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Sum of the First Natural Numbers and their Powers

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Abstract : *We consider here the sum of the first n natural numbers and their powers. In order to achieve this, we derive their summation in terms of n and with varying values of n, 1 , 2 , 3 , . . .and so on ,we obtain the sum for the first n terms.*

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I. Introduction

The sum

$\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$ (1)

Is common in mathematics. We meet also, though, to a less extent the series

$\sum_{r=1}^{n} r^2 = \frac{1}{6}\{2n^3 + 3n^3 + n\} = \frac{n}{6}\{2n^2 + 3n + 1\} = \frac{n}{6}(n+1)(2n+1)$ (2)

However, for sum

$\sum_{r=1}^{n} r^x$, where x > 2 (3)

we will develop enabling equations to carry out the summations.

II. Literature Review

It is common to meet arithmetical and geometric series in practice in many mathematical computations, especially in finance. The summation of these series are very easy. The summation of series for the first n positive integers, first n squares and first n cubes have been derived and applied . These can be seen in Youse (1970).

III. Methodology

We derive the enabling equations in the form:

$r^2 - (r-1)^2 = 2r - 1$ (4)

$r^3 - (r-1)^3 = 3r^2 - 3r + 1$ (5)

$r^4 - (r-1)^4 = 4r^3 - 6r^2 + 4r - 1$ (6)

$r^5 - (r-1)^5 = 5r^4 - 10r^3 + 10r^2 - 5r + 1$ (7)

$r^6 - (r-1)^6 = 6r^5 - 15r^4 + 20r^3 - 15r^2 + 6r - 1$ (8)

$r^7 - (r-1)^7 = 7r^6 - 21r^5 + 35r^4 - 35r^3 + 21r^2 - 7r + 1$ (9)

$r^8 - (r-1)^8 = 8r^7 - 28r^6 + 56r^5 - 70r^4 + 56r^3 - 28k^2 + 8r - 1$ (10)

Equations (4) to (10) are used to derive, respectively the sums (11) to (17) below:

$\sum_{r=1}^{n} r = \frac{n}{2}(n+1) = \frac{1}{2}(n^2+n)$ (11)

$\sum_{r=1}^{n} r^2 = \frac{1}{6}\{2n^3 + 3n^3 + n\} = \frac{n}{6}\{2n^2 + 3n + 1\} = \frac{n}{6}(n+1)(2n+1)$ (12)

$\sum_{r=1}^{n} r^3 = \frac{1}{4}\{n^4 + 2n^3 + n^2\} = \frac{1}{4}\{n^2 + n\}^2 = \frac{n^2}{4}\{n+1\}^2$ (13)

$$\sum_{r=1}^{n} r^4 = \frac{1}{30}\{6n^5 + 15n^4 + 10n^3 - n\}$$

$= \frac{n}{30}\{6n^4 + 15n^3 + 10n^2 - 1\}$

$= \frac{n}{30}\{(n+1)(2n+1)(3n^2 + 3n - 1)$ (14)

$\sum_{r=1}^{n} r^5 = \frac{1}{12}(2n^6 + 6n^5 + 5n^4 - n^2) = \frac{n^2}{12}(n+1)^2\ (2n^2 + 2n - 1)$ (15)

$$\sum_{r=1}^{n} r^6 = \frac{n}{42}\{6n^6 + 21n^5 + 21n^4 - 7n^2 + 1\}$$

$= \frac{1}{42}(6n^7 + 21n^6 + 21n^5 - 7n^3 + n)$ (16)

$\sum_{r=1}^{n} r^7 = \frac{1}{24}\{3n^8 + 12n^7 + 14n^6 - 7n^4 + 44n^2 - 42n\}$ (17)

We now take the cases, one after the other. For example, to prove that

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$$

we use

$r^2 - (r-1)^2 = 2r - 1$

to generate n equations , by setting r = 1, 2 , 3, . . . , n as follows:

$n^2 - (n-1)^2 = 2n - 1$

$(n-1)^2 - (n-2)^2 = 2(n-1) - 1$

$(n-2)^2 - (n-3)^2 = 2(n-2) - 1$

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$1^2 - (n-n)^2 = 2(1) - 1$

Adding the last n equations, gives

$$n^2 = 2\sum_{r=1}^{n} r - n$$

Therefore

$$\sum_{r=1}^{n} r = \frac{1}{2}[n^2 + n] = \frac{n}{2}(n+1)$$

To prove that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)$ (2n+1)

We use $r^3 - (r-1)^3 = 3r^2 - 3r + 1$ for r =1,2,3……., n to obtain

$n^3 - (n-1)^3 = 3n^2 - 3n + 1$

$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$

$(n-2)^3 - (n-2)^3 = 3(n-2)^2 - 3(n-2) + 1$

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$1^3 - 0 = 3(1)^2 - 3(1) + 1$

Adding the last n equations, we have

$$n^3 = 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r + 1$$

⇨ $3\sum_{r=1}^{n} r^2 = n^3 + 3\sum r - n$

⇨ $\sum_{r=1}^{n} r^2 = \frac{1}{3}\{n^3 + \frac{3n}{2}(n+1) - n\}$

$= \frac{1}{6}\{2n^3 + 3n^2 - 2n + 3n\}$

$= \frac{1}{6}\{2n^3 + 3n^2 + n\} = = \frac{n}{6}\{2n^2 + 3n + 1\}$

$= \frac{n}{6}(n+1)$(2n+1)

To prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}(n^2 + n)^2 = \frac{n^2}{4}[n+1]^2$

we use the equation $r^4 - (r-1)^4 = 4r^3 - 6r^2 + 4r - 1$, r = 1,2,………,n to generate n equations as follows:

$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$

$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$

$(n-2)^4 - (n-3)^4 = 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1$

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$1^4 + 0^4 = 4(1) - 6(1) + 4(1) - 1$

Adding the last n equations we have

$n^4 + 0 = 4\sum_{r=1}^{n} r^3 - 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - n$

i.e. $4\sum_{r=1}^{n} r^3 = n^4 + 6\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + n$

$= n^4 + 6\left\{\frac{n}{6}(n+1)(2n+1)\right\} - 4\frac{n}{2}(n+1) + n$

$= n^4 + 6\left\{\frac{n}{6}\, n(n+1)(2n+1)\right\} - 2n^2 - 2n + n$

$= n^4 + n[2n^2 + 3n + 1] - 2n^2 - n$

$= n^4 + 2n^3 + 3n^2 + n - 2n^2 - n$

$= n^4 + 2n^3 + n^2 = [n^2 + n]^2$

i.e. $4\sum_{r=1}^{n} r^3 = (n^2 + n)^2$

Therefore;

$\sum_{r=1}^{n} r^3 = \frac{1}{4}[n^2 + n]^2$

To prove that

$\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2 + 3n - 1) = \frac{n}{30}[6n^4 + 15n^3 + 10n^2 - 1]$

we use $r^5 - (r-1)^5 = 5r^4 - 10r^3 + 10r^2 - 5r + 1$ to generate the following n equations (r =1, 2, ……..n)

$n^5 - (n-1)^5 = 5n^4 - 10n^3 + 10n^2 - 5n + 1$

$(n-1)^5 - (n-2)^5 = 5(n-1)^4 - 10(n-1)^3 + 10(n-1)^2 - 5(n-1) + 1$

$(n-2)^5 - (n-3)^5 = 5(n-2)^4 - 10(n-2)^3 + 10(n-2)^2 - 5(n-2) + 1$

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$1^5 - 0^5 = 5(1)^4 - 10(1)^3 + 10(1)^2 - 5(1) + 1$

Adding the last n equations, we have

$n^5 - 0 = 5\sum_{r=1}^{n} r^4 - 10\sum_{r=1}^{n} r^3 + 10\sum_{r=1}^{n} r^2 - 5\sum_{r=1}^{n} r + n$

i.e $\sum_{r=1}^{n} r^4 = \frac{1}{5}\{n^5 + 10\sum r^3 - 10\sum r^2 + 5\sum r - n\}$

$= \frac{1}{5}\left\{n^5 + \frac{10}{4}(n^2 + n^2)^2 - \frac{10n}{6}(n+1)(2n+1) + \frac{5n}{2}(n+1) - n\right\}$

$= \frac{1}{120}\{24n^5 + 60(n^4 + 2n^3 + n^2) - 40(2n^3 + n^2 + 2n^2 + n) + 60(n^2 + n) - 24n)$

$= \frac{1}{120}\{24n^5 + 60n^4 + 120n^3 + 60n^2 - 80n^3 - 120n^2 - 40n + 60n^2 + 60n - 24n)$

$= \frac{1}{120}\{24n^5 + 60n^4 + 40n^3 - 4n\}$

$= \frac{n}{30}\{6n^4 + 15n^3 + 10n^2 - 1\} = \frac{n}{30}(n+1)(2n+1)(3n^2 + 3n - 1)$

To prove that;

$\sum_{r=1}^{n} r^5 = \frac{n^2}{12}(n+1)^2(2n^2 + 2n - 1) = \frac{1}{12}(2n^6 + 6n^5 + 5n^4 - n^2)$

Actually $\sum r^5 = \frac{1}{12}[2n^6 + 6n^5 + 5n^4 - n^2]$

$= \frac{n^2(n+1)}{12} 2n^3 + 4n^2 + n - 1)$

$= \frac{1}{12} n^2(n+1)^2(2n^2 + 2n - 1)$

We use the equation $r^6 - (r-1)^6 = 6r^5 - 15r^4 + 20r^3 - 15r^2 + 6r - 1$ to generate the following n equations (r = 1, 2, …, n)

$n^6 - (n-1)^6 = 6n^5 - 15n^4 + 20n^3 - 15n^2 + 6n - 1$

$(n-1)^6 - (n-2)^6 = 6(n-1)^5 - 15(n-1)^4 + 20(n-1)^3 - 15(n-1)^2 + 6(n-1) - 1$

$(n-2)^6 - (n-3)^6 = 6(n-2)^5 - 15(n-2)^4 + 20(n-2)^3 - 15(n-2)^2 + 6(n-2) - 1$

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$(1)^6 - 0(1)^6 = 6(1)^5 - 15(1)^4 + 20(1)^3 - 15(1)^2 + 6(1) - 1$
Adding these n equations, we have;
$n^6 = 6\sum r^5 - 15\sum r^4 + 20\sum r^3 - 15\sum r^2 + 6\sum r - n$
Therefore, $\sum r^5 = \frac{1}{6}\{15\sum r^4 - 20\sum r^3 + 15\sum r^2 - 6\sum r + n^6 + n\}$
$= \frac{1}{6}\{\frac{15}{30}(6n^5 + 15n^4 + 10n^3 - n) - \frac{20}{4}(n^4 + 2n^3 + n^2) + \frac{15}{6}(2n^3 + 3n^2 + n) - \frac{6}{2}(n^2 + n) + n + n^6\}$
$= \frac{1}{12}(2n^6 + 6n^5 + 5n^4 - n^2)$
$= \frac{1}{12}n^2(n+1)^2(2n^2 + 2n - 1)$

To prove that $r^7 - (r-1)^7 = 7r^6 - 21r^5 + 35r^4 - 35r^3 + 21r^2 - 7r + 1$
$n^7 - (n-1)^7 = 7n^6 - 21n^5 + 35n^4 - 35n^3 + 21n^2 - 7n + 1$
$(n-1)^7 - (n-1)^7 = 7n^6 - 21n^5 + 35n^4 - 35n^3 + 21n^2 - 7n + 1$
$(n-2)^7 - (n-3)^7 = 7(n-2)^6 - 21(n-2)^5 + 35(n-2)^4 - 35(n-2)^3 + 21(n-2)^2 - 7(n-2) + 1$
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$(1)^7 - 0 = 7(1)^6 - 21(1)^5 + 35(1)^4 - 35(1)^3 + 21(1)^2 - 7(1) + 1$
Adding these equations, we have
$n^7 = 7\sum r^6 - 21\sum r^5 + 35\sum r^4 - 35\sum r^3 + 21\sum r^2 - 7\sum r - n$
Hence,
$\sum r^6 = \frac{1}{7}[n^7 + 21\sum r^5 - 35\sum r^4 + 35\sum r^3 - 21\sum r^2 + 7\sum r - n]$
$= \frac{1}{7}\{n^7 + \frac{21}{12}(2n^6 + 6n^5 + 5n^4 - n^2) - \frac{35}{30}(6n^5 + 15n^4 + 10n^3 - n) + \frac{35}{4}(n^4 + 2n^3 + n^2) - \frac{21}{6}(2n^3 +$
3n2+n + 72n2+n−n}

$= \frac{1}{168}\{24n^7 + 42(2n^6 + 6n^5 + 5n^4 - n^2) - 28(6n^5 + 15n^4 + 10n^3 - n) + 210(n^4 + 2n^3 + n^2) -$
$84(2n^3 + 3n^2 + n) + 84(n^2 + n) - 24n\}$
$= \frac{1}{168}\{24n^7 + 42(2n^6 + 6n^5 + 5n^4 - n^2) - 28(6n^5 + 15n^4 + 10n^3 - n) + 210(n^4 + 2n^3 + n^2) -$
$84(2n^3 + 3n^2 + n) + 84(n^2 + n) - 24n\}$
$= \frac{1}{168}\{24n^7 + 84n^6 + 252n^5 + 210n^4 - 42n^2 - 168n^5 - 420n^4 - 280n^3 + 28n + 210n^4 + 420n^3 +$
$210n^2 - 168n^3 - 252n^2 - 84n + 84n^2 + 84n - 24n\}$

$= \frac{1}{168}(24n^7 + 84n^6 + 84n^5 - 28n^3 + 4n)$
$= \frac{1}{42}(6n^7 + 21n^6 + 21n^5 - 7n^3 + n)$
$= \frac{n}{42}\{6n^6 + 21n^5 + 21n^4 - 7n^2 + 1\}$

To prove that $\sum_{r=1}^{n} r^7 = \frac{1}{24}\{3n^8 + 12n^7 + 14n^6 - 7n^4 + 44n^2 - 42n\}$
We use the equation
$r^8 - (r-1)^8 = 8r^7 - 28r^6 + 56r^5 - 70r^4 + 56r^3 - 28r^2 7r + 8r - 1$
and arguments as in previous summation to obtain the equation
$n^8 = 8\sum r^7 - 28\sum r^6 + 56\sum r^5 - 70\sum r^4 + 56\sum r^3 - 28\sum r^2 + 8\sum r - n$
On re-arranging the last equation, we have
$\sum_{r=1}^{n} r^7 = \frac{1}{8}[n^8 + 28\sum r^6 - 56\sum r^5 + 70\sum r^4 - 56\sum r^3 + 28\sum r^2 - 8\sum r + n]$
$= \frac{1}{8}\{n^8 + \frac{28}{42}(6n^7 + 21n^6 + 21n^5 - 7n^3 + n) - \frac{56}{12}(2n^6 + 6n^5 + 5n^4 - n^2) + \frac{70}{30}(6n^5 + 15n^4 + 10n^3 -$
n−564n4+2n3+n2+ 2862n3+3n2+n−82n2+n+n}

$$= \frac{1}{8}\{ n^8 + \frac{2}{3}(6n^7 + 21n^6 + 21n^5 - 7n^3 + n) - \frac{14}{3}(2n^6 + 6n^5 + 5n^4 - n^2) + \frac{7}{3}(6n^5 + 15n^4 + 10n^3 - n) - 14\,(n^4 + 2n^3 + n^2) + \frac{14}{3}(2n^3 + 3n^2 + n) - 4(n^2 + n + n\}$$

$$= \frac{1}{8}\{ 3n^8 + 12n^7 + 42n^6 + 42n^5 - 14n^3 + 2n - 28n^6 - 84n^5 - 70n^4 + 14n^2 + 42n^5 + 105n^4 + 70n^3 - 7n - 42n^4 - 84n^3 - 42n^2 + 28n^3 + 42n^2 + 14n - 12n^2 - 12n + 3n\}$$

$$= \frac{1}{24}(\, 3n^8 + 12n^7 + 14n^6 - 7n^4 + 2n^2)$$

$$= \frac{n^2}{24}(n+1)^2(3n^4 + 6n^3 - n^2 - 4n + 2)$$

IV. Results

The results for the first n natural numbers, first n squares, first n cubes, etc are as follows: (using their earlier numbering)

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1) = \frac{1}{2}(n^2 + n) \qquad (11)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}\{2n^3 + 3n^3 + n\} = \frac{n}{6}\{2n^2 + 3n + 1\} = \frac{n}{6}(n+1)(2n+1) \qquad (12)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}\{n^4 + 2n^3 + n^2\} = \frac{1}{4}\{n^2 + n)^2 = \frac{n^2}{4}\{n+1)^2 \qquad (13)$$

$$\sum_{r=1}^{n} r^4 = \frac{1}{30}\{6n^5 + 15n^4 + 10n^3 - n\}$$

$$= \frac{n}{30}\{6n^4 + 15n^3 + 10n^2 - 1\}$$

$$= \frac{n}{30}\{(n+1)(2n+1)(3n^2 + 3n - 1) \qquad (14)$$

$$\sum_{r=1}^{n} r^5 = \frac{1}{12}(2n^6 + 6n^5 + 5n^4 - n^2) = \frac{n^2}{12}(n+1)^2\,(2n^2 + 2n - 1) \qquad (15)$$

$$\sum_{r=1}^{n} r^6 = \frac{n}{42}\{6n^6 + 21n^5 + 21n^4 - 7n^2 + 1\}$$

$$= \frac{1}{42}(6n^7 + 21n^6 + 21n^5 - 7n^3 + n) \qquad (16)$$

$$\sum_{r=1}^{n} r^7 = \frac{1}{24}\{3n^8 + 12n^7 + 14n^6 - 7n^4 + 2n^2\}$$

$$= \frac{n^2}{24}(n+1)^2(3n^4 + 6n^3 - n^2 - 4n + 2)(17)$$

Reference

[1]. Youse , B.K**. (1970)*An Introduction to Mathematics*, Allyn and Bacon**

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