IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 5 Ser. V (Sep. – Oct. 2020), PP 07-10
www.iosrjournals.org

Moving Space Curve in Minkowski 3-Space and Soliton Equations

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Abstract:
In this paper we introduce a relationship between curve evolution and the soliton equations in Minkowski 3-space in case of space-like curve with space-like principal normal.
MSC: 53A05, 53A17.
***Key Word**: Moving space curve; Minkowski 3-space; Soliton equations.*

Date of Submission: 15-10-2020 Date of Acceptance: 31-10-2020

I. Introduction

The study of possible links between intrinsic kinematics of space curves [23] and integrablesoliton bearing equations [1] deserves attention because of a wide variety of applications of moving curves such as vortex filament motion in fluids [13]. dynamics of continuum spin chain [17], interface dynamics [11], etc. The pioneering work by Hasimoto [13] on the motion of a vortex filament in a fluid was the first to suggest such a link [3].

After the work of Hasimoto, several authors [4,5,10,15,16,17,18, 20,21, 22, 25] studied the connection between thy integrable nonlinear Schrodinger equation and non-stretching vortex filament equation. Ding and Inoguchi also presented this connection in Minkowski 3-space [6,7,8,12].

The present work is aimed to study the relationship between moving space curve in Minkowski 3-space in case of space-like curve and soliton equations. The paper is organized as follows: In section 2 we discuss the basic geometry of a curve in Minkowski 3-space and introduce the Frenet-Serret equations which describe it. In section 3 we derive the relationship between curve evolution and soliton equations in Minkowski 3-space in case of space-like curve with space-like principal normal.

II. Preliminaries

The Minkowski 3-space$E_1^3$is the Euclidean 3-space E^3 provided with the standard flat metric given by

$$ds^2 = -dx_1^2 + dx_2^2 + dx_3^2,$$

where(x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since ds^2 *is* an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian causal characters: it can be space-like vector if $ds^2\ (u, v\ ) > 0\ or\ v = 0$, timelike vector if $ds^2(u, v) < 0$ and null (light-like) vector if $ds^2(u, v) = 0$and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)\ in\ E_1^3$ can locally be space-like curve, time-like curve or null (light-like)curve, if all of its velocity vectors α' (s)are respectively space-like, time-like or null (light-like) vectors. Denote by $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ the moving Frenet frame along the curve α (s)in the space E_1^3 [9, 14, 24].

III. Moving space-like curves with a space-like normal

A space curve embedded in three-dimensions may be described using the usual Frenet-Serret equations. In the case of space-like curve with a space-like principal normal, the usual Frenet equations read as the following [14]:

$$\boldsymbol{t_s} = \kappa\boldsymbol{n}, \qquad \boldsymbol{n_s} = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}, \qquad \boldsymbol{b_s} = \tau\boldsymbol{n}(3.1)$$

where$\kappa, \tau$and $s$are curvature, torsion and arc-length of $\alpha\ (s)$and

$$\langle t, t\rangle = \langle n, n\rangle = -\langle b, b\rangle = 1, \qquad \langle t, n\rangle = \langle n, b\rangle = -\langle b, t\rangle = 0$$

To describe the time evolution of the triad $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$, from above relations we have the following set of equations:

$$\boldsymbol{t_u} = g\boldsymbol{n} + h\boldsymbol{b}, \quad \boldsymbol{n_u} = -g\boldsymbol{t} + f\boldsymbol{b}, \quad \boldsymbol{b_u} = h\,\boldsymbol{t} + f\boldsymbol{n}, \qquad (3.2)$$

where the functions $g = g(s, u),\ h = h(s, u)$ and $f = f(s, u)$determine the motion of the curve with respect to time u. In the case of non-stretching curves, requiring that the unit traid satisfy the compatibility conditions:

$$\boldsymbol{t}_{su} = \boldsymbol{t}_{us}, \qquad \boldsymbol{n}_{su} = \boldsymbol{n}_{us}, \qquad \boldsymbol{b}_{su} = \boldsymbol{b}_{us}. \qquad (3.3)$$

By substituting in the compatibility conditions (3.3) from (3.1) and (3.2), we obtain the following relations:

$$\kappa_u = g_s + h\tau, \qquad \tau_u = \kappa h + f_s, \qquad h_s = \kappa f - gh. \tag{3.4}$$

Now, we will study three formulations to the relation between moving space-like curve with space-like normal and soliton equations as follows:

Formulation (3.1)

The second and third equations of the set (3.1) are combined to yield

$$(\boldsymbol{n} + \boldsymbol{b})_s - \tau(\boldsymbol{n} + \boldsymbol{b}) = -\kappa\, \boldsymbol{t}, \qquad (\boldsymbol{n} - \boldsymbol{b})_s + \tau(\boldsymbol{n} - \boldsymbol{b}) = -\kappa\, \boldsymbol{t}\,. \tag{3.5}$$

This immediately suggests the definition of certain complex vectors

$$N_1 = \frac{1}{\sqrt{2}}(n+b)\exp\left(-\int \tau\, ds\right) N_2 = \frac{1}{\sqrt{2}}(n-b)\exp\left(\int \tau\, ds\right) \tag{3.6}$$

Then we have a new frame $\{\boldsymbol{t}, N_1, N_2\}$ satisfying the following conditions:

$$\langle t,t\rangle = \langle N_1, N_2\rangle = 1, \qquad \langle t, N_1\rangle = \langle t, N_2\rangle = \langle N_1, N_1\rangle = \langle N_2, N_2\rangle = 0$$

Differentiating equation (3.6)with respect to s and using equation (3.5), we will get:

$$N_{1s} = -\frac{1}{\sqrt{2}}\kappa \exp\left(-\int \tau\, ds\right) t, \qquad N_{2s} = -\frac{1}{\sqrt{2}}\kappa\exp\left(\int \tau\, ds\right). \tag{3.7}$$

Thus the functions Ψ_1 and Ψ_2 appear in a natural fashion in the above equations

$$\Psi_1 = \frac{1}{\sqrt{2}}\kappa\exp\left(-\int \tau\, ds\right)\Psi_2 = \frac{1}{\sqrt{2}}\kappa\exp\left(\int \tau\, ds\right). \tag{3.8}$$

By using the definitions of $N_1$and N_2 in (3.6), we get the following system

$$\begin{cases} t_s = \Psi_2 N_1 + \Psi_1 N_2, & N_{1s} = -\Psi_1 t, \qquad N_{2s} = -\Psi_2 t, \\ t_u = \gamma_2 N_1 + \gamma_1 N_2, & N_{1u} = -\gamma_1 t + R_1 N_1, \qquad N_{2u} = -\gamma_2 t - R_1 N_2, \end{cases} \tag{3.9}$$

where

$$\gamma_1 = \frac{1}{\sqrt{2}}(g-h)\exp\left(-\int \tau\, ds\right), \qquad \gamma_2 = \frac{1}{\sqrt{2}}(g-h)\exp\left(\int \tau\, ds\right), \quad R_1 = f - \int \tau_u ds$$

On imposing the compatibility conditions $\boldsymbol{t}_{su} = \boldsymbol{t}_{us}$, $N_{1su} = N_{1us}$, $N_{2su} = N_{2us}$ and using (3.9), we obtain

$$\Psi_{1u} - \gamma_{1s} - R_1\Psi_1 = 0, \qquad \Psi_{2u} - \gamma_{2s} + R_2\Psi_2 = 0, \qquad R_{1s} = \gamma_1\Psi_2 - \gamma_2\Psi_1 \tag{3.10}$$

Or

$$\Psi_u - \gamma_s - R_1\Psi^* = 0, \qquad R_{1s} = -\frac{i}{\sqrt{2}}(\gamma\Psi^* - \gamma\Psi^*) \tag{3.11}$$

where $\Psi = \Psi_1 + i\Psi_2$, $\gamma = \gamma_1 + i\gamma_2$, $\Psi^* = \Psi_1 - i\Psi_2$ and $\gamma^* = \gamma_1 - i\gamma_2$. the two equations (3.11) are the system of soliton equations because it is an integrable system (of soliton type) [19] corresponding to moving space-like curve with space-like normal in formulation (I).

It is worth noting that: as noted by Lamb [18], the structure of the two Equations in (3.11) which arose from compatibility conditions on curve evolution suggests a possible relationship with soliton-bearing equations, via the Ablowitz-Kaup-Newel1-Segur (AKNS) formalism [1,2]. In other wording: If we put $\Psi_1 = r$, $\Psi_2 = q$, $\gamma_1 = -r_s$, $\gamma_2 = q_s$and $R_1 = -R$where $r = r(s,u)$, $q = q(s,u)$and $R = R(s,u)$are real functions of the variables $s$and u, we get the following soliton equations:

$$r_u + r_{ss} + rR = 0, \quad q_t - q_{ss} - qR = 0, \quad R_s = rq_s + r_s q, \tag{3.12}$$

which are introduced by Ding and Inoguchi in [8]. It is worth noting that: the third equation of (3.12) implies that Rhas the form $R(s,u) = rq + R_0(u)$, where $R_0(u)$is a function depending only on u. Then under the transformations $q \mapsto q\, exp\left(\int R_0 du\right), r \mapsto r exp\left(-\int R_0 du\right)$,we obtain

$$q_u = q_{ss} + q^2 r, \quad r_u = -r_{ss} - r^2 q \tag{3.13}$$

This system is just the second AKNS hierarchies of real type (2) by a scaling transformation

$$q \to \sqrt{2}\, q \text{ and } r \to \sqrt{2}\, r.$$

Formulation (3.2)

Combining the first and second equations of the set (3.1), we get

$$(\boldsymbol{n} + i\,\boldsymbol{t})_s - i\kappa(\boldsymbol{n} + i\,\boldsymbol{t}) = \tau\boldsymbol{b}. \tag{3.14}$$

The above equation suggests the definition of second complex vector

$$M = \frac{1}{\sqrt{2}}(n + it)\exp\left(-i\int \kappa ds\right) \tag{3.15}$$

Then we have the new moving curve $\{\boldsymbol{b}, M, M^*\}$ satisfy

$$-\langle b,b\rangle = \langle M, M^*\rangle = 1, \qquad \langle b, M\rangle = \langle b, M^*\rangle = \langle M, M\rangle = \langle M^*, M^*\rangle = 0$$

Differentiating equation (3.15) with respect to s and using equation (3.14), we get:

$$M_s = \frac{1}{\sqrt{2}}\tau \exp\left(-i\int \kappa\, ds\right) b \qquad (3.16)$$

Thus the functions Φ appears in a natural fashion in the above equation.

$$\Phi = \frac{1}{\sqrt{2}}\tau \exp\left(-i\int \kappa\, ds\right) \qquad (3.17)$$

By using the definitions of M, we can get the following system of differential equations

$$\begin{cases} b_s = \Phi^* M + \Phi M^*, & M_s = \Phi \mathrm{b}, \\ b_s = \beta^* M + \beta M^*, & M_u = -\mathrm{i}R_2 M + \beta b, \end{cases} \qquad (3.18)$$

where

$$\beta = \frac{1}{\sqrt{2}}(f + ih)\exp\left(-i\int \kappa ds\right), \quad R_2 = \int \kappa_u ds - g.$$

Now, from compatibility condition $M_{su} = M_{us}$ and equating the coefficients of $\boldsymbol{b}, M$ and M^* we get

$$\Phi_u - \beta_s + iR_2\Phi = 0 \qquad R_{2s} = i(\beta^*\Phi - \beta\Phi^*) (3.19)$$

The system of equations (3.19) are the AKNS-heirarchy which is known to be a universal model in integrable systems since almost all the famous equations coming from varied physical backgrounds, such as the NLS, KdV, mKdV and so on, belong to this hierarchy [2].

Formulation (3.3)

From the first and third equations of (3.1), we get

$$(\boldsymbol{t} + i\boldsymbol{b})_s = (\kappa + i\tau)\boldsymbol{n}. \qquad (3.20)$$

This suggests the definition of a third complex vector

$$P = \frac{1}{\sqrt{2}}(\boldsymbol{t} + i\boldsymbol{b}). \qquad (3.21)$$

Then we have the new moving curve $\{n, p, p^*\}$satisfy

$$\langle n,n\rangle = \langle P,P\rangle = \langle P^*,P^*\rangle = 1 \qquad \langle n,P\rangle = \langle n,P^*\rangle = \langle P,P^*\rangle = 0.$$

Differentiating equation (3.21) with respect to $s$and using equation (3.20), we get:

$$P_s = \frac{1}{\sqrt{2}}(\kappa + i\tau)\,\boldsymbol{n} \qquad (3.22)$$

If we put $\chi = \frac{1}{\sqrt{2}}(\kappa - i\tau)$ we can get the following system

$$\begin{cases} n_s = -(\chi^* P + \chi P^*), & P_s = \chi^* n, \\ n_u = -(\gamma_3^* P + \gamma_3 P^*), & P_u = iR_3 P^* + \gamma_3^* n, \end{cases} \qquad (3.23)$$

where$\gamma_3 = \frac{1}{\sqrt{2}}(g - if)$and $R_3 = h$. Then from equations (3.23). we can obtain

$$\chi_u - \gamma_{3s} + iR_3\chi^* = 0, \qquad iR_{3s} = (\gamma_3^*\chi - \gamma_3\chi^*) \qquad (3.24)$$

as an associated integrable system by the AKNS formulation.

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