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Dyadic Rationals and Surreal Number Theory

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Abstract
We establish a number of properties of the dyadic rational numbers associated with surreal number theory. In particular, we show that a two parameter function of dyadic rationals can give all the trees of n-days in surreal number formalism.

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I. Introduction

In mathematics, from number theory history [1], one learns that historically, roughly speaking, the starting point was the natural numbers N and after a centuries of though evolution one ends up with the real numbers $\mathbb{R}$ from which one constructs the differential and integral calculus. Surprisingly in 1973 Conway [2] (see also Ref. [3]) developed the surreal numbers structure $\mathcal{S}$ which contains no only the real numbers $\mathbb{R}$, but also the hypereals and other numerical structures.
Consider the set

$$x = \{X_L \mid X_R\} \tag{1}$$

and call X_L and X_R the left and right sets of x, respectively. Surreal numbers are defined in terms of two axioms:

Axiom 1. Every surreal number corresponds to two sets X_L and X_R of previously created numbers, such that no member of the left set $x_L \in X_L$ is greater or equal to any member x_R of the right set X_R.

Let us denote by the symbol $\not\geq$ the notion of no greater or equal to. So the axiom establishes that if x is a surreal number then for each $x_L \in X_L$ and $x_R \in X_R$ one has $x_L \not\geq x_R$. This is
denoted by $X_L \not\geq X_R$.

Axiom 2. One number $x = \{X_L \mid X_R\}$ is less than or equal to another number $y = \{Y_L \mid Y_R\}$ if and only if the two conditions $X_L \ngeq y$ and $x \ngeq Y_R$ are satisfied.

This can be simplified by saying that $x \leq y$ if and only if $X_L \ngeq y$ and $x \ngeq Y_R$.

Observe that Conway definition relies in an inductive method; before a surreal number x is introduced one needs to know the two sets X_L and X_R of surreal numbers. Using Conway algorithm one finds that at the l_2-day one obtains $2^{l_2+1}-1$ numbers, all of which are of form

$$x = \frac{m}{2^n}, \tag{2}$$

where m is an integer and n is a natural number. Of course, the numbers (2) are dyadic rationals which are dense in the real $\mathbb{R}$. It is also possible to show that the real numbers $\mathbb{R}$ are contained in the surreals $\mathcal{S}$ (see Ref. [2,3] for details). Of course, in some sense the prove relies on the fact that the dyadic numbers (2) are dense in the real $\mathbb{R}$.

In 1986, Gonshor [4] introduced a different but equivalent definition of surreal numbers.

II. Dyadic numbers in the surreal number theory

As we mentioned earlier, in [4] Gonshor provided a surreal number definition equivalent to the one given by Conway; in this note we will work with the Gonshor's definition, so we begin by recalling it.

Definition 2.1 A *surreal number* is a function f from initial segment of the ordinals into the set $\{+,-\}$.

For instance, if f is the function so that $f(1)=+$, $f(2)=+$, $f(3)=-$, $f(4)=+$ then f is the surreal number $(++-+)$. In the Gonshor approach one obtains the sequence:

1-day

$$-1 = (-), \qquad (+) = +1, \tag{3}$$

in the 2-day

$$-2=(--), \quad -\frac{1}{2}=(-+), \quad (+-)=+\frac{1}{2}, \qquad (++)=+2, \tag{4}$$

and 3-day

$$\begin{gathered}-3=(---), \quad -\tfrac{3}{2}=(--+), \quad -\tfrac{3}{4}=(-+-), \quad -\tfrac{1}{4}=(-++), \\ (+--)=+\tfrac{1}{4}, \quad (+-+)=+\tfrac{3}{4}, \quad (++-)=+\tfrac{3}{2}, \qquad (+++)=+3,\end{gathered} \tag{5}$$

respectively.

Here, we would like to propose that the different dyadic numbers in the surreal number theory can be obtained from the two parameter function:

$$\mathcal{J}(l_1,l_2)=\left\{\begin{array}{ll}(I)\ l_1\varepsilon_0, & \text{if } l_2-l_1=0, \\ (II)\ l_1\varepsilon_0-\frac{\varepsilon_1}{2}, & \text{if } l_2-l_1=1 \\ (III)\ l_1\varepsilon_0-\frac{\varepsilon_1}{2}\pm\sum\limits_{k=1}^{l_2-(l_1+1)}\frac{\varepsilon_{k+1}}{2^{k+1}}\ , & \text{if } l_2-l_1>1\end{array}\right\}. \tag{6}$$

Here, $l_1,l_2\in\{1,2,...\}$, $\varepsilon_0,\varepsilon_1,\varepsilon_2,...,\varepsilon_p\in\{+,-\}$ and $\varepsilon_0\neq\varepsilon_1$. The positive sector of (6), with $\varepsilon_0=+1$ and therefore $\varepsilon_1=-1$, becomes

$$\mathcal{J}_{(+)}(l_1,l_2)=\left\{\begin{array}{ll}(I)\ l_1, & \text{if } l_2-l_1=0, \\ (II)\ l_1-\frac{1}{2}, & \text{if } l_2-l_1=1 \\ (III)\ l_1-\frac{1}{2}\pm\sum\limits_{k=1}^{l_2-(l_1+1)}\frac{1}{2^{k+1}}\ , & \text{if } l_2-l_1>1\end{array}\right\}, \tag{7}$$

while the negative sector, with $\varepsilon_0=-1$ and therefore $\varepsilon_1=+1$, is given by

$$\mathcal{J}_{(-)}(l_1,l_2)=\left\{\begin{array}{ll}(I)\ -l_1, & \text{if } l_2-l_1=0, \\ (II)\ -l_1+\frac{1}{2}, & \text{if } l_2-l_1=1, \\ (III)\ -l_1+\frac{1}{2}\mp\sum\limits_{k=1}^{l_2-(l_1+1)}\frac{1}{2^{k+1}}\ , & \text{if } l_2-l_1>1.\end{array}\right\}. \tag{8}$$

Observe that

$$\mathcal{J}_{(-)}(l_1, l_2) = -\mathcal{J}_{(+)}(l_1, l_2). \tag{9}$$

Moreover, it is worth mentioning that Gonshor [4] derived the formula

$$\acute{\mathcal{J}} = l_1 \varepsilon_0 + \frac{\varepsilon_1}{2} + \sum_{i=2}^{l_2-(l_1+1)} \frac{\varepsilon_i}{2^i}, \tag{10}$$

which corresponds to (III) in (6).

Example 2.2 Let us consider the Gonshor surreal number $(++-+-+)$. One gets

$$(++-+-+) = 2 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = \frac{27}{16}. \tag{11}$$

By defining the order $x < y$ if $x(\alpha) < y(\alpha)$, where α is the first place where x and y differ and the convention $- < 0 < +$, it is possible to show that the Conway and Gonshor definitions of surreal numbers are equivalent (see Ref. [4] for details).

Let us focus in (7) with $\varepsilon_0 = +$ and therefore $\varepsilon_1 = -$. Also, we write ε_i explicitly as $\pm$. Notice that according to (6) one always has $l_2 - l_1 > 0$. The first thing that one observe is that (I) in (7) and (8) gives the integer numbers $\mathbb{Z}_{(+)}$ and $\mathbb{Z}_{(-)}$, respectively. By completeness one sets $\mathcal{J}_{(\pm)}(0,0) = 0$. While (II) provides with the dyadic rationals $\frac{m}{2}$ where m is an odd element in the rationals $\mathbb{Q}_{(+)}$ and $\mathbb{Q}_{(-)}$. So, this suggests that both integer $\mathbb{Z}$ and rational $\mathbb{Q}$ numbers are contained in the surreal numbers $\mathcal{S}$.

Assume $l_1 = 1$. In this case (7) becomes

$$\mathcal{J}_{(+)}(1, l_2) = \left\{ \begin{array}{ll} (I)\ 1, & \text{if } l_2 = 1, \\ (II)\ \frac{1}{2}, & \text{if } l_2 = 2 \\ (III)\ \frac{1}{2} \pm \sum_{k=1}^{l_2-2} \frac{1}{2^{k+1}}\ , & \text{if } l_2 > 2 \end{array} \right\}. \tag{12}$$

This implies that from (I) and (II) one gets $\mathcal{J}_{(+)}(1,1) = 1$, $\mathcal{J}_{(+)}(1,2) = \frac{1}{2}$ and from (III) one obtains $\mathcal{J}_{(+)}(1,3) = \{\frac{1}{4}, \frac{3}{4}\}$, $\mathcal{J}_{(+)}(1,4) = \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$ and so on. Since

$\mathcal{J}_{(-)}(l_1,l_2) = -\mathcal{J}_{(+)}(l_1,l_2)$ one also has $\mathcal{J}_{(-)}(1,1) = -1$, $\mathcal{J}_{(-)}(1,2) = -\frac{1}{2}$ and $\mathcal{J}_{(-)}(1,3) = \{-\frac{1}{4}, -\frac{3}{4}\}$, $\mathcal{J}_{(+)}(1,4) = \{-\frac{1}{8}, -\frac{3}{8}, -\frac{5}{8}, -\frac{7}{8}\}$ and so on.

There must be many interesting combinations between (12) and (7) (and (8), but perhaps one of the most attractive is

Proposition 2.3 *The functions* $\mathcal{J}_{(+)}(l_1,l_2)$ *and* $\mathcal{J}_{(+)}(1,l_2)$ *are related by*

$$\mathcal{J}_{(+)}(l_1,l_2) = \mathcal{J}_{(+)}(1,l_2) + (l_1 - 1). \tag{13}$$

This means that the tree $\mathcal{J}_{(+)}(1,l_2)$ (and $\mathcal{J}_{(-)}(1,l_2)$) plays the role of a main building block; any other tree $\mathcal{J}_{(+)}(l_1,l_2)$ with $l_1 > 1$ can be obtained from (13). Surprisingly, $\mathcal{J}_{(+)}(1,l_2)$ has been studied in the context of Zeno algorithm [5], Tompson group [6], Minkowski's question mark function [7] among others. In some sense if $\mathcal{J}_{(-)}(1,l_2)$ were added to $\mathcal{J}_{(+)}(1,l_2)$ and (3) was used the surreal numbers terms of dyadic rationals could be discovered for another routes, different than game theory [2].

Another interesting aspect of the tree structure $\mathcal{J}_{(+)}(l_1,l_2)$, $\mathcal{J}_{(-)}(l_1,l_2)$ and $\mathcal{J}_{(\pm)}(0,0)$ is that one can derive, in an alternative way, how many numbers are created in the l_2-day. It is worth to mention that this notion of "day" is used by the mathematicians, in spire of their development of surreal numbers theory is considered only in the mathematical context. First, let us use Gonshor formalism to answer this question. In the 0-day one starts with the number 0 and in the 1-day the numbers -1 and $+1$ are created, namely $(-)$ and $(+)$. While in the 2-day 4 numbers are created, namely $(++) = 2$, $(+-) = \frac{1}{2}$, $(--) = -2$, $(-+) = -\frac{1}{2}$, and so on.

First, we shall need the proposition

Proposition 2.4 *The identity*

$$2^{l_2+1} = 2 + 2 + 4 + 8 + 16 + ... + 2^{l_2}, \tag{14}$$

holds.

Proof. By induction one assumes that (14) holds for an integer l_2 and proves that also holds for $l_2 + 1$. Thus, one needs to prove that

$$2^{l_2+2} = 2 + 2 + 4 + 8 + 16 + ... + 2^{l_2+1} \tag{15}$$

is true. But (15) implies that

$$2^{l_2+2} = 2 + 2 + 4 + 8 + 16 + ... + 2^{l_2} + 2^{l_2+1}. \tag{16}$$

For assumption (14) holds and therefore (16) becomes

$$2^{l_2+2} = 2^{l_2+1} + 2^{l_2+1}, \tag{17}$$

which is an identity. □

Proposition 2.5 *The total number of surreal numbers created at the l_2-day are*

$$t = 2^{l_2+1} - 1. \tag{18}$$

Proof.
The series $1 + 2(1 + 2 + 4 + 8 + ... + 2^{l_2-1})$ determines t. So, from the identity (14) one sees that (18) holds. □

Remark 2.6 Since the function $\mathcal{J}_{(\pm)}(l_1, l_2)$ is two a parameter function l_1 and l_2, if one fixes l_1 and change l_2 one moves vertically producing the corresponding tree, as $\mathcal{J}_{(\pm)}(1, l_2)$. While if one fixes l_2 and change l_1 one is moving horizontally. In this sense l_2 determines the day parameter used by the mathematician.

Example 2.7 Let us set $l_2 = 3$. From (7) one obtains $\mathcal{J}_{(+)}(3, 3) = 3$, $\mathcal{J}_{(+)}(2, 3) = \frac{3}{2}$, $\mathcal{J}_{(+)}(1, 3) = \{\frac{1}{4}, \frac{3}{4}\}$, and the corresponding negatives. So, in the 3-day we have 8 numbers and so one discovers the series $t = 1 + 2 + 4 + 8... + 2^{l_2}$ which is what one obtains with Gonshor approach.

The ω-day is defined as the limit when the surreal numbers reproduce the real numbers.

Proposition 2.8 *In the ω-day the tree $\mathcal{J}_{(\pm)}(1,l_2)$ take values in the interval*

$$-1 \leq \mathcal{J}_{(\pm)}(1,l_2) \leq 1, \tag{19}$$

over the real $\mathbb{R}$.

Proof. Let us first proof that $0 \leq \mathcal{J}_{(+)}(1,l_2) \leq 1$. From III in (7), one has

$$\mathcal{J}_{(+)}(1,l_2) = \frac{1}{2} \pm \frac{1}{2^2} \pm ... \pm \frac{1}{2^n}, \tag{20}$$

with $n = l_2 - 2$ and $l_2 > 2$. The maximum of (20) is obtained when one takes only the positive number value in each term. In this case (20) becomes

$$\begin{aligned} &\mathcal{J}_{(++)}(1,l_2) \equiv \tfrac{1}{2} + \tfrac{1}{2^2} + ... + \tfrac{1}{2^n} \\ &= \tfrac{1}{2^n}(2^{n-1} + 2^{n-2} + ... + 2 + 1) \qquad , \\ &= \tfrac{1}{2^n}(2^{n-1} + 2^{n-2} + ... + 2 + 2 - 1) \end{aligned} \tag{21}$$

which leads to

$$\mathcal{J}_{(++)}(1,l_2) = \frac{1}{2^n}(2^n - 1) = 1 - \frac{1}{2^n}. \tag{22}$$

But, since $\frac{1}{2^n} \to 0$ when $n \to \infty$, one has $\mathcal{J}_{(++)}(1,l_2) \to 1$.

Now the minimum value of $\mathcal{J}_{(+)}(1,l_2)$ is obtained when one takes all the negative values (20). So, one has

$$\begin{aligned} &\mathcal{J}_{(+-)}(1,l_2) = \tfrac{1}{2} - \tfrac{1}{2^2} - ... - \tfrac{1}{2^n} \\ &= \tfrac{1}{2^n}(2^{n-1} - 2^{n-2} - ... - 2 - 1) \\ &= \tfrac{1}{2^n}(2^{n-1} - 2^{n-2} - ... - 2 - 2 + 1). \end{aligned} \tag{23}$$

This implies

$$\mathcal{J}_{(+-)}(1,l_2) = \frac{1}{2^n}(2^{n-1} - 2^{n-1} + 1) = \frac{1}{2^n}. \tag{24}$$

This means that $\mathcal{J}_{(+-)}(1, l_2) \to 0$ when $n \to \infty$. Since $\mathcal{J}_{(-)}(1, l_2) = -\mathcal{J}_{(+)}(1, l_2)$ one sees that the proposition is verified. $\square$

Corollary 2.9 *In the ω-day the function $\mathcal{J}_{(\pm)}(l_1, l_2)$ takes values in the interval*

$$-l_1 \leq \mathcal{J}_{(\pm)}(l_1, l_2) \leq l_1 \tag{25}$$

over the real $\mathbb{R}$.

Proof. According to (13) one has $\mathcal{J}_{(+)}(l_1, l_2) = \mathcal{J}_{(+)}(1, l_2) + (l_1 - 1)$. Since $0 \leq \mathcal{J}_{(+)}(1, l_2)$ and $0 \leq (l_1 - 1)$ one sees that $0 \leq \mathcal{J}_{(+)}(l_1, l_2)$. On the other hand, since $\mathcal{J}_{(+)}(1, l_2) \leq 1$ one learns that $\mathcal{J}_{(+)}(1, l_2) + (l_1 - 1) \leq 1 + (l_1 - 1)$ and therefore $\mathcal{J}_{(+)}(1, l_2) + (l_1 - 1) \leq l_1$ which means that $\mathcal{J}_{(+)}(l_1, l_2) \leq l_1$. This prove that $0 \leq \mathcal{J}_{(+)}(l_1, l_2) \leq l_1$. The other part of the proof follows from the fact that $\mathcal{J}_{(-)}(l_1, l_2) = -\mathcal{J}_{(+)}(l_1, l_2)$. $\square$

A connection between oriented matroid theory [8] (see also Refs. [9]-[15] and references therein) and surreal number theory has been developed [16]. So, one may expect that, in the context of surreal number theory, the mathematical notions of this article may be useful for further developing of oriented matroid theory .

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