p_{th} = 3 and u = 37, 42, 47; a = 40, when p_{th} = 3 and u = 52, 57, 62; a = 50, when p_{th} = 3 and u = 67. If p = 3+4k, then a = 10, when p_{th} = 6 and u = 3; a = 20, when p_{th} = 5 and u = 5; a = 20, when p_{th} = 4 and u = 8; a = 30, when p_{th} = 4 and u = 10, 13; a = 10, when p_{th} = 3 and u = 15, 18, 20, 23, 25, 28, 30, 33, 35, 38, 40; a = 20, when p_{th} = 3 and u = 43, 45, 48, 50; a = 30, when p_{th} = 3 and u = 53, 55, 58, 60; a = 40, when p_{th} = 3 and u = 63, 65, 68, 70. For m = 3 we have the following results. If p = 5+4k, then a = 10, when p_{th} = 4 and u = 1; a = 20, when p_{th} = 3 and u = 3, 5; a = 30, when p_{th} = 3 and u = 6, 8, 10, 11; a = 40, when p_{th} = 3 and u = 13. If p = 3+k, then a = 10, when p_{th} = 3 and u = 2, 7, 12. If p = 3+2k, then a = 10, when p_{th} = 3 and u = 4, 9. For triplets of the form (14, y, z) for m = 2, the following results are obtained. If p = 4+4k, then a = 10, when p_{th} = 5 and u = 1; a = 20, when p_{th} = 3 and u = 11; a = 30, when p_{th} = 3 and u = 21; a = 40, when p_{th} = 3 and u = 31. If p = 6+4k, then a = 30, when p_{th} = 3 and u = 9; a = 50, when p_{th} = 3 and u = 19; a = 60, when p_{th} = 3 and u = 29. For m = 4 we have the following results. If p = 3+4k, then a = 10, when p_{th} = 3 and u = 1, 7. If p = 5+4k, then a = 30, when p_{th} = 3 and u = 3. A direct verification shows that for additional triplets the root is located between $x + a - 10$ and $x + a$. So, Lemma 3 is completely proved. In our case, a more accurate determination of the position of the root is not required. The main conclusion from the analysis performed is that increasing the base of a triplet by the number $a = 10l$ does not change the class (type) of solutions of equation (6b) or that the same of equation (1), so that they are not natural (rational) numbers. Since an arbitrary triplet can be obtained from initial or additional triplets by increasing the bases of the triplets by the number $a = 10l$, this conclusion is valid in the general case.

VI. Proof of the theorem

We now prove Fermat's theorem by the method of induction with respect to the parameters a and p. According to Lemmas 1 and 2, the theorem is valid for all initial and additional triplets. We prove the induction transition. The proof of the theorem consists of two parts: the proof for all permissible triplets (induction by a) and the proof for all permissible exponents (induction by p). *Induction by a.* The value of p is fixed, although it is chosen arbitrarily. We prove that Fermat's theorem is valid for any permissible triplet. It is enough to prove that the theorem is valid for any base x when it increases by an arbitrary number a. Let a = 0. It follows from Corollary 1 that for all initial and additional triplets with an arbitrary permissible exponent p, we have $F(p; x, y, z) = F(p; x, u, v) > 0$ if $p < p_{th}$ or $F(p; x, u, v) < 0$ if $p \geq p_{th}$. The transition of the function $F(p; x, u, v)$ through 0 does not occur for a natural or rational value of x, so equation (6b) has a real positive root. If we successively increase the base x by 10, then this procedure, on the one hand, allows us to obtain permissible triplets, and, on the other hand, to determine the position of the root of the initial equation (6b) using the Descartes' rule. Let us suppose that for $a = a_l = 10l$, the equation (6a), namely, $F(p; x + a_l, u, v) = 0$ does not turn into 0 for the natural value of x, u and v. Equation (6a) corresponds to equation (6b), in which the change of variable is made $x \rightarrow x + a_l$; therefore, equation (6b) has a real positive root. From the definition of a_l, it follows that we can choose the value of l so that a_l is the smallest number for which the inequality $F(p; x + a_l, u, v) > 0$ holds. Then the root of the function $F(p; x, u, v)$ or that the same of equation (6b) is located between $x + a_l - 10$ and $x + a_l$, and it is not a natural (rational) number. We put $a_{l+1} = 10(l + 1) = a_l + 10$, which corresponds to the change of variables $x \rightarrow x + a_{l+1}$ or, that the same, $x + a_l \rightarrow x + a_{l+1}$. Then *a fortiori* $F(p; x + a_{l+1}, u, v) > 0$. So, the real positive root of the function $F(p; x, u, v)$ does not change its position, namely, it is located between $x + a_l - 10$ and $x + a_l$, and it remains real positive, that is, cannot be a natural number. Therefore, there are no permissible bases (natural numbers), which are solutions of equation (6b) or equivalent equation (1). Induction transition by the parameter a is proved. *Induction by p.* The value of a is fixed, although it is chosen arbitrarily. We prove that Fermat's theorem is valid for any admissible exponent p. We put $p = p_{min}$, where p_{min} can take the values 3, 4, 5, or 6 (see above). To avoid confusion, we denote x_0 – the base of the initial or additional triplet; x_0 takes the values 5, 6, ..., 14; u and v take the corresponding permissible values (see above). Then, for a fixed a, an arbitrary base is represented as $x = x_0 + a$. When a = 0, for most initial and additional triplets $p_{min} \geq p_{th}$, then $F(p_{min}; x_0, u, v) < 0$. For triplets that are exceptions $p_{min} < p_{th}$, then $F(p_{min}; x_0, u, v) > 0$ and for arbitrary a, we have $F(p_{min}; x_0 + a, u, v) > 0$. For these triplets, if we put $p = p_0 = p_{min} + b$, where b is period (b = 1, 2 or 4), then $F(p_0; x_0, u, v) < 0$. It follows from Lemmas 1 and 2 that for all initial and additional triplets, the root of the function $F(p_{min}|p_0; x_0, u, v)$ is not a natural number. It follows from Lemma 3 that for all initial and additional triplets, the root of equation (6b), in which $p = p_{min}$ or $p = p_0 = p_{min} + b$, is located between $x_0 + a - 10$ and $x_0 + a$, where a = 10 ... 80. (Of course, a is different for different triplets as well as p_{min} and p_0). A more accurate determination of the position of the root is not required. Therefore, $F(p_{min}|p_0; x_0 + a - 10, u, v) < 0$, but $F(p_{min}|p_0; x_0 + a, u, v) > 0$. Since in the given interval between $x_0 + a - 10$ and $x_0 + a$ there are no permissible triplets (permissible values of x_0, u, v), the root of the function $F(p_{min}|p_0; x_0, u, v)$ is a real positive and cannot be a natural number. *A fortiori* this is true for

$a > 80$, since in this case for an arbitrary a we have $F(p_{\min}|p_0; x_0 + a, u, v) > 0$. Thus, the Fermat theorem is valid for the exponents $p_{\min}$ and p_0. If we successively increase the exponent p by period b_k, then this procedure, on the one hand, allows us to obtain all permissible exponents, and on the other hand, to determine groups of triplets described by equation (6b) of the same (given) degree. Let us assume that for $p = p_k = p_{\min} + b_k$, where $b_k = k$, $2k$ or $4k$ (see above) equation (6b), namely $F(p_k; x, u, v) = 0$ has no natural solutions, so its root is a real positive number. We use $p_{\min}$, since $p_0 = p_{\min} + b$ and therefore there is no need to consider p_0 separately. From the definition of p_k, it follows that we can choose the value of k so that p_k is the smallest number for which the inequality $p_k \geq p_{\text{th}}(a)$ is valid, i.e. the inequality $F(p_k; x, u, v) < 0$ holds. We put $p = p_{k+1} = p_k + b$, where $b = 1$, 2 or 4. Then *a fortiori* $F(p_{k+1}; x, u, v) < 0$. Therefore, the root of the equation $F(p_{k+1}; x, u, v) = 0$ is not a natural number. Therefore, there is no equation (6b) or that the same equation (1) with permissible exponent, the root of which is a natural number. Induction transition by the parameter p is proved. Thus, a change in the base and the exponent does not change the class of solutions of equation (6b) or, it is the same, of equation (1). The root of the equation remains real positive and is not a natural number. This implies the validity of the Fermat theorem.

VII. Connection of Fermat's theorem with Beal conjecture

Beal conjecture consists in the statement that the equation $x^p + y^q = z^r$ has no solution in positive integers x, y, z, p, q and r with p, q and r at least 3 and x, y, and z coprime. If $p = q = r$, then this equation turns into equation (1) and then, of course, the validity of Beal conjecture follows from Fermat's theorem, but the opposite is not true. Let us show that in this case the validity of the Beal conjecture follows from our method of proving Fermat's theorem. Indeed. For all initial and additional triplets, the Beal conjecture is true, since the bases of these triplets (natural numbers) are coprime. In addition, in each initial or additional triplet, one base is an even number, and the other two bases are odd numbers. The difference of the parameters v and u, namely $m = v - u$, takes values 1, 2 or 3 for the initial triplets (see table 3), 1 or 3 for additional triplets of the form (11, y, z) and (13, y, z) and 2 or 4 for additional triplets of the form (12, y, z) and (14, y, z). All permissible triplets are obtained from the initial and additional triplets by increasing all the bases of the triplet (initial or additional) simultaneously by $a = 10l$. If we simultaneously increase the bases of the triplet by $a = 10l$, then the properties noted above are remain unchanged (saved), namely, the parity of the bases does not change and the parameters v, u and $m = v - u$ do not change their values. Therefore, the bases of triplets cannot have an even number as a common divisor. The common divisor also cannot be the number 3, although the two bases can be divided by 3. Since $m \leq 4$, other divisors may not be considered. It follows that the permissible bases remain mutually prime (coprime) numbers, which proves the Beal conjecture. In the general case, when p, q and r are different, the analysis technique used in the proof of Fermat's theorem can be used to prove Beal conjecture. In particular, the second restriction on permissible solutions established for equation (1) remains valid for the Beal equation. Let's give an example. We can assume, without loss of generality, that $x < y < z$. If $p < q < r$, then the Beal equation obviously has no solutions not only for coprime numbers x, y and z, but, also for numbers having a common divisor. If $p > q > r$, then the permissible values of exponents p, q and r and bases x, y and z are determined using the second restriction. In particular, it follows from the second restriction that three cases are possible: 1) x, y and z are even numbers; 2) x and y are odd, and z is an even number; 3) x and y have different parity, and z is odd. Table 1 can be used to select permissible combinations of endings and exponents. Proving this conjecture and analyzing possible options in these cases requires a lot of time due to the generality of the problem statement.

VIII. Conclusion

1. The above proof of the theorem uses only the characteristic properties of natural numbers and some general theorems on the roots of algebraic equations.
2. Fermat's theorem has an obvious geometric interpretation. For $p = 1$, equation (1) always has a solution, i.e. the sum of two integer segments is always an integer segment. For $p = 2$, equation (1) has a solution only in some cases, i.e. the sum of the areas of two squares with integer sides is only sometimes equal to the area of the square with integer sides. For $p = 3$, equation (1) has no solution, i.e. the volume of a cube with integer sides is never the sum of the volumes of two cubes with integer sides. This is true *a fortiori* for hypercube.
3. The analysis technique used in the proof of Fermat's theorem can be used to prove Beal conjecture. In the case when $p = q = r$, Beal conjecture is one of the consequences of Fermat's theorem.

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Table 1
Admissible ends of powers for elementary bases

Number	Last digit of number	Number	Last digit of number	Number	Last digit of number	Number	Last digit of number
2^3	8	3^3	7	4^3	4	5^3	5
2^4	6	3^4	1	4^4	6	5^4	5 (repeat)
2^5	2	3^5	3	4^5	4 (repeat)		
2^6	4	3^6	9				
2^7	8 (repeat)	3^7	7 (repeat)				
Number	Last digit of number	Number	Last digit of number	Number	Last digit of number	Number	Last digit of number
6^3	6	7^3	3	8^3	2	9^3	9
6^4	6 (repeat)	7^4	1	8^4	6	9^4	1
		7^5	7	8^5	8	9^5	9 (repeat)
		7^6	9	8^6	4		
		7^7	3 (repeat)	8^7	2 (repeat)		
Number	Last digit of number	Number	Last digit of number				
0^3	0	1^3	1				
0^4	0 (repeat)	1^4	1(repeat)				

Table 2
Combination of ends for elementary bases permissible under basic restrictions

$2^{3+2k}+3^{3+2k}=5^{3+2k}$	$3^{3+4k}+4^{3+4k}=1^{3+4k}$	$7^{3+4k}+4^{3+4k}=3^{3+4k}$	$8^{3+4k}+4^{3+4k}=6^{3+4k}$
$2^{4+2k}+4^{4+2k}=0^{4+2k}$	$3^{5+4k}+4^{5+4k}=7^{5+4k}$	$7^{5+4k}+4^{5+4k}=1^{5+4k}$	$8^{5+4k}+4^{5+4k}=2^{5+4k}$
$2^{5+4k}+4^{5+4k}=6^{5+4k}$	$3^{6+4k}+4^{6+4k}=5^{6+4k}$	$7^{6+4k}+4^{6+4k}=5^{6+4k}$	$8^{6+4k}+4^{6+4k}=0^{6+4k}$
$2^{3+k}+5^{3+k}=7^{3+k}$	$3^{3+k}+5^{3+k}=8^{3+k}$	$7^{3+k}+5^{3+k}=2^{3+k}$	$8^{3+4k}+5^{3+4k}=3^{3+4k}$
$2^{4+4k}+5^{4+4k}=1^{4+4k}$	$3^{4+4k}+5^{4+4k}=6^{4+4k}$	$7^{4+4k}+5^{4+4k}=6^{4+4k}$	$8^{4+4k}+5^{4+4k}=1^{4+4k}$
$2^{6+4k}+5^{6+4k}=3^{6+4k}$	$3^{6+4k}+5^{6+4k}=2^{6+4k}$	$7^{4+4k}+5^{4+4k}=4^{4+4k}$	$8^{4+2k}+5^{4+2k}=7^{4+2k}$
$2^{3+4k}+6^{3+4k}=4^{3+4k}$	$3^{3+4k}+6^{3+4k}=7^{3+4k}$	$7^{4+2k}+5^{4+2k}=8^{4+2k}$	$8^{3+k}+5^{3+k}=3^{3+k}$
$2^{5+4k}+6^{5+4k}=8^{5+4k}$	$3^{5+4k}+6^{5+4k}=9^{5+4k}$	$7^{3+4k}+6^{3+4k}=9^{3+4k}$	$8^{6+4k}+5^{6+4k}=7^{6+4k}$
$2^{6+4k}+6^{6+4k}=0^{6+4k}$	$3^{6+4k}+6^{6+4k}=5^{6+4k}$	$7^{5+4k}+6^{5+4k}=3^{5+4k}$	$8^{3+4k}+6^{3+4k}=2^{3+4k}$
$2^{3+4k}+7^{3+4k}=1^{3+4k}$	$3^{3+2k}+7^{3+2k}=0^{3+2k}$	$7^{6+4k}+6^{6+4k}=5^{6+4k}$	$8^{5+4k}+6^{5+4k}=4^{5+4k}$
$2^{5+4k}+7^{5+4k}=9^{5+4k}$	$3^{3+4k}+8^{3+4k}=9^{3+4k}$	$7^{3+2k}+8^{3+2k}=5^{3+2k}$	$8^{6+4k}+6^{6+4k}=0^{6+4k}$
$2^{3+2k}+8^{3+2k}=0^{3+2k}$	$3^{5+4k}+8^{5+4k}=1^{5+4k}$	$7^{3+4k}+9^{3+4k}=8^{3+4k}$	$8^{3+4k}+9^{3+4k}=1^{3+4k}$
$2^{3+4k}+9^{3+4k}=3^{3+4k}$	$3^{3+4k}+9^{3+4k}=6^{3+4k}$	$7^{5+4k}+9^{5+4k}=6^{5+4k}$	$8^{5+4k}+9^{5+4k}=7^{5+4k}$
$2^{5+4k}+9^{5+4k}=1^{5+4k}$	$3^{5+4k}+9^{5+4k}=2^{5+4k}$	$7^{6+4k}+9^{6+4k}=0^{6+4k}$	$8^{6+4k}+9^{6+4k}=5^{6+4k}$
$2^{6+4k}+9^{6+4k}=5^{6+4k}$	$3^{6+4k}+9^{6+4k}=2^{6+4k}$	$7^{4+4k}+0^{4+4k}=1^{4+4k}$	$8^{4+2k}+0^{4+2k}=2^{4+2k}$
$2^{3+4k}+1^{3+4k}=9^{3+4k}$	$3^{6+4k}+9^{6+4k}=0^{6+4k}$	$7^{6+4k}+0^{6+4k}=3^{6+4k}$	$8^{4+4k}+0^{4+4k}=6^{4+4k}$
$2^{4+4k}+0^{4+4k}=4^{4+4k}$	$3^{4+2k}+0^{4+2k}=7^{4+2k}$	$7^{4+4k}+0^{4+4k}=9^{4+4k}$	$8^{3+4k}+1^{3+4k}=7^{3+4k}$
$2^{4+2k}+0^{4+2k}=8^{4+2k}$	$3^{4+4k}+0^{4+4k}=9^{4+4k}$	$7^{6+4k}+4^{6+4k}=5^{6+4k}$	$8^{5+4k}+1^{5+4k}=9^{5+4k}$
$2^{5+4k}+1^{5+4k}=3^{5+4k}$	$3^{3+4k}+1^{3+4k}=2^{3+4k}$	$7^{6+4k}+1^{6+4k}=0^{6+4k}$	$8^{6+4k}+1^{6+4k}=5^{6+4k}$

$2^{6+4k}+1^{6+4k}=5^{6+4k}$	$3^{5+4k}+1^{5+4k}=4^{5+4k}$	$7^{5+4k}+1^{5+4k}=8^{5+4k}$	$8^{4+4k}+5^{4+4k}=9^{4+4k}$
×	$3^{6+4k}+1^{6+4k}=0^{6+4k}$	×	×
$4^{3+4k}+5^{3+4k}=9^{3+4k}$	$3^{3+4k}+3^{3+4k}=4^{3+4k}$	$5^{4+2k}+6^{4+2k}=9^{4+2k}$	$6^{3+4k}+1^{3+4k}=3^{3+4k}$
$4^{4+4k}+5^{4+4k}=3^{4+4k}$	$3^{4+4k}+0^{4+4k}=1^{4+4k}$	$5^{3+k}+6^{3+k}=1^{3+k}$	$6^{5+4k}+1^{5+4k}=7^{5+4k}$
$4^{4+4k}+5^{4+4k}=7^{4+4k}$	×	$5^{4+4k}+1^{4+4k}=2^{4+4k}$	×
$4^{3+4k}+6^{3+4k}=0^{3+4k}$	$9^{3+k}+5^{3+k}=4^{3+k}$	$5^{3+k}+1^{3+k}=6^{3+k}$	$0^{4+4k}+1^{4+4k}=3^{4+4k}$
$4^{3+4k}+9^{3+4k}=7^{3+4k}$	$9^{4+2k}+5^{4+2k}=6^{4+2k}$	$5^{4+4k}+6^{4+4k}=3^{4+4k}$	$0^{4+4k}+1^{4+4k}=7^{4+4k}$
$4^{4+4k}+0^{4+4k}=2^{4+4k}$	$9^{4+4k}+5^{4+4k}=2^{4+4k}$	$5^{4+4k}+6^{4+4k}=7^{4+4k}$	$0^{4+2k}+1^{4+2k}=9^{4+2k}$
$4^{4+4k}+0^{4+4k}=8^{4+4k}$	$9^{4+4k}+5^{4+4k}=8^{4+4k}$	$5^{4+4k}+1^{4+4k}=8^{4+4k}$	×
$4^{3+4k}+1^{3+4k}=5^{3+4k}$	$9^{3+2k}+6^{3+2k}=5^{3+2k}$	×	
×	$9^{4+2k}+0^{4+2k}=1^{4+2k}$		
	$9^{4+4k}+0^{4+4k}=3^{4+4k}$		
	$9^{4+4k}+0^{4+4k}=7^{4+4k}$		
	$9^{3+2k}+1^{3+2k}=0^{3+2k}$		

Note. The relationships given in table 2 do not mean actual equality and they are symbolic notation representing the fulfillment of the basic restrictions necessary for equation (1), namely, the coincidence of the exponents of all components and the coincidence of the last digit, to which the left and right sides of equation (1) end.

Table 3
Permissible triplets of the form (*x*, *y*, *z*), in which *x* is an elementary base

Form of triplet	Permissible triplets of this form	Degree, permissible by basic restrictions, *p*	The number of permissible triplets	Boundary triplet	p_{th}	*u*	*v*	*m*
(1, *y*, *z*)	-	-	0	-				
(2, *y*, *z*)	-	-	0	-				
(3, *y*, *z*)	-	-	0	(3, 4, 5)				
(4, *y*, *z*)	-	-	0	-				
(5, *y*, *z*)	(5, 6, 7)	4+4*k*	4	(5, 12, 13)	3	1	2	1
	(5, 7, 8)	4+2*k*			3	2	3	1
	(5, 8, 9)	4+4*k*			3	3	4	1
	(5, 11, 12)	4+4*k*			3	6	7	1
(6, *y*, *z*)	(6, 7, 9)	3 + 4*k*	1	(6, 8, 10)	3	1	3	2
(7, *y*, *z*)	(7, 9, 10)	6+4*k*	6	(7, 24, 25)	4	2	3	1
	(7, 10, 11)	4+4*k*			4	3	4	1
	(7, 14, 15)	6+4*k*			3	7	8	1
	(7, 15, 16)	4+4*k*			3	8	9	1
	(7, 19, 20)	6+4*k*			3	12	13	1
	(7, 20, 21)	4+4*k*			3	13	14	1
(8, *y*, *z*)	(8, 9, 11)	3 + 4*k*	1	(8, 15, 17)	3	1	3	2
(9, *y*, *z*)	(9, 10, 11)	4+2*k*	13	(9, 40, 41)	5	1	2	1
	(9, 12, 13)	3 + 4*k*			4	3	4	1
	(9, 15, 16)	4+2*k*			4	6	7	1
	(9, 17, 18)	3 + 4*k*			3	8	9	1
	(9, 20, 21)	4+2*k*			3	11	12	1
	(9, 22, 23)	3 + 4*k*			3	13	14	1
	(9, 25, 26)	4+2*k*			3	16	17	1
	(9, 27, 28)	3 + 4*k*			3	18	19	1
	(9, 30, 31)	4+2*k*			3	21	22	1
	(9, 32, 33)	3 + 4*k*			3	23	24	1
	(9, 35, 36)	4+2*k*			3	26	27	1
	(9, 37, 38)	3 + 4*k*			3	28	29	1
	(9, 10, 13)	4+4*k*		(9, 12, 15)	3	1	4	3
(10, *y*, *z*)	(10, 11, 13)	4+4*k*	4	(10, 24, 26)	4	1	3	2
	(10, 17, 19)	4+4*k*			3	7	9	2
	(10, 19, 21)	4+2*k*			3	9	11	2
	(10, 21, 23)	4+4*k*			3	11	13	2

Note. To determine the degree *p*, we used tables 1, 2. Threshold exponent p_{th} is the value of the exponent *p*, at which the difference between the left and right sides of equation (1) changes sign from plus to minus.