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Approximating Second-Order Linear Dirichlet and Neumann Boundary-Value Problems in Ordinary Differential Equations by Laguerre Collocation Method

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Abstract: *Time and again, systems described by differential equations are so complex that purely analytical solutions of the equations are not very easy to come by. Therefore, in this paper, we develop a collocation method using Laguerre polynomials as basis function to approximate two-point second-order linear boundary value problems with Dirichlet and Neumann boundary conditions in ordinary differential equations. The collocation method developed is implemented in MAPLE 17 in conjunction with MATLAB R2014a through six illustrative examples. Absolute errors are equally estimated. From the result, we observed that the accuracy of the collocation method constructed increases with the use of more terms of the Laguerre polynomials as basis function. Based on the careful observations from the numerical experiment, it may be concluded here that the collocation method developed is more efficient, effective and applicable in terms of accuracy for approximating boundary value problems with Dirichlet boundary condition. Therefore, this method is highly recommended as a way of application for approximating many models in sciences and engineering that appear in form of second order boundary value problems with Dirichlet boundary condition as well as Neumann boundary condition.*

Key words: *Linear Boundary Value Problems, Collocation, Laguerre polynomials.*

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I. Introduction

Fundamentally, all systems that undergo change can be described by differential equations. Therefore, [15] assert that Ordinary Differential Equations (ODEs) of the Initial Value Problem (IVP) or Boundary Value Problems (BVPs) type can model phenomena in wide range of fields including science, engineering, economics, social science, biology, business, healthcare among others.To be more specific, [7],[23]and [22] opine that Boundary Value Problems (BVPs) in Ordinary Differential Equations (ODEs) are used to model many physical phenomena in engineering, sciences especially physics and other related areas such asbiology, spring problem, buoyance problem, electrical problem, boundary layer theory, astronomy, heat transfer, Sturm-Liouville problem, diffusion process, electromagnetism as well as deflection in cables.Moreover, Boundary Value Problems play an important role in many fields such as physics, chemistry and engineering. The Two-point Boundary Value Problems occur in a variety of problems, including the modelling of chemical reaction, heat transfer, diffusion, and the solution of optimal control problems ([9]).

Time and again, systems described by differential equations are so complex that purely analytical solutions of the equations are not very easy to come by. Consequent upon this,numerical techniques for solving differential equations form the nucleus of concern.

There are several types of boundary value problems (BVPs) and some of them depend on the boundary condition itself ([13]; [9]). In this work, we consider the following second order linear two-point boundary value problems:

$$P(x)y''(x) + Q(x)y'(x) + R(x)y = G(x), \quad [a,b] \tag{1}$$

with the Dirichlet boundary condition:

$$y(a) = y_a, \quad y(b) = y_b; \tag{2}$$

and Neumann boundary condition:

$$y'(a) = y_a, \quad y'(b) = y_b \tag{3}$$

.

Some of the most prominentmethods for solving boundary value problems are given in the works of[12],[5], [2], [24], [9], in the papers of [13], [19], [20], [11],[18], [14], as well as in the publications of [8], [3],[16], [22] and [21].

[19] in [12] summarised the above prominent methods of solving boundary value problems into four traditional methods, namely: finite difference method, shooting method, collocation method and finite element method. In this paper, we developeda collocation technique using Laguerre polynomial as basis function and applied it to second-order Dirichlet and Neumann boundary-value problems of ordinary differential equations. In the words of [12], “Collocation method is a method which involves the determination of an approximate solution to an equation using a suitable set of functions, sometimes called trial or basis functions. The approximate solution is required to satisfy the governing equation and its supplementary conditions at certain points in the range of interest called collocation points.”

Both [3] and [19] stated that monomial and polynomial functions as well as spline function among others may be used to develop a collocation method. Nevertheless, [19] unequivocally encouraged the use of orthogonal polynomials as basis functions since polynomial functions are vulnerable to Runge phenomenon and monomial elements as non-orthogonal functions can make the coefficient matrix of the linear equations ill-conditioned.Buttressing[19], [17]clearly opined that “Many scientists over the years have given special attention to applications of orthogonal polynomials because its important role played in different fields of human endeavour. These orthogonal polynomial include Laguerre polynomials, Legendre polynomial, Hermite polynomial, Chebyshev polynomial among others. These polynomial series deal with various problems in engineering and science. They are used in solving systems of ordinary differential equations with boundary conditions to obtain very accurate approximations. The main characteristic of these applications is that they reduce these problems to those of solving a system of algebraic equation by greatly simplifying the problem”.

Every so often, researchers have applied orthogonal polynomials as basis functions for developing approximate methods to solve different forms of ordinary differential equations. In the light of this, [4] and [25] used Chebyshev polynomials and Legendre polynomials respectively as basis functions to develop collocation methods for approximating ordinary differential equations with accurate numerical solutions. [1] used Hermite polynomials to develop continuous linear multistep methods for approximating initial value problem of ordinary differential equations. In a related development, [12] used the Probabilist’s Hermite polynomials of degree eight (8) as basis function to construct a collocation method for approximating second order linear boundary value problems of ordinary differential equations with Dirichlet, Neumann and Robin boundary conditions.In another development, [10] used Laguerre polynomials as basis function to construct a collocation method called “Continuous Implicit Linear Multistep Methods for the Solution of Initial Value Problems of First-Order Ordinary Differential Equations”.

From the foregoing and other available literature, we are made to understand that orthogonal polynomials have been widely used effectively as basis functions to construct so many numerical methods for the approximations of initial value and boundary value problems of ordinary differential equations, even partial differential equations as seen in the work of [16]where the solution of second order partial differential equation using the Hermite polynomials as basis functions was approximated. Therefore, in this paper, the researchers develop a collocation technique using Laguerre polynomials which are orthogonal polynomials as basisfunctions for approximating second-order linear Dirichlet and Neumann boundary-value problems of ordinary differential equations.

II. Formulation of Method

Laguerre polynomial is used as basis function to construct a collocation technique for approximating second-order linear Dirichlet and Neumann boundary-value problems of ordinary differential equations in this section. The formulation of the method is partly based on the procedure in [12].

2.1 Second Order Boundary Value Problems (BVPs)

In this Section, we shall consider Equations (1) – (3) in which we assume x and y to denote the independent and dependent variables respectively.

2.2 Laguerre Polynomials

In Mathematics, the Laguerre polynomials:

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(k!)^2 (n-k)!} x^k \quad (4)$$

named after Edmond Laguerre (1834-1886) are solutions of Laguerre’s equation:

$$xy'' + (1-x)y' + ny = 0 \qquad (5)$$

which is a second-order linear differential equation. The first eleven terms of Laguerre polynomials can be generated from Equation (4) and presented as follows:

$$\left.\begin{aligned}
L_0(x) &= 1\\
L_1(x) &= 1 - x\\
L_2(x) &= 1 - 2x + \frac{1}{2}x^2\\
L_3(x) &= 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3\\
L_4(x) &= 1 - 4x + 3x^2 - \frac{2}{3}x^3 + \frac{1}{24}x^4\\
L_5(x) &= 1 - 5x + 5x^2 - \frac{5}{3}x^3 + \frac{5}{24}x^4 - \frac{1}{120}x^5\\
L_6(x) &= 1 - 6x + \frac{15}{2}x^2 - \frac{10}{3}x^3 + \frac{5}{8}x^4 - \frac{1}{20}x^5 + \frac{1}{720}x^6\\
L_7(x) &= 1 - 7x + \frac{21}{2}x^2 - \frac{35}{6}x^3 + \frac{35}{24}x^4 - \frac{7}{40}x^5 + \frac{7}{720}x^6 - \frac{1}{5040}x^7\\
L_8(x) &= 1 - 8x + 14x^2 - \frac{28}{3}x^3 + \frac{35}{12}x^4 - \frac{7}{40}x^5 + \frac{7}{180}x^6 - \frac{1}{630}x^7 + \frac{1}{40320}x^8
\end{aligned}\right\} \qquad (6)$$

2.3Collocation Method for Approximating Boundary Value Problems

The general idea behind collocation method is to reduce a boundary value problem to a set of solvable algebraic equations ([12]). Here, we choose$\varphi_1(x), \ \ldots, \ \varphi_N(x)$ as the set of polynomial basis functions to obtain approximate solution. Next, to solve a boundary value problem using a collocation method, we consider the possible solution to the boundary value problem in Equations (1) – (3) to be:

$$y(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + \ \ldots + a_N\varphi_N(x) \qquad (7)$$

or

$$y(x) = \sum_{j=1}^{N} a_j\varphi_j(x) \qquad (8)$$

where N is the number of terms of a basis or trial function ([12]).

2.3.1 Collocation Method using Laguerre Polynomials as Basis Functions

Here, we wish to approximate the possible solution $y(x)$in Equation (8) by Laguerre polynomial of degree $n$with the formula:

$$y(x) = \sum_{j=1}^{n+1} s_j L_{j-1}(x) \qquad (9)$$

where $L_{j-1}(x)$ are Laguerre polynomials generated by Equation (4) and$n+1$ is the truncated number of terms of the Laguerre polynomials which is equivalent to a polynomial of degree n.

The first nine (9) terms of $L_n(x)$ are given in Equation (6). Applying equation (9) to approximate a boundary value problem in Equations (1) – (3), the first and the last terms corresponding to $j = 1$ and $j = n + 1$respectively are boundary conditions; the remaining $n-1$ equations are obtained from the given equation by differentiating Equation (9) the required number of times and then evaluating x_i for $2 \le i \le n$, where x_i is defined similar to[12] as:

$$x_i = a + \frac{i-1}{n}(b-a), \quad i = 1, 2, \quad \ldots, n. \qquad (10)$$

2.3.2 Approximating the Solution of BVPs with Laguerre Polynomials

Approximating the solution of BVPs in Equations (1) – (3) using a collocation method with Laguerre polynomials as basis functions comprises two stages: finding solution at the boundary mesh points; and finding solution at the interior mesh points**.**

2.3.3 Solution at the Boundary Mesh Points with Dirichlet Boundary Condition

The boundary value problem in Equation (1) with its associated Dirichlet boundary condition in Equation (2) at both ends (i.e., the first and the last mesh points) assumes the following form:

$$y(a) = \sum_{j=1}^{n+1} s_j L_{j-1}(a)\,; \text{ and } y(b) = \sum_{j=1}^{n+1} s_j L_{j-1}(b) \quad (11)$$

2.3.4 Solution at the Boundary Mesh Points with Neumann Boundary Condition

The boundary value problem in Equation (1) with its corresponding Dirichlet boundary condition in Equation (3) at both ends (i.e., the first and the last mesh points) assumes the following form:

$$y'(a) = \sum_{j=1}^{n+1} s_j L'_{j-1}(a)\,; \text{ and } y'(b) = \sum_{j=1}^{n+1} s_j L'_{j-1}(b) \quad (12)$$

2.3.5 Solution at the Interior Mesh Points

Since the first and the last boundary mesh points has been taken care of in the preceding subsection (i.e., Equations (11) and (12)), the remaining $n-1$ equations are obtained from the differential equation evaluated at x_i.Now, let's consider Equation (1):

$$P(x)y''(x) + Q(x)y'(x) + R(x)y = G(x).$$

Finding the first and second derivatives of equation (9) will return:

$$y'(x) = \sum_{j=1}^{n+1} s_j L'_{j-1}(x)\,; \text{ and } y''(x) = \sum_{j=1}^{n+1} s_j L''_{j-1}(x) \quad (13)$$

respectively.The $n-1$ equations on the interior mesh points are obtained from

$$\sum_{j=1}^{n+1} s_j (P(x_i)L''_{j-1}(x_i) + Q(x_i)L'_{j-1}(x_i) + R(x_i)L_{j-1}(x_i) = G(x_i)) \quad (14)$$

where x_i is defined by Equation (10).

From the discretization of the boundary points (i.e., first and last points) and interior points above, we now obtain an $(n+1)\times(n+1)$ matrix of the form:

$$\begin{pmatrix} m_{1,1} & \cdots & m_{1,n+1} \\ \vdots & \ddots & \vdots \\ m_{n+1,1} & \cdots & m_{n+1,n+1} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{n+1} \end{pmatrix}, \quad (15)$$

where the coefficient matrixm_{ij} and the column matrix c_i are constants. But m_{ij} is defined by:

$$m_{ij} = \begin{cases} \sum_{j=1}^{n+1} L_{j-1}(a) \text{or} \sum_{j=1}^{n+1} L'_{j-1}(a), & i = 1, \\ \sum_{j=1}^{n+1} (P(x_i)L''_{j-1}(x_i) + Q(x_i)L'_{j-1}(x_i) + R(x_i)L_{j-1}(x_i)), & 2 \le i \le n, \\ \sum_{j=1}^{n+1} L_{j-1}(b) \text{ or } \sum_{j=1}^{n+1} L'_{j-1}(b), & i = n+1 \end{cases} \quad (16)$$

and

$$c_i = \begin{cases} \alpha_1, & i = 1, \\ G(x_i), & 2 \le i \le n \\ \alpha_2 i = n+1 \end{cases} \quad (17)$$

Solving Equation (15) yields the values of s_j, $1 \le j \le n+1$. These values $s_j$are substituted into Equation (9) to get the required approximate series solution.

III. Numerical Experiments

In this Section, we consider six (6) numerical examples. From the six numerical examples, three of them are second-order boundary value problems with the Dirichlet boundary condition and the remaining three are boundary value problems with the Neumann boundary conditions. The approximate solutions are in form of series solutions and are compared with the exact solutions at some selected mesh points within the given interval and the results and absolute errors are displayed in Tables (1) – (6). The coefficients for Examples 1-6 are provided in the Appendices A – F.These coefficients are substituted into Equation (9) to get the various required approximate solutions in series form. All computations are carried out usingMAPLE 17in conjunction with MATLAB R2014asoftware. These examples are somewhat artificial in the sense that the exact solutions of the differential equations are known in advance. That notwithstanding, such an approach is needed to examine the accuracy, the simplicity, the effectiveness and the applicability of the newly constructed method.

3.1 Boundary Value Problems with Dirichlet Boundary Condition

Problem 1

Consider the boundary value problem:

$$y'' = 4y, \quad [0,1];$$

with Dirichlet boundary condition: $y(0) = 1$ and $y(1) = 3$; and

Exact Solution:

$$y(x) = \frac{e^{-2x+4} + 3e^{2+2x} - 3e^{-2x+2} - e^{2x}}{e^4 - 1}.$$

Problem 2

Consider the boundary value problem:

$$y'' = y + \cos(x), \quad [0,1];$$

with Dirichlet boundary condition: $y(0) = 0$ and $y(1) = 1$; and

Exact Solution:

$$y(x) = -\frac{1}{2}\frac{-e^{x+1}\cos(1) + \cos(x)\,e^2 + e^{1-x}\cos(1) + e^x - 2e^{x+1} - e^{2-x} + 2e^{1-x} - \cos(x)}{e^2 - 1}$$

Problem 3

Consider the boundary value problem:

$$y'' = -y + sin^2(\pi x) - \pi^2 \sin(\pi x), \quad [0,1];$$

with Dirichlet boundary condition: $y(0) = 0$ and $y(1) = 0$; and

Exact Solution:

$$y(x) = \frac{2\sin(x)(\cos(1) - 1)\pi^2}{(4\pi^2 - 1)\sin(1)} - \frac{2\cos(x)\,\pi^2}{4\pi^2 - 1} + \frac{8\pi^4 \sin(\pi x) + 4\pi^4 + \pi^2\cos(2\pi x) - 2\pi^2\sin(\pi x) - 5\pi^2 - \cos(2\pi x) + 1}{8\pi^4 - 10\pi^2 + 2}$$

3.1.1 Solutions to Problems 1 – 3

Using the formulas in Equations (9) – (17), the results [exact solutions and the approximate solutions which correspond to Laguerre polynomial of degrees 3 – 8 (LPD3, LPD4, LPD5, LPD6, LPD7 and LPD8)] together with the absolute errors (er1, er2, er3, er4, er5 and er6)for Problems 1 – 3 are presented in Tables 1 – 3 respectively.

Table 1: Exact and Approximate Solutions for Problem 1 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

n	x_n	Exact Solution $y(x_n)$	Approximate Solutions (Laguerre Polynomials) LPD3	LPD4	LPD5	LPD6	LPD7	LPD8	Absolute Errors er1	er2	er3	er4	er5	er6
0	0.0	1.0000e+00	1.0000e+00	1.0000e+00	1.0000e+00	1.0000e+00	1.0000e+00	1.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
1	0.1	9.7776e-01	1.2000e+00	9.7671e-01	9.7809e-01	9.7774e-01	9.7776e-01	9.7776e-01	2.2224e-01	1.0418e-03	3.3054e-04	1.7068e-05	3.3734e-06	3.1030e-06
2	0.2	9.9475e-01	1.4000e+00	9.9397e-01	9.9505e-01	9.9474e-01	9.9475e-01	9.9475e-01	4.0525e-01	7.8126e-04	2.9349e-04	1.0046e-05	2.7428e-06	3.1102e-06
3	0.3	1.0517e+00	1.6000e+00	1.0515e+00	1.0519e+00	1.0517e+00	1.0517e+00	1.0517e+00	5.4833e-01	2.0449e-04	2.4460e-04	4.6340e-06	2.5800e-06	3.5971e-06
4	0.4	1.1508e+00	1.8000e+00	1.1510e+00	1.1510e+00	1.1508e+00	1.1508e+00	1.1508e+00	6.4920e-01	2.1869e-04	2.4831e-04	8.2755e-07	2.5158e-06	4.1000e-06
5	0.5	1.2961e+00	2.0000e+00	1.2966e+00	1.2964e+00	1.2961e+00	1.2961e+00	1.2961e+00	7.0389e-01	4.4318e-04	2.6386e-04	4.1109e-06	2.4770e-06	4.8434e-06
6	0.6	1.4934e+00	2.2000e+00	1.4941e+00	1.4937e+00	1.4934e+00	1.4934e+00	1.4934e+00	7.0656e-01	6.8889e-04	2.6345e-04	9.1943e-06	2.6315e-06	5.6827e-06
7	0.7	1.7507e+00	2.4000e+00	1.7519e+00	1.7510e+00	1.7507e+00	1.7507e+00	1.7507e+00	6.4930e-01	1.2068e-03	2.7604e-04	1.3370e-05	2.8116e-06	5.7105e-06
8	0.8	2.0782e+	2.6000e+00	2.0802e+00	2.0786e+00	2.0783e+00	2.0782e+00	2.0782e+00	5.2177e-01	1.9608e-03	3.4733e-04	1.9897e-05	3.1023e-06	5.1091e-06
9	0.9	2.4892e+00	2.8000e+00	2.4914e+00	2.4896e+00	2.4892e+00	2.4892e+00	2.4892e+00	3.1083e-01	2.2139e-03	4.0319e-04	2.8692e-05	3.9155e-06	5.5892e-06
10	1.0	3.0000e+00	3.0000e+00	3.0000e+00	3.0000e+00	3.0000e+00	3.0000e+00	3.0000e+00	0.0000e+00	0.0000e+00	4.4409e-16	4.4409e-16	0.0000e+00	0.0000e+00

Table 2: Exact and Approximate Solutions for Problem 2 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

n	x_n	Exact Solution $y(x_n)$	Approximate Solutions (Laguerre Polynomials)						Absolute Errors					
			LPD3	LPD4	LPD5	LPD6	LPD7	LPD8	er1	er2	er3	er4	er5	er6
0	0.0	0.0000e+00	1.1102e-15	-1.7764e-15	1.9895e-13	1.5348e-11	-8.7311e-11	5.0670e-08	1.1102e-15	1.7764e-15	1.9895e-13	1.5348e-11	8.7311e-11	5.0670e-08
1	0.1	4.7498e-02	3.2397e-01	4.7449e-02	4.7504e-02	4.7498e-02	4.7498e-02	4.7498e-02	2.7647e-01	4.9263e-05	5.5859e-06	1.0085e-07	3.5313e-09	2.7526e-08
2	0.2	1.0542e-01	6.0225e-01	1.0538e-01	1.0543e-01	1.0542e-01	1.0542e-01	1.0542e-01	4.9683e-01	4.0491e-05	5.3391e-06	6.6846e-08	3.3040e-09	8.8561e-09
3	0.3	1.7420e-01	8.3332e-01	1.7418e-01	1.7421e-01	1.7420e-01	1.7420e-01	1.7420e-01	6.5912e-01	1.7781e-05	4.7806e-06	3.9376e-08	3.4439e-09	6.2044e-09
4	0.4	2.5428e-01	1.0156e+00	2.5428e-01	2.5428e-01	2.5428e-01	2.5428e-01	2.5428e-01	7.6137e-01	7.0642e-07	4.9052e-06	1.9370e-08	3.6004e-09	1.7924e-08
5	0.5	3.4611e-01	1.1477e+00	3.4612e-01	3.4612e-01	3.4611e-01	3.4611e-01	3.4611e-01	8.0158e-01	8.8064e-06	5.1084e-06	5.8679e-09	3.7065e-09	2.6623e-08
6	0.6	4.5018e-01	1.2279e+00	4.5020e-01	4.5019e-01	4.5018e-01	4.5018e-01	4.5018e-01	7.7774e-01	1.8457e-05	4.9828e-06	3.1144e-08	3.9358e-09	3.2589e-08
7	0.7	5.6701e-01	1.2548e+00	5.6705e-01	5.6702e-01	5.6701e-01	5.6701e-01	5.6701e-01	6.8779e-01	3.6383e-05	4.9391e-06	5.1192e-08	4.1094e-09	3.6095e-08
8	0.8	6.9717e-01	1.2268e+00	6.9723e-01	6.9717e-01	6.9717e-01	6.9717e-01	6.9717e-01	5.2963e-01	6.0741e-05	5.6019e-06	7.9232e-08	4.3141e-09	3.7420e-08
9	0.9	8.4127e-01	1.1424e+00	8.4133e-01	8.4127e-01	8.4127e-01	8.4127e-01	8.4127e-01	3.0111e-01	6.8017e-05	5.9295e-06	1.1420e-07	4.9874e-09	3.6758e-08
10	1.0	1.0000e+00	1.0000e+00	1.0000e+00	1.0000e+00	1.0000e+00	1.0000e+00	1.0000e+00	1.1102e-15	2.6645e-15	1.3034e-13	1.0241e-11	1.8485e-10	3.4955e-08

Table 3: Exact and Approximate Solutions for Problem 3 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

n	x_n	Exact Solution $y(x_n)$	Approximate Solutions (Laguerre Polynomials)						Absolute Errors					
			LPD3	LPD4	LPD5	LPD6	LPD7	LPD8	er1	er2	er3	er4	er5	er6
0	0.0	1.1102e-16	-1.7764e-15	2.8422e-14	-1.4211e-14	-1.0118e-10	-5.2296e-12	-7.0315e-08	1.8874e-15	2.8311e-14	1.4322e-14	1.0118e-10	5.2297e-12	7.0315e-08
1	0.1	3.1606e-01	3.9509e-01	3.1278e-01	3.1409e-01	3.1593e-01	3.1612e-01	3.1642e-01	7.9025e-02	3.2794e-03	1.9718e-03	1.3301e-04	5.2578e-05	3.5195e-04
2	0.2	5.9980e-01	7.0238e-01	5.9620e-01	5.9800e-01	5.9985e-01	6.0010e-01	6.0035e-01	1.0258e-01	3.5927e-03	1.7954e-03	5.2423e-05	3.0209e-04	5.5273e-04
3	0.3	8.2348e-01	9.2187e-01	8.2015e-01	8.2201e-01	8.2373e-01	8.2401e-01	8.2426e-01	9.8388e-02	3.3350e-03	1.4705e-03	2.4753e-04	5.2930e-04	7.7706e-04
4	0.4	9.6623e-01	1.0536e+00	9.6312e-01	9.6491e-01	9.6668e-01	9.6698e-01	9.6723e-01	8.7341e-02	3.1082e-03	1.3164e-03	4.4797e-04	7.5192e-04	1.0007e-03
5	0.5	1.0151e+00	1.0975e+00	1.0122e+00	1.0140e+00	1.0158e+00	1.0161e+00	1.0163e+00	8.2349e-02	2.9083e-03	1.1595e-03	6.5880e-04	9.7147e-04	1.2159e-03
6	0.6	9.6581e-01	1.0536e+00	9.6312e-01	9.6491e-01	9.6668e-01	9.6698e-01	9.6723e-01	8.7756e-02	2.6926e-03	9.0072e-04	8.6360e-04	1.1675e-03	1.4163e-03
7	0.7	8.2269e-01	9.2187e-01	8.2015e-01	8.2201e-01	8.2373e-01	8.2401e-01	8.2426e-01	9.9184e-02	2.5390e-03	6.7447e-04	1.0436e-03	1.3253e-03	1.5731e-03
8	0.8	5.9869e-01	7.0238e-01	5.9620e-01	5.9800e-01	5.9985e-01	6.0010e-01	6.0035e-01	1.0369e-01	2.4853e-03	6.8804e-04	1.1597e-03	1.4094e-03	1.6601e-03
9	0.9	3.1474e-01	3.9509e-01	3.1278e-01	3.1409e-01	3.1593e-01	3.1612e-01	3.1642e-01	8.0344e-02	1.9604e-03	6.5278e-04	1.1860e-03	1.3716e-03	1.6710e-03
10	1.0	-1.4070e-03	0.0000e+00	-8.5265e-14	0.0000e+00	1.0162e-10	1.2548e-11	-1.9778e-08	1.4070e-03	1.4070e-03	1.4070e-03	1.4070e-03	1.4070e-03	1.4069e-03

3.2 Boundary Value Problems with Neumann Boundary Condition

Problem 4

Consider the boundary value problem:

$$-y'' = (2-4x^2)y, \quad [0,1];$$

with Neumann boundary condition: $y'(0) = 0$ and $y'(1) = \frac{-2}{e}$; and

Exact Solution:

$$y(x) = e^{-x^2}.$$

Problem 5

Consider the boundary value problem:

$$y'' = y' + 2y, \quad [0,1];$$

with Neumann boundary condition: $y'(0) = 1$ and $y'(1) = \frac{2e^2+e^{-1}}{3}$; and

Exact Solution:

$$y(x) = \frac{e^{2x} - e^{-x}}{3}.$$

Problem 6

Consider the boundary value problem:

$$y'' = 2y, \quad [0,1];$$

with Neumann boundary condition: $y'^{(0)} = -1$ and $y'(1) = \frac{-1}{4}$; and

Exact Solution:

$$y(x) = -\frac{1}{8}\frac{\sqrt{2}(-4e^{\sqrt{2}(x-1)} - 4e^{-\sqrt{2}(x-1)} + e^{\sqrt{2}x} + e^{-\sqrt{2}x})e^{\sqrt{2}}}{e^{2\sqrt{2}-1}}.$$

3.2.1 Solutions to Problems 4 – 6

Using the formulas in Equations (9) – (17), the results [exact solutions and the approximate solutions which correspond to Laguerre polynomials of degrees 3 – 8 (LPD3, LPD4, LPD5, LPD6, LPD7 and LPD8)] together with the absolute errors (er1, er2, er3, er4, er5 and er6) for Problems 4 – 6 are presented in Tables 4 – 6 respectively.

Table 4: Exact and ApproximateSolutions for Problem 4 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

n	x_n	Exact Solution $y(x_n)$	Approximate Solutions (Laguerre Polynomials)						Absolute Errors					
			LPD3	LPD4	LPD5	LPD6	LPD7	LPD8	er1	er2	er3	er4	er5	er6
0	0.0	1.0000e+00	1.5079e+01	9.9106e-01	9.9497e-01	1.0009e+00	1.0006e+00	9.9996e-01	1.4079e+01	8.9366e-03	5.0291e-03	8.6532e-04	5.8451e-04	4.3593e-05
1	0.1	9.9005e-01	1.3657e+01	9.7937e-01	9.8514e-01	9.9106e-01	9.9064e-01	9.9000e-01	1.2667e+01	1.0682e-02	4.9051e-03	1.0077e-03	5.8530e-04	5.0045e-05
2	0.2	9.6079e-01	1.2213e+01	9.4649e-01	9.5615e-01	9.6202e-01	9.6137e-01	9.6073e-01	1.1253e+01	1.4300e-02	4.6377e-03	1.2338e-03	5.7857e-04	5.8449e-05
3	0.3	9.1393e-01	1.0751e+01	8.9577e-01	9.0965e-01	9.1536e-01	9.1449e-01	9.1387e-01	9.8374e+00	1.8161e-02	4.2825e-03	1.4330e-03	5.6078e-04	6.5550e-05
4	0.4	8.5214e-01	9.2746e+00	8.3060e-01	8.4829e-01	8.5375e-01	8.5268e-01	8.5207e-01	8.4225e+00	2.1544e-02	3.8538e-03	1.6038e-03	5.3385e-04	7.1641e-05
5	0.5	7.7880e-01	7.7869e+00	7.5441e-01	7.7543e-01	7.8056e-01	7.7930e-01	7.7872e-01	7.0081e+00	2.4390e-02	3.3730e-03	1.7568e-03	4.9981e-04	7.6750e-05
6	0.6	6.9768e-01	6.2919e+00	6.7068e-01	6.9481e-01	6.9957e-01	6.9814e-01	6.9760e-01	5.5942e+00	2.6994e-02	2.8652e-03	1.8926e-03	4.6061e-04	8.1100e-05
7	0.7	6.1263e-01	4.7931e+00	5.8294e-01	6.1029e-01	6.1464e-01	6.1305e-01	6.1254e-01	4.1804e+00	2.9691e-02	2.3397e-03	2.0143e-03	4.1900e-04	8.5020e-05
8	0.8	5.2729e-01	3.2942e+00	4.9474e-01	5.2550e-01	5.2943e-01	5.2767e-01	5.2720e-01	2.7669e+00	3.2554e-02	1.7947e-03	2.1399e-03	3.7720e-04	8.8868e-05
9	0.9	4.4486e-01	1.7988e+00	4.0970e-01	4.4358e-01	4.4714e-01	4.4519e-01	4.4476e-01	1.3539e+00	3.5155e-02	1.2733e-03	2.2817e-03	3.3660e-04	9.3403e-05
10	1.0	3.6788e-01	3.1055e-01	3.3149e-01	3.6688e-01	3.7025e-01	3.6819e-01	3.6778e-01	5.7332e-02	3.6392e-02	9.9504e-04	2.3732e-03	3.1183e-04	9.7127e-05

Table 5: Exact and ApproximateSolutions for Problem 5 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

n	x_n	Exact Solution $y(x_n)$	Approximate Solutions (Laguerre Polynomials)						Absolute Errors					
			LPD3	LPD4	LPD5	LPD6	LPD7	LPD8	er1	er2	er3	er4	er5	er6
0	0.0	0.0000e+00	2.4106e-01	-2.5627e-02	7.2148e-03	-5.5680e-04	1.0364e-04	-6.0863e-06	2.4106e-01	2.5627e-02	7.2148e-03	5.5680e-04	1.0364e-04	6.0863e-06
1	0.1	1.0552e-01	3.4231e-01	8.1342e-02	1.1249e-01	1.0501e-01	1.0562e-01	1.0552e-01	2.3679e-01	2.4180e-02	6.9715e-03	5.1543e-04	9.9036e-05	5.5660e-06
2	0.2	2.2436e-01	4.5150e-01	2.0327e-01	2.3092e-01	2.2392e-01	2.2446e-01	2.2436e-01	2.2714e-01	2.1099e-02	6.5600e-03	4.4711e-04	9.2869e-05	4.8495e-06
3	0.3	3.6043e-01	5.7676e-01	3.4278e-01	3.6666e-01	3.6005e-01	3.6052e-01	3.6043e-01	2.1633e-01	1.7650e-02	6.2246e-03	3.8239e-04	8.8273e-05	4.1813e-06
4	0.4	5.1841e-01	7.2623e-01	5.0402e-01	5.2442e-01	5.1809e-01	5.1849e-01	5.1840e-01	2.0783e-01	1.4388e-02	6.0168e-03	3.2069e-04	8.5053e-05	3.5256e-06
5	0.5	7.0392e-01	9.0805e-01	6.9258e-01	7.0983e-01	7.0366e-01	7.0400e-01	7.0391e-01	2.0414e-01	1.1335e-02	5.9156e-03	2.5793e-04	8.3229e-05	2.8767e-06
6	0.6	9.2377e-01	1.1304e+00	9.1557e-01	9.2968e-01	9.2357e-01	9.2385e-01	9.2377e-01	2.0659e-01	8.2005e-03	5.9090e-03	1.9398e-04	8.3036e-05	2.2203e-06
7	0.7	1.1862e+00	1.4013e+00	1.1816e+00	1.1922e+00	1.1861e+00	1.1863e+00	1.1862e+00	2.1509e-01	4.6493e-03	6.0277e-03	1.2879e-04	8.4515e-05	1.5400e-06
8	0.8	1.5012e+00	1.7290e+00	1.5006e+00	1.5076e+00	1.501176872	1.5013e+00	1.5012e+00	2.2775e-01	6.2437e-04	6.3201e-03	5.7615e-05	8.7926e-05	8.2611e-07
9	0.9	1.8810e+00	2.1216e+00	1.8843e+00	1.8878e+00	1.8810e+00	1.8811e+00	1.8810e+00	2.4055e-01	3.2558e-03	6.7566e-03	2.1456e-05	9.3838e-05	3.1828e-08
10	1.0	2.3404e+00	2.5872e+00	2.3456e+00	2.3474e+00	2.3405e+00	2.3405e+00	2.3404e+00	2.4682e-01	5.2134e-03	7.0463e-03	7.1964e-05	9.8910e-05	5.7055e-07

Table 6: Exact and ApproximateSolutions for Problem 6 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

n	x_n	Exact Solution $y(x_n)$	Approximate Solutions (Laguerre Polynomials)						Absolute Errors					
			LPD3	LPD4	LPD5	LPD6	LPD7	LPD8	er1	er2	er3	er4	er5	er6
0	0.0	7.0459e-01	7.2886e-01	7.0611e-01	7.0494e-01	7.0461e-01	7.0459e-01	7.0459e-01	2.4264e-02	1.5186e-03	3.4760e-04	1.4040e-05	2.4184e-06	7.0095e-08
1	0.1	6.1132e-01	6.3496e-01	6.1270e-01	6.1165e-01	6.1133e-01	6.1132e-01	6.1132e-01	2.3641e-02	1.3893e-03	3.3296e-04	1.2336e-05	2.2907e-06	5.9857e-08
2	0.2	5.3029e-01	5.5259e-01	5.3141e-01	5.3060e-01	5.3030e-01	5.3029e-01	5.3029e-01	2.2301e-02	1.1287e-03	3.0927e-04	9.6273e-06	2.1279e-06	4.6178e-08
3	0.3	4.5988e-01	4.8074e-01	4.6073e-01	4.6017e-01	4.5989e-01	4.5988e-01	4.5988e-01	2.0860e-02	8.5382e-04	2.9081e-04	7.1616e-06	2.0144e-06	3.3799e-08
4	0.4	3.9869e-01	4.1841e-01	3.9930e-01	3.9897e-01	3.9869e-01	3.9869e-01	3.9869e-01	1.9721e-02	6.0859e-04	2.7975e-04	4.9006e-06	1.9411e-06	2.2007e-08
5	0.5	3.4548e-01	3.6458e-01	3.4587e-01	3.4576e-01	3.4548e-01	3.4548e-01	3.4548e-01	1.9102e-02	3.9229e-04	2.7437e-04	2.6932e-06	1.9055e-06	1.0682e-08
6	0.6	2.9920e-01	3.1826e-01	2.9938e-01	2.9947e-01	2.9920e-01	2.9920e-01	2.9920e-01	1.9062e-02	1.8397e-04	2.7361e-04	5.3932e-07	1.9095e-06	4.2869e-10
7	0.7	2.5891e-01	2.7843e-01	2.5887e-01	2.5918e-01	2.5890e-01	2.5891e-01	2.5891e-01	1.9521e-02	3.6188e-05	2.7840e-04	1.5609e-06	1.9506e-06	1.1574e-08
8	0.8	2.2380e-01	2.4408e-01	2.2353e-01	2.2409e-01	2.2380e-01	2.2380e-01	2.2380e-01	2.0278e-02	2.6826e-04	2.9015e-04	3.7471e-06	2.0307e-06	2.2866e-08
9	0.9	1.9318e-01	2.1421e-01	1.9270e-01	1.9349e-01	1.9318e-01	1.9318e-01	1.9318e-01	2.1029e-02	4.7643e-04	3.0696e-04	6.0541e-06	2.1573e-06	3.4953e-08
10	1.0	1.6643e-01	1.8781e-01	1.6586e-01	1.6675e-01	1.6642e-01	1.6643e-01	1.6643e-01	2.1380e-02	5.7489e-04	3.1760e-04	7.4595e-06	2.2596e-06	4.3776e-08

Next, for Tables 1 – 6 above, different graphs each are presented as follows:

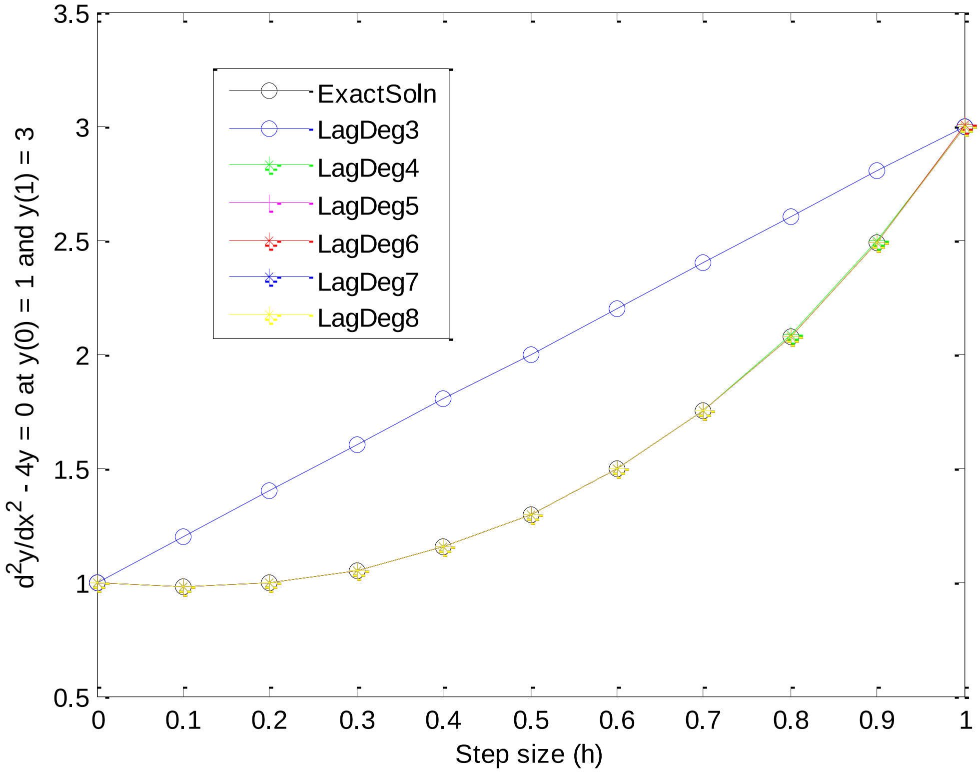

Figure 1: Plot of Exact and Approximate Solutions against the Step Sizefor Problem 1 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

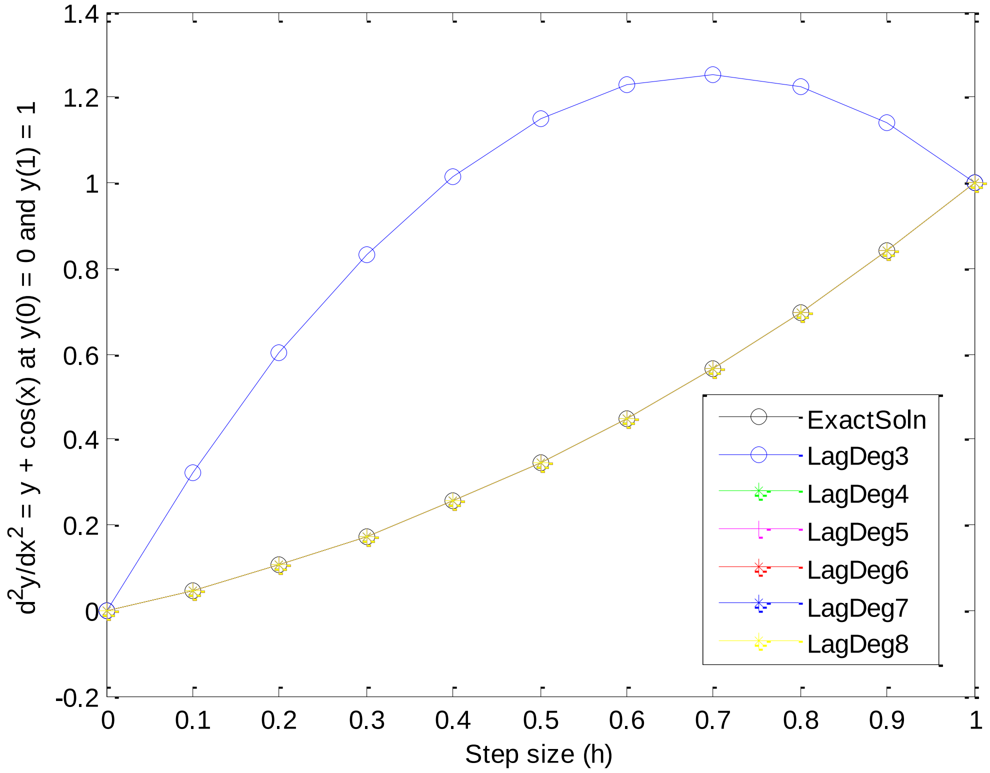

Figure 2: Plot of Exact and Approximate Solutions against the Step Sizefor Problem 2 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

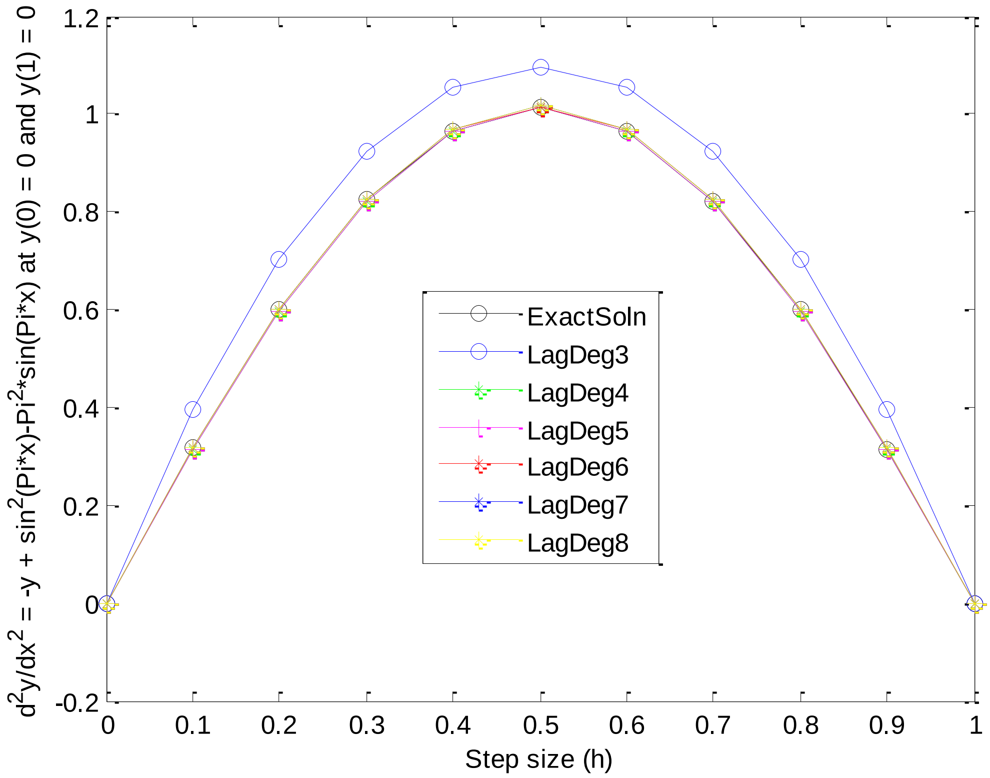

Figure 3: Plot of Exact and Approximate Solutions against the Step Sizefor Problem 3 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

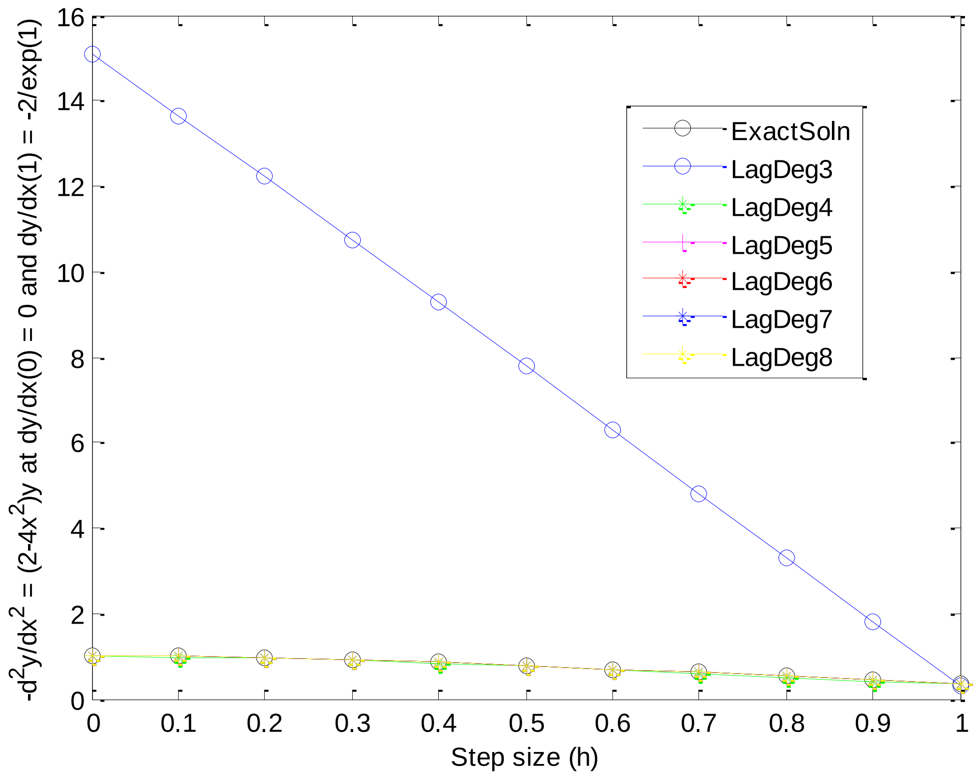

Figure 4: Plot of Exact and Approximate Solutions against the Step Sizefor Problem 4 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

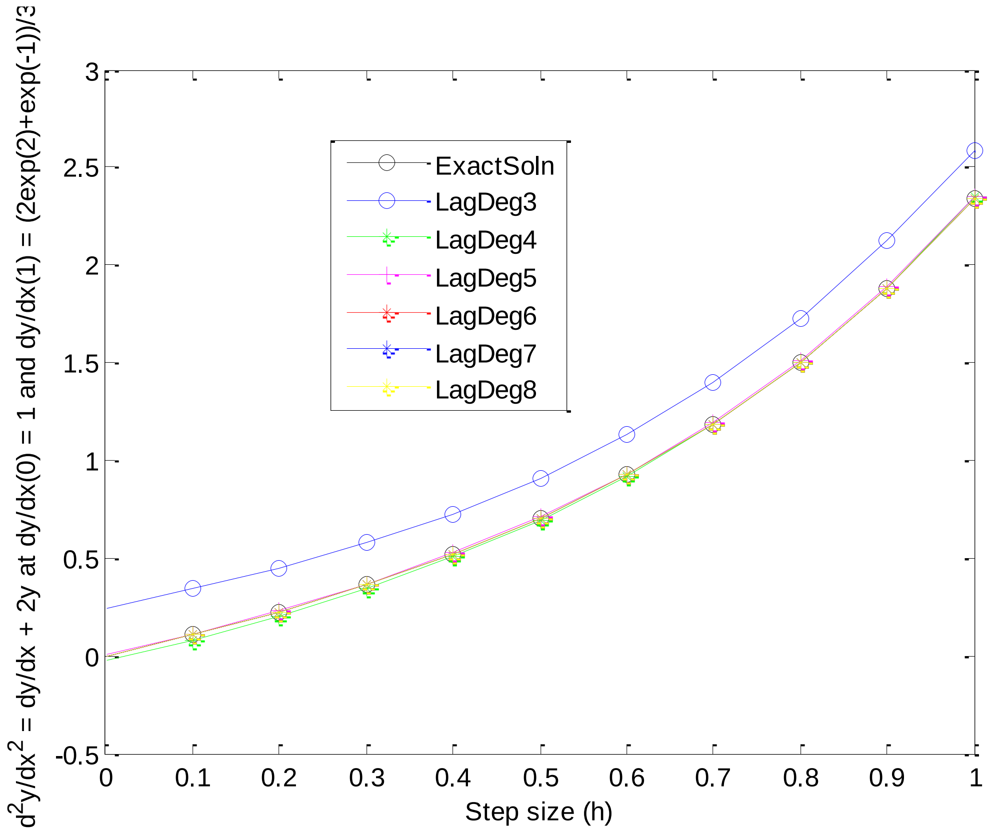

Figure 5: Plot of Exact and Approximate Solutions against the Step Sizefor Problem 5 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

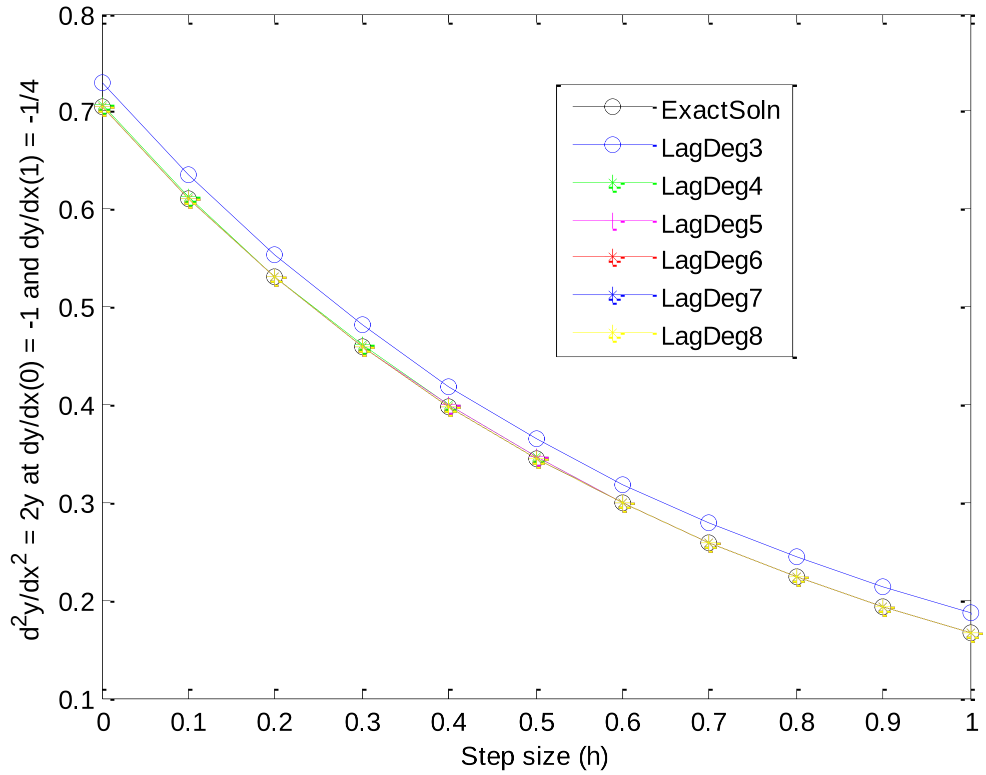

Figure 6: Plot of Exact and Approximate Solutions against the Step Sizefor Problem 6 Using Laguerre Polynomial of Degrees 3, 4, 5, 6, 7 and 8

IV. Discussion of Results

In Tables 1 to 3, numerical solutions obtained by collocation method constructed with Laguerre polynomials of degrees 3 to 8 for the solution of linear BVPs of ODEs for Problems 1 to 3 respectively which have Dirichlet boundary conditions at both end-points are compared with the respective exact solutions. The observed absolute errors between the respective exact solutions and that obtained by the collocation method constructed with Laguerre polynomials of degrees 3 to 8 at various values of the mesh points are given. It is equally observed from the results that the collocation method constructed with Laguerre polynomials of degrees 3 to 8 shows a progressive increase in the accuracy of the constructed method measured in terms of their absolute errors. This is pictorially observed in Figures 1 to 3.

By the same token, in Tables 4 to 6, numerical solutions obtained by collocation method constructed with Laguerre polynomials of degrees 3 to 8 for the solution of linear BVPs of ODEs for Problems 4 to 6 respectively which have Neumann boundary conditions at both end-points are compared with the respective exact solutions. The observed absolute errors between the respective exact solutions and that obtained by the collocation method constructed with Laguerre polynomials of degrees 3 to 8 at various values of mesh points are given. It is correspondingly observed from the results that the collocation method constructed with Laguerre polynomials of degrees 3 to 8 shows a progressive increase in the accuracy of the constructed method measured in terms of their absolute errors. This is pictorially confirmed in Figures 4 to 6.

By the way of comparison, it is keenly observed from the results obtained by the collocation method constructed with Laguerre polynomials of degrees 3 to 8 for the solution of linear BVPs of ODEs for Problems 1 to 3 respectively which have Dirichlet boundary conditions at both end-points are more accurate than the ones obtained by collocation method constructed with Laguerre polynomials of degrees 3 to 8 for the solution of linear BVPs of ODEs for Problems 4 to 6 respectively which have Neumann boundary conditions at both end-points.

V. Conclusion

In this work, we implemented the collocation method via Laguerre polynomials of degrees 3 to 8 for the solution of linear BVPs of ODEs with Dirichlet boundary conditions at both end-points and Neumann boundary conditions at both end-points. With the help of six illustrative examples, the accuracy, the simplicity, the efficiency, the effectiveness and the applicability of the newly constructed method was demonstrated. Tables 1 to 6 together with the plots (Figures 1 to 6) meticulously presented to us the nature and the behaviour of the newly constructed method. Based on the careful observations from the computed results, it may be concluded here that the collocation method developed is more efficient, effective and applicable in terms of accuracy for approximating boundary value problems with Dirichlet boundary condition. Therefore, this method is highly recommended as a way of application for approximating many models in sciences and engineeringthat appear in form of second order boundary value problems with Dirichlet boundary condition as well as Neumann boundary condition.

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Appendices

Appendix A: Coefficients of the Laguerre Polynomial of Degrees 3 to 8 for Problem 1

LPD	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
3	3	-2	0	0					
4	$\frac{88249}{3915}$	$-\frac{11126}{135}$	$\frac{31568}{261}$	$-\frac{106048}{1305}$	$\frac{3072}{145}$				
5	$\frac{297300478}{7553201}$	$-\frac{1322911777}{7553201}$	$\frac{2460059000}{7553201}$	$-\frac{2323339500}{7553201}$	$\frac{1106295000}{7553201}$	$-\frac{150000}{5399}$			
6	$\frac{22159493051}{245358795}$	$-\frac{41471886292}{81786265}$	$\frac{60228093908}{49071759}$	$-\frac{26415443880}{16357253}$	$\frac{19850969664}{16357253}$	$-\frac{8060383872}{16357253}$	$\frac{1244160}{14723}$		
7	$\frac{475506870943201}{3022697749031}$	$-\frac{345056509546494134}{341564845640503}$	$\frac{972613139939160580}{341564845640503}$	$-\frac{1542075537513471576}{341564845640503}$	$\frac{1478741058112273200}{341564845640503}$	$-\frac{854754429281934720}{341564845640503}$	$\frac{274982686598289600}{341564845640503}$	$-\frac{4743607680}{42817169}$	
8	$\frac{1004937954882009299134751}{26473268541494340107213}$	$\frac{1021541646999825086}{26473268541494340107213}$	$-\frac{3380524399167733244329456}{3781895505927762872459}$	$\frac{12873842567457883484595840}{3781895505927762872459}$	$-\frac{23247868876455795310468096}{3781895505927762872459}$	$\frac{24086980451900258243112960}{3781895505927762872459}$	$-\frac{14714156087938848737525760}{3781895505927762872459}$	$\frac{4952944340865555683082240}{3781895505927762872459}$	$-\frac{710998812073057541160960}{3781895505927762872459}$

Appendix B: Coefficients of the Laguerre Polynomial of Degrees 3 to 8 for Problem 2

LPD	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
3	$-\frac{47}{52}-\frac{873}{52}\cos\left(\frac{2}{3}\right)+\frac{639}{52}\cos\left(\frac{1}{3}\right)$	$\frac{263}{52}+\frac{2565}{52}\cos\left(\frac{2}{3}\right)-\frac{1863}{52}\cos\left(\frac{1}{3}\right)$	$-\frac{135}{26}-\frac{765}{26}\cos\left(\frac{2}{3}\right)+\frac{531}{26}\cos\left(\frac{1}{3}\right)$	$\frac{27}{26}-\frac{81}{26}\cos\left(\frac{2}{3}\right)+\frac{81}{26}\cos\left(\frac{1}{3}\right)$					
4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				
5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]			
6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		
7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Appendix C: Coefficients of the Laguerre Polynomial of Degrees 3 to 8 for Problem 3

LPD	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
3	$-\frac{9}{32}\pi^2\sqrt{3}+\frac{27}{64}$	$\frac{27}{32}\pi^2\sqrt{3}-\frac{81}{64}$	$-\frac{9}{16}\pi^2\sqrt{3}+\frac{27}{32}$	0					
4	$\frac{26204}{1355}\pi^2-\frac{13024}{1355}\pi^2\sqrt{2}-\frac{2636}{271}$	$-\frac{119444}{1355}\pi^2+\frac{58784}{1355}\pi^2\sqrt{2}+\frac{12132}{271}$	$\frac{40920}{271}\pi^2-\frac{19904}{271}\pi^2\sqrt{2}-\frac{21016}{271}$	$\frac{75264}{1355}\pi^2\sqrt{2}-\frac{155904}{1355}\pi^2+\frac{16128}{271}$	$-\frac{21504}{1355}\pi^2\sqrt{2}+\frac{44544}{1355}\pi^2-\frac{4608}{271}$				
5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0			
6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$-\frac{736384770}{25709}\pi^2\sqrt{3}+\frac{1290096720}{25709}\pi^2-\frac{94102965}{25709}$	$\frac{133888140}{25709}\pi^2\sqrt{3}-\frac{234563040}{25709}\pi^2+\frac{17109630}{25709}$		
7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	
8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Appendix D: Coefficients of the Laguerre Polynomial of Degrees 3 to 8 for Problem 4

LPD	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
3	$\frac{93224}{16677e}$	$-\frac{29092}{1853e}$	$\frac{41924}{1853e}$	$-\frac{18252}{1853e}$					
4	$\frac{1065485}{172932e}$	$-\frac{790076}{43233e}$	$\frac{1183054}{43233e}$	$-\frac{198496}{14411e}$	$\frac{17536}{14411e}$				
5	$-\frac{61116963818637}{1462511797000e}$	$\frac{362544138708}{1462511797e}$	$-\frac{3297397475451}{5850047188e}$	$\frac{1881385511865}{2925023594e}$	$-\frac{533151510945}{1462511797e}$	$\frac{119336475000}{1462511797e}$			
6	$-\frac{283282667546716}{44104237078091e}$	$\frac{812970792599924}{44104237078091e}$	$\frac{2529407753847476}{44104237078091e}$	$-\frac{11196427488801588}{44104237078091e}$	$\frac{16097908052678112}{44104237078091e}$	$-\frac{10369373689160640}{44104237078091e}$	$\frac{2528788733533440}{44104237078091e}$		
7	$\frac{489001114484945704496413952}{8951904630564573096281 03e}$	$-\frac{1079444689744230398688416}{2609884731943024226321e}$	$\frac{3509600424759340119373 6576}{2609884731943024226321e}$	$-\frac{6326883900699441903657052 8}{2609884731943024226321e}$	$\frac{68283510366934988116717152}{2609884731943024226321e}$	$-\frac{44084792416118639332993440}{2609884731943024226321e}$	$\frac{15751145167895992052802240}{2609884731943024226321e}$	$-\frac{2401142138754849999684000}{2609884731943024226321e}$	
8	$-\frac{226992159929692229714765 5}{456683545561966346968 4e}$	$\frac{7725593474634396023034 04}{1631012662721308382 03e}$	$-\frac{2234952644418516912614628 2}{1141708863904915867421e}$	$\frac{7504734393982184863444896}{163101266272130838203e}$	$-\frac{7681685982268266699231552 0}{1141708863904915867421e}$	$\frac{7156509154863475479527424 0}{1141708863904915867421e}$	$-\frac{4147872052467292350234624 0}{1141708863904915867421e}$	$\frac{195290322173310562467840 0}{163101266272130838203e}$	$-\frac{280111392621917173186560}{163101266272130838203e}$

Appendix E: Coefficients of the Laguerre Polynomial of Degrees 3 to 8 for Problem 5

LPD	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
3	$\frac{2501}{1809}e^2+\frac{2501}{3618}e^{-1}-\frac{10}{9}$	$-\frac{700}{201}e^2-\frac{350}{201}e^{-1}+1$	$\frac{674}{201}e^2+\frac{337}{201}e^{-1}-1$	$-\frac{72}{67}e^2-\frac{36}{67}e^{-1}$					
4	$-\frac{105337}{49768}+\frac{64803}{24884}e^2+\frac{64803}{49768}e^{-1}$	$\frac{35853}{6221}-\frac{172852}{18663}e^2-\frac{86426}{18663}e^{-1}$	$-\frac{57421}{6221}+\frac{248762}{18663}e^2+\frac{124381}{18663}e^{-1}$	$\frac{39360}{6221}-\frac{54336}{6221}e^2-\frac{27168}{6221}e^{-1}$	$-\frac{11328}{6221}+\frac{13696}{6221}e^2+\frac{6848}{6221}e^{-1}$				
5	$-\frac{1208107777}{38472050}+\frac{983406884}{19236025}e^2+\frac{491703442}{19236025}e^{-1}$	$\frac{1756731}{1538881}-\frac{10672164}{4616643}e^2-\frac{5336082}{4616643}e^{-1}$	$-\frac{33451606}{1538881}+\frac{203145712}{4616643}e^2+\frac{101572856}{4616643}e^{-1}$	$\frac{31051140}{1538881}-\frac{65744760}{1538881}e^2-\frac{32872380}{1538881}e^{-1}$	$-\frac{14620380}{1538881}+\frac{32406920}{1538881}e^2+\frac{16203460}{1538881}e^{-1}$	$\frac{2625000}{1538881}-\frac{6450000}{1538881}e^2-\frac{3225000}{1538881}e^{-1}$			
6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$-\frac{16566145920}{312291439}+\frac{26580545280}{312291439}e^2+\frac{13290272640}{312291439}e^{-1}$		
7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Appendix F: Coefficients of the Laguerre Polynomial of Degrees 3 to 8 for Problem 6

LPD	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
3	$-\frac{5}{201}$	$\frac{203}{134}$	$-\frac{237}{134}$	$\frac{135}{134}$					
4	$\frac{148409}{194464}$	$-\frac{26669}{12154}$	$\frac{112191}{24308}$	$-\frac{23556}{6077}$	$\frac{144}{103}$				
5	$-\frac{322632397}{748768000}$	$\frac{206421}{46798}$	$-\frac{29999521}{2995072}$	$\frac{18489315}{1497536}$	$-\frac{5697045}{748768}$	$\frac{46875}{23399}$			
6	$\frac{578843245}{466374766}$	$-\frac{3028781477}{466374766}$	$\frac{9157397169}{466374766}$	$-\frac{14311382805}{466374766}$	$\frac{6424581420}{233187383}$	$-\frac{3108815640}{233187383}$	$\frac{349920}{125437}$		
7	$-\frac{90705439229705}{77936529161491}$	$\frac{2631482043791}{227220201637}$	$-\frac{8774314211462}{227220201637}$	$\frac{16736587413070}{227220201637}$	$-\frac{19235378968540}{227220201637}$	$\frac{13429838457140}{227220201637}$	$-\frac{5272613822280}{227220201637}$		
8	$\frac{7135411959179910 5}{3257412430023900 8}$	$-\frac{103679705131352159}{6107648306294814}$	$\frac{827954370066514373}{12215296612589628}$	$-\frac{155683066513185988}{1017941384382469}$	$\frac{220967308011355696}{1017941384382469}$	$-\frac{201674695557959680}{1017941384382469}$	$\frac{115943448627793920}{1017941384382469}$	$-\frac{38452846903296000}{1017941384382469}$	$\frac{330301440}{59288693}$