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The proof of 3x+1 problem

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Abstract: *First, the 3x+1 problem is transformed into the decision of the terms valued 1 of the all-odd 3x+1 sequences. Then, the property of the all-odd 3x+1 sequences is used to obtain the equations with equal terms and the sufficient and necessary conditions for the all-odd 3x+1 sequences to have equal terms. Then, the uniqueness of the characteristic solutions of the equations with equal terms is proved. At last, based on the above results the 3x+1 problem is proved to be true.*

Keywords: *the 3x+1 problem; the all–odd 3x+1 sequences; equations with equal terms; characteristic solutions*
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I. Introduction

The 3x+1 problem is a conjecture proposed by L. Collatz in 1937, also known as Collatz problem, 3x+1 mapping, Hasse's algorithm, Kakutani problem, etc.[1]. The schemes put forward by M.R.Feix and J.L.Rouet[2], M.Chamberland[3] are ineffective. E.Belaga[4] even doubt the provability of the 3x+1 problem. This paper is different from the above approaches. In this paper the 3x+1 problem is equivalently transformed into the all-odd 3x+1 sequences, by which the equations with equal terms are introduced, then mathematical induction is applied to the equations with equal terms, at last the 3x+1 problem is proved.

First, we make the following conventions for the terminologies and symbols used in this paper:
(1)The word "sequence" used in this paper denotes infinite sequences, unless otherwise specified.
(2) The lower case italic Latin letters used as variables denote positive integers, unless otherwise specified.
(3) Sequences in this paper are denoted by upper case italic Latin letters (or with primes or with subscripts). Their corresponding lower case italic Latin letters denote the general terms of the sequences in question. For example, the general terms of the sequences A and B' are a_n and b'_n.
(4) N and N_o are two special symbols. N denotes the set of positive integers, N_o denotes the set of positive odd numbers.

1. 3x+1 problem and all-odd 3x+1 sequences

The 3x+1 problem can be described by natural language as follows: Starting from any given positive integer n, if it is an even number, then it is divided by 2; if it is an odd number, then it is multiplied by 3 and plus 1; for the result obtained repeat the above operations; at last number 1 is bound to be obtained. For example, suppose n=52, then the result is: 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.
We call the above description the original description of the 3x+1 problem. In order to describe it clearly and solve it at last, we introduce the concept of deevenization (deevenization is a new English word coined by the authors).
Suppose $n = 2^i r$, where $r \in N_o$, $i \geq 0$. We call the process from n to r a deeven process or deeven operation. We use "$\beta()$" to denote the deeven function, i.e.
$\beta(n) = \beta(2^i r) = (2^i r) / 2^i = r$. When $n \in N_o$, $\beta(n) = n$.
According to the above definition we know that for any positive integer n, we must have $\beta(n) \in N_o$.
Therefore, we can simplify the 3x+1 problem. Although the object of study of the 3x+1 problem is any positive integer, from the original description of the 3x+1 problem we know that starting from the given positive integer n, when $n = 2^i r$ ($r \in N_o$), n should be divided by 2 i times to obtain r, i.e. the deeven process should be performed first. Therefore, we can change the object of study from n ($n \in N$) to r ($r \in N_o$), And this change will make no difference on the nature of the 3x+1 problem.
According to the concept of deevenization, the 3x+1 problem can be stated as: start from any positive odd number r, repeat the operations of multiplying 3, adding 1, and deevenization, we are bound to obtain the odd

number 1 in finite steps.
Obviously, when we record the results of multiplying 3, adding 1, and deevenization successively, we are bound to obtain a sequence (or series) of which all terms are positive odd numbers. This sequence is the all-odd 3x+1 sequence.
Definition 1: If the sequence E satisfies the follow relation:

$$e_1 \in N_o,\quad e_2 = \beta(3e_1+1), \ldots, e_{n+1} = \beta(3e_n+1), \ldots,$$

then E is called an all-odd 3x+1 sequence.
Example 1: Suppose the first term of the all-odd 3x+1 sequence E is $e_1=11$, then we have
$e_2 = \beta(3e_1+1) = \beta(3\times11+1) = \beta(2\times17) = 17$;
$e_3 = \beta(3e_2+1) = \beta(3\times17+1) = \beta(2^2\times13) = 13$;
$e_4 = \beta(3e_3+1) = \beta(3\times13+1) = \beta(2^3\times5) = 5$;
$e_5 = \beta(3e_4+1) = \beta(3\times5+1) = \beta(2^4\times1) = 1$;
$e_6 = \beta(3e_5+1) = \beta(3\times1+1) = \beta(2^2\times1) = 1$;
……
That is, the all-odd 3x+1 sequence E is：11，17，13，5，1，1，…
Please note that e_5，e_6 and their successors repeat the pattern: 1 times 3 plus1 equals 4, 4 being divided by 2^2 equals 1.

It is easy to see that from the fifth term onward, the successors of the above all-odd 3x+1 sequence E are all 1.

When the 3x+1 problem is transformed into the all-odd 3x+1 sequence, the key to solve it lies in proving the proposition: all of the all-odd 3x+1 sequences are bound to have a term valued 1.
Definition 2: If term b_i and term $b_j (i \neq j)$ in the sequence B equals each other, then b_i and b_j are called the equal terms in the sequence B; if the sequence B has equal terms, then B is a sequence with equal terms.
From this definition we learn that if the term b_i is an equal term in the sequence B, then there must exists a term b_j $(i \neq j)$ such that $b_i = b_j$.

2. Equations with equal terms

Now, we study the sufficient and necessary condition for an all-odd 3x+1 sequence to have equal terms. For readability, we give necessary condition and sufficient condition separately as two theorems. First, we give the following definition.
Definition 3: suppose $a_0, a_1, \ldots, a_k \in N_o$, and

$$a_1 = (3a+1)/2^{i_1},\ a_2 = (3a_1+1)/2^{i_2}, \ldots, a_k = (3a_{k-1}+1)/2^{i_k}.$$

Then we call $i_1, i_2, \ldots, i_k$ the k successive exponents of the term a_0 of the all-odd 3x+1 sequence, k successive exponent of a_0 for short .
Example 2: (i) Find the 3 successive exponents of 3; (ii) Find the k successive exponents of 1.

Solution: (i) Let $a=3$, then, $a_1 = (3a+1)/2^{i1} = (3\times3+1)/2^1 = 5 \in N_o$. Thus, $i_1 = 1$.
Likewise, $i_2 = 4$, $i_3 = 2$.
(ii) Similar to (i) we can find the k successive exponents of 1 as : $i_1 = i_2 = \cdots = i_k = 2$.
From Example 2 we learn that, when an odd number is given, its k successive exponents are also given at the same time. And the k successive exponents of any odd number a are all positive numbers. As all of the terms of the all-odd 3x+1 sequences are odd numbers, we have:
Fact 1: The k successive exponents of all of the terms of the all-odd 3x+1 sequences are positive integers.
(In Note 2 the importance of Fact 1 is discussed).
Theorem 1: Suppose the k successive exponents of the ith term e_i of an all-odd 3x+1 sequence E are $i_1, i_2, \ldots, i_k$, if $e_i = e_{i+k}$, then

$$e_i = (3^{k-1} + 3^{k-2}\cdot2^{i_1} + \cdots + 3\cdot2^{i_1+i_2+\cdots+i_{k-2}} + 2^{i_1+i_2+\cdots+i_{k-1}})/(2^{i_1+i_2+\cdots+i_k} - 3^k)\ . \qquad (1)$$

Proof: From the definition of the k successive exponents, we know

$$e_{i+1} = (3e_i+1)/2^{i_1} \qquad (2)$$

$$e_{i+2} = (3e_{i+1}+1)/2^{i_2} \qquad (3)$$

……

$$e_{i+k} = (3e_{i+k-1}+1)/2^{i_k}$$

Substituting (2) into (3), we obtain:

$$e_{i+2} = (3^2 e_i + 3 + 2^{i_1})/2^{i_1+i_2}$$

Likewise,

$$e_{i+3} = (3^3 e_i + 3^2 + 3\cdot2^{i_1} + 2^{i_1+i_2})/2^{i_1+i_2+i_3}$$

……

$e_{i+k}=(3^k e_i +3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}})/ 2^{i_1+i_2+\cdots+i_k}$.

From $e_i=e_{i+k}$, we know

$e_i=(3^k e_i +3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}})/ 2^{i_1+i_2+\cdots+i_k}$.

$2^{i_1+i_2+\cdots+i_k} e_i=3^k e_i +3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}}$.

$2^{i_1+i_2+\cdots+i_k} e_i-3^k e_i=3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}}$.

$e_i=(3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}}) /(2^{i_1+i_2+\cdots+i_k}-3^k)$.

. Q.E.D.

Theorem 2: If term e_i in an all-odd 3x+1 sequence E satisfies the following relation:

$e_i=(3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}})/(2^{i_1+i_2+\cdots+i_k}-3^k)$

where i_1, i_2, …, i_k are the k successive exponents of e_i , then $e_i=e_{i+k}$.

Proof: From

$e_i=(3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}})/(2^{i_1+i_2+\cdots+i_k}-3^k)$

we obtain

$(2^{i_1+i_2+\cdots+i_k}-3^k) e_i=3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}}$

$2^{i_1+i_2+\cdots+i_k} e_i-3^k e_i=3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}}$

$2^{i_1+i_2+\cdots+i_k} e_i=3^k e_i +3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}}$

$e_i=(3^k e_i +3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}})/ 2^{i_1+i_2+\cdots+i_k}$. (4)

Besides, because the sequence E is an all-odd 3x+1 sequence, and i_1, i_2, …, i_k are the k successive exponents of e_i (That is, i_1, i_2,…,i_k are the numbers of factor 2 in $3e_i+1$, $3e_{i+1}+1$,…, $3e_{i+k-1}+1$ respectively)， we obtain the following k equations,

$e_{i+1}=(3e_i +1) / 2^{i_1}$

$e_{i+2}=(3e_{i+1}+1) / 2^{i_2}$

……

$e_{i+k}=(3e_{i+k-1}+1) / 2^{i_k}$

Similar to the inference in Theorem 1, we obtain

$e_{i+k}=(3^k e_i +3^{k-1}+3^{k-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}})/ 2^{i_1+i_2+\cdots+i_k}$. (5)

From (4) and (5), we obtain

$e_i=e_{i+k}$. Q.E.D.

Now, we use the symbol "x" to replace e_i. Thus, formula (1) becomes formula (6):

$x=(3^{n-1}+3^{n-2}\cdot 2^{i_1}+\ldots+3\cdot 2^{i_1+i_2+\cdots+i_{n-2}}+2^{i_1+i_2+\cdots+i_{n-1}}) /(2^{i_1+i_2+\cdots+i_n}-3^n)$ (6)

From Theorem 1 and Theorem 2 we can obtain Corollary 1 directly.

Corollary 1: An all-odd 3x+1 sequence E has an equal term x if and only if x is given by formula (6).

Corollary 1 gives the sufficient and necessary condition for any all-odd 3x+1 sequence to have equal terms. It is correct. But it is not clearly expressed. So, we make it more clear by the following statements.

Because (6) is an equation, we can view it as an equation with x and i_1, i_2, …, i_k as its variables (called the equation with equal terms). And we call the solution with $x\in N_o$ and i_1, i_2,…,i_k being the k successive exponents of x the characteristic solution.

From Theorem 1 we know that if any (or every) all-odd 3x+1 sequence E has an equal term x, then x can necessarily be expressed by the equation (6). This means that if any all-odd 3x+1 sequence E has an equal term, then the equation with equal terms has a characteristic solution. Likewise, Theorem 2 tells us that if the equation with equal terms has a characteristic solution, then any all-odd 3x+1 sequence E has an equal term. Thus, we have Corollary 2

Corollary 2: The sufficient and necessary condition for any all-odd 3x+1 sequence to have an equal term is that the equation with equal terms has a characteristic solution (and x in the characteristic solution is the equal term).

(If you doubt Corollary 2, please refer to "The discussions on the solving of application problems by listing equations" in Note 1)

Obviously, Corollary 1 and Corollary 2 are equivalent, but Corollary 2 is more clear.

It is worth mentioning that the equation with equal terms is similar to the ratio of the circumference of a circle to its diameter $\pi=c/d$ (c is the circumference, d is the diameter of a circle). The most important parameter of a circle is its radius r. But π is irrelevant with r. Therefore, the ratio of the circumference that would have been of certain circle to its diameter becomes the ratio of the circumferences of all circles to their diameters. Similarly, the most important parameter of a sequence is its terms. But the equation with equal terms is irrelevant with the terms of any all-odd 3x+1 sequence. Therefore, whether the equations with equal terms have characteristic solutions that would have been the decision condition of whether certain all-odd 3x+1 sequence has an equal term become the decision condition of whether all of the all-odd 3x+1 sequences have equal terms. This result profoundly reveals the importance of Corollary 2, and paves the way for solving the 3x+1 problem.

3. The final proofs

We know that, the range of the function $f(i_1,\ldots, i_n)=(3^{n-1}+3^{n-2}\cdot 2^{i_1}+\ldots+3\cdot 2^{i_1+i_2+\cdots+i_{n-2}}+2^{i_1+i_2+\cdots+i_{n-1}}) /(2^{i_1+i_2+\cdots+i_n}-3^n)$ is an infinite range. But it worth notice that in this range there is only one element which is a positive integer. That is,

Lemma 1. When $f(i_1,\ldots, i_n)\in N$, $f(i_1,\ldots, i_n)=1$ uniquely. Or, $f(i_1,\ldots, i_n)=1\in N$ uniquely.

Proof: We use mathematical induction.

Step 1: Verify that, when $f(i_1)\in N$, $f(i_1)=1$ uniquely. Because $f(i_1)=1 /(2^{i_1}-3)$, there is and only is $i_1=2$ such that $f(i_1)=1$. That is $f(i_1)=1\in N$ uniquely. Hence when $n=1$ Lemma 1 holds.

In order not to confuse with the result inferred by Step 3, here we first point out the relevance between $f(i_1,\ldots, i_{k+1})$ and $f(i_1,\ldots, i_k)$.

$$f(i_1,\ldots, i_k)=(3^{k-1}+3^{k-2}\cdot 2^{i_1}+\ldots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}}) /(2^{i_1+i_2+\cdots+i_k}-3^k) \qquad (7)$$

Therefore,

$$3^{k-1}+3^{k-2}\cdot 2^{i_1}+\ldots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}}=(2^{i_1+i_2+\cdots+i_k}-3^k)\cdot f(i_1,\ldots, i_k) \qquad (8)$$

Besides, $f(i_1,\ldots, i_{k+1})=(3^k+3^{k-1}\cdot 2^{i_1}+\ldots+3\cdot 2^{i_1+i_2+\cdots+i_{k-1}}+2^{i_1+i_2+\cdots+i_k}) /(2^{i_1+i_2+\cdots+i_{k+1}}-3^{k+1})$, thus,

$$f(i_1,\ldots, i_{k+1})=(3(3^{k-1}+3^{k-2}\cdot 2^{i_1}+\ldots+3\cdot 2^{i_1+i_2+\cdots+i_{k-2}}+2^{i_1+i_2+\cdots+i_{k-1}})+2^{i_1+i_2+\cdots+i_k}) /(2^{i_1+i_2+\cdots+i_{k+1}}-3^{k+1}) \ (9)$$

Substituting formula (8) into formula (9) we obtain

$$f(i_1,\ldots, i_{k+1})=(3(2^{i_1+i_2+\cdots+i_k}-3^k)\cdot f(i_1,\ldots, i_k)+2^{i_1+i_2+\cdots+i_k})/(2^{i_1+i_2+\cdots+i_{k+1}}-3^{k+1}) \qquad (10)$$

(The above process shows that, when $f(i_1,\ldots, i_k)$ is given by formula (7), $f(i_1,\ldots, i_{k+1})$ in formula (10) and formula (9) is the same function. It worth notice that, formula (10) let us see that $f(i_1,\ldots, i_k)$ is a "component" of $f(i_1,\ldots, i_{k+1})$. The relationship between $f(i_1,\ldots, i_{k+1})$ and $f(i_1,\ldots, i_k)$ paves the way for Step 3 induction. Therefore, the existence of formula (10) is the fundamental reason for Lemma 1 being able to be proved by mathematical induction.)

Step 2: Suppose $f(i_1,\ldots, i_k)=1\in N$ uniquely.

Step 3: Prove $f(i_1,\ldots, i_{k+1})=1\in N$ uniquely.

From the induction supposition and formula (10) we know,

$$f(i_1,\ldots, i_{k+1})=(3(2^{i_1+i_2+\cdots+i_k}-3^k)+2^{i_1+i_2+\cdots+i_k}) /(2^{i_1+i_2+\cdots+i_{k+1}}-3^{k+1})$$

$$f(i_1,\ldots, i_{k+1})=(4\cdot 2^{i_1+i_2+\cdots+i_k}-3^{k+1}) /(2^{i_1+i_2+\cdots+i_{k+1}}-3^{k+1})$$

$$f(i_1,\ldots, i_{k+1})=(2^{i_1+i_2+\cdots+i_k+2}-3^{k+1}) /(2^{i_1+i_2+\cdots+i_{k+1}}-3^{k+1}) \qquad (11)$$

From formula (11) we know that, when $i_{k+1}>2$, $f(i_1,\ldots, i_{k+1})<1$. When 当 $i_{k+1}=1$,

$$f(i_1,\ldots, i_{k+1})=(2^{i_1+i_2+\cdots+i_k+2}-3^{k+1}) /(2^{i_1+i_2+\cdots+i_k+1}-3^{k+1})$$

$$=(2\cdot 2^{i_1+i_2+\cdots+i_k+1}-3^{k+1}) /(2^{i_1+i_2+\cdots+i_k+1}-3^{k+1})$$

$$=((2^{i_1+i_2+\cdots+i_k+1}-3^{k+1})+2^{i_1+i_2+\cdots+i_k+1}) /(2^{i_1+i_2+\cdots+i_k+1}-3^{k+1})$$

$$=1+2^{i_1+i_2+\cdots+i_k+1}/(2^{i_1+i_2+\cdots+i_k+1}-3^{k+1})\notin N.$$

Now, there is and only is $i_{k+1}=2$ such that $f(i_1,\ldots, i_{k+1})=1\in N$. That is, $f(i_1,\ldots, i_{k+1})=1\in N$ uniquely.

Thus, we prove that, when $f(i_1,\ldots, i_k)=1\in N$ uniquely, $f(i_1,\ldots, i_{k+1})=1\in N$ uniquely. Q.E.D.

From the above proof it is not hard to see that, $f(i_1,\ldots, i_n)=1$ if and only if $i_1=i_2=\ldots=i_n=2$. Hence we know that the following theorem holds.

Theorem 3: The equations with equal terms:

$$x=(3^{n-1}+3^{n-2}\cdot 2^{i_1}+\cdots+3\cdot 2^{i_1+i_2+\cdots+i_{n-2}}+2^{i_1+i_2+\cdots+i_{n-1}})/(2^{i_1+i_2+\cdots+i_n}-3^n)$$

only has the following characteristic solutions:

$$x=1, \text{ and } i_1=i_2=\cdots=i_n=2. \qquad (12)$$

Theorem 4: 3x+1 problem is true.

We give the following two proofs.

Proof 1: From the proof of Theorem 1 we know that, so long as we suppose an all-odd 3x+1 sequence E has an equal term x and the k successive exponents of x are i_1, i_2, $\cdots$, i_k, then we can obtain the equation with equal terms. From Corollary 1 and Theorem 3 we know that sequence E has the equal term 1. As sequence E is general, we obtain that any all-odd 3x+1 sequence necessarily has the equal term 1. Thus, the theorem holds and 3x+1 problem is true.
Proof 2: Theorem 3 tells us that the equation with equal terms has necessarily the characteristic solution given by formula (12). From Corollary 2 we know that any all-odd 3x+1 sequence necessarily has equal terms. Theorem 3 also tells us that all of the equal terms equal 1. Therefore, any all-odd 3x+1 sequence has a term valued 1. Thus, the theorem holds and the 3x+1 problem is true.
Q.E.D.
Note 1: The answers to some doubts.
This paper is doubted by some reviewers and scholars. Although these doubts are not correct, they reflect some deeply rooted problems. The followings are the answers to these doubts.
(1) The discussions on the solving of application problems by listing equations
Doubt 1: From Theorem 2 we cannot obtain "If the equation with equal terms has a characteristic solution, then the all-odd 3x+1 sequence E has equal terms".
In order for the obtaining of the argument "If the equation with equal terms has a characteristic solution, then the all-odd 3x+1 sequence E has equal terms" more clear, we do not depend on Theorem 2 but on the following listable equations approach to obtain the argument so as to answer the doubt.
Let us investigate the contents of "the list of equations to solve the application problems" in middle school mathematics textbook. The application problems in middle school mathematics textbook have a common feature: for a given problem, one or more equations can always be listed corresponding to it. Therefore, the problem is called the listable equation problem.
When we list a equation corresponding to a application problem (or a listable equation problem), we face two objects. One is the application problem given (called the original problem), the other is the equation listed. Thus, two questions arise: 1. Why can the solution of the original problem be obtained by solving the equation listed? 2. Which solutions of the equation listed are the solutions of the original problem? In order to answer these questions, let us look at an example first.

Problem 1: The sum of the square of an integer and a positive integer equals 3, find out the two numbers.
Solution: Suppose the integer is x, the positive integer is y. According to the problem we can list

$$x^2+y=3 \tag{13}$$

Here, Problem 1 is the original problem, equation (13) is the equation listed. At first glance, they are quite different. Yet, they refer to the same thing. Because x is supposed to be the integer, x^2 can be read as "the square of the integer". Likewise, y can be read as "the positive integer". Thus, equation (13) can be read as "the sum of the square of the integer and the positive integer equals 3". Hence we see that the original problem and the equation listed refer to the same thing, i.e., the equation listed is a re-description of the original problem. Thus we say that, the original problem and the equation listed are "identical" and call this fact "the principle of identity". The principle of identity tells us that, the finding of the solution of the original problem can be realized by finding the solution of the equation listed. This is the fundamental reason for "the list of equation to solve the application problems" being a classical mathematical method. .
Besides, from the angle of equation listed, x and y in (13) can be any real number or complex number. But, in order for equation (13) and Problem 1 to refer to the same thing, x must be the integer, y must be the positive integer. Here, x and y are variables. The conditions set to variables x and y are called constraints. Precisely speaking, only all of the variables satisfy the constraints can the equation listed and the original problem refer to the same thing. Since in this case the two refer to the same thing, the solutions of the two are necessarily the same. The remaining thing for us to do is that we should make sure what kind of solutions are the solutions of the equations listed.
The so-called solution of an equation, formally speaking, is an assignment to its relevant variables. As to the equation listed, if the assignment to each variable satisfies the constraints of the variable in question, then the solution (called the effective solution) is one whose variables of the equation listed satisfy the constraints. Thus, we know that all of the effective solutions of the equation listed are the solutions of the original problem.

Solving equation (13), we can obtain 3 effective solutions of the equation:

$$\begin{matrix} x=1 \\ y=2, \end{matrix}\left\{ \quad \begin{matrix} x=0 \\ y=3, \end{matrix}\left\{ \quad \begin{matrix} x=-1 \\ y=2\ . \end{matrix}\left\{ \right.\right.\right.$$

It is not hard to verify that these effective solutions are all solutions of Problem 1. Obviously, the following conclusion holds:
Conclusion 1: The original problem has a solution, if and only if the equation listed has an effective solution.

According to the above discussion, let us prove the argument "if an equation with equal terms has a characteristic solution, then the all-odd $3x$+1 sequence E has equal terms" again.
Proof: From the proof of Theorem 1 we know that, so long as we suppose that the all-odd $3x$+1 sequence E has the equal term x, and the k successive exponents of x are i_1, i_2,…, i_k, we can obtain the equation with equal terms. This fact tells us that the problem of finding the equal terms of the all-odd 3x+1 sequence E is a listable equation problem (i.e., application problem). At this time, finding the equal terms of the all-odd 3x+1 sequence E is the original problem, and the equation with equal terms is the equation listed. From Conclusion 1 we know that, if the equation with equal terms has an effective solution, then the all-odd 3x+1 sequence E has equal terms. And the characteristic solution of the equation with equal terms is just its effective solution. Thus, the argument holds.
Q.E.D.
Through the above proof (It is in fact the second proof of Corollary 2) we see clearly the sameness between x in the characteristic solution of the equation with equal terms and the equal terms of the all-odd 3x+1 sequence E, that is, if the equation with equal terms has a characteristic solution, then the all-odd 3x+1 sequence E has equal terms. As E generally refers to any all-odd 3x+1 sequence, thus, if the equation with equal terms has a characteristic solution, then any all-odd 3x+1 sequence E has equal terms. It is groundless to doubt this conclusion.

Note 2: "A counterexample" and the principles of supposition
In this note we mainly discuss the problem that the result of our proof of the 3x+1 problem has the "counterexample".
First, we give the principles that must be followed when making supposition:
The principles of supposition: If the proposition A is necessarily true, then we cannot suppose it is false; if the proposition A is necessarily false, then we cannot suppose it is true.
When the proposition A being true (false) is proved logically (or by a fact), then we say that A is necessarily true (false), we also say that A is a conclusion or a theorem. The difference between a proposition and a conclusion lies in that, the former is a judgement whose truth value is unknown, while the latter is a judgement whose truth value is known. Therefore, to the former we can suppose it being true, we can also suppose it being false, while to the latter we cannot make the opposite supposition. (In fact, this is an alternative expression of the principles of supposition)
(Note: Because there is a fundamental difference between a proposition and a conclusion, their expressions should be different. For example, proposition I "6 can be divided by 3 exactly" corresponds to conclusion I "that 6 can be divided by 3 exactly is true"; proposition II "5 can be divided by 3 exactly" corresponds to conclusion II "that 5 can be divided by 3 exactly is false". But people are accustomed to omit "is true" and "is false". They usually express conclusion I as "6 can be divided by 3 exactly", express conclusion II as "5 cannot be divided by 3 exactly". Although these omissions usually do not result in misunderstanding, but we must not confuse the propositions with the conclusions.)
The correctness of the principles of suppositions is self-evident. For example, in the axiomatic system of number theory, we cannot suppose "3+2–5=1" or "3+2–5≠0". If we made such suppositions, we would obtain 0=1，0=n 和 0≠0，$n≠n$ etc., so as to cause an unbearable chaos in the axiomatic system of number theory. Likewise, in the axiomatic system of Euclidean geometry we cannot suppose the sum of the three internal angles of a triangle not to equal to 180°， while in the axiomatic system of non-Euclidean geometry we cannot suppose the sum of the three internal angles of a triangle to equal to 180° etc..
Now, we show some concepts related to the "counterexample".
Definition 4: Suppose the positive integer b satisfies the relations: $a^n \mid b$，$a^{n+1} \nmid b$，$0\le n$. Then,we denote b/a^n as $\beta(b)_a$，i.e., $\beta(b)_a = b/a^n$. We call $\beta()_a$ the de-a-factor operator. And we denote $\beta(\ )_2$ as $\beta(\ )$ briefly.
For example, $\beta(45)_3 = 45/3^2 = 5$, $\beta(40) = 40/2^3 = 5$.
Definition 5: If the sequence E satisfies the relations: $e_1 \in N (3 \nmid e_1)$, $e_2 = \beta(2e_1+1)_3$, …, $e_{n+1} = \beta(2e_n+1)_3$, …, then we call E an anti-3x+1 sequence.
Definition 6: Suppose $3 \nmid a_i \in N$, $0\le i\le k$. And suppose $a_1 = (2a_0+1)/3^{i1}$, $a_2 = (2a_1+1)/3^{i2}$,…, $a_k = (2a_{k-1}+1)/3^{ik}$. Then, we call i_1, i_2,…, i_k the k successive exponents of the term a_0 of the anti-3x+1 sequence, the k successive exponents of a_0 for short.
According to Definition 5, we can obtain the following anti-3x+1 sequences.
Sequence 1: 787,175,13, 1, 1, 1，…
Sequence 2: 10, 7, 5,11,23,47，…
Sequence 1 is a sequence that has the equal term 1, its property is like the all-odd 3x+1 sequences. Sequence 2 is a sequence that tends to infinity without an equal term.
Now, let's calculate the k successive exponents of the first three terms of sequence 1.

The k successive exponents of the first term 787 is: $i_1=2$, $i_2=3$, $i_3=3$, $i_4=\ldots=i_k=1$.
The k successive exponents of the second term 175 is: $i_1'=3$, $i_2'=3$, $i_3'=\ldots=i_k'=1$.
The k successive exponents of the third term 13 is: $i_1''=3$, $i_2''=\ldots=i_k''=1$.
Then let's calculate the k successive exponents of the first three terms of sequence 2.
The k successive exponents of the first term 10 is: $i_1=1$, $i_2=1$, $i_3=\ldots=i_k=0$.
The k successive exponents of the second term 7 is: $i_1'=1$, $i_2'=\ldots=i_k'=0$.
The k successive exponents of the third term 5 is: $i_1''=\ldots=i_k''=0$.
From Definition 3 and Definition 6 we know that the all-odd 3x+1 sequences and the anti-3x+1 sequences have their own k successive exponents.
Definition 7: If in the k successive exponents $i_1, i_2,\ldots, i_k$ of a_0 there exists $i_q=0(1\le q\le k)$, then we call a_0 a term of k successive exponents with zero. If the k successive exponents of a_0 $i_1, i_2,\ldots, i_k\in N$, then we call a_0 a term of k successive exponents without zero. If all of the terms of a sequence are terms of k successive exponents without zero, then we call the sequence a sequence of k successive exponents without zero. If all of the terms of a sequence are terms of k successive exponents with zero, then we call the sequence a sequence of k successive exponents with zero.
Obviously, sequence 1 is a sequence of k successive exponents without zero, sequence 2 is a sequence of k successive exponents with zero. (From Fact 1 we know that the all-odd 3x+1 sequences are sequences of k successive exponents without zero.)
From the relationships of the k successive exponents of various terms of sequence 1 and sequence 2 we can obtain:
Suppose x' is the successor of the term x of an anti-3x+1 sequence, the k+1 successive exponents of x are i_1, i_2, …, i_{k+1}, the k successive exponents of x' are i_1', i_2', …, i_k'. Then, $i_1'=i_2$, $i_2'=i_3$, …, $i_k'= i_{k+1}$. Thus we know that, so long as the k successive exponents of the first term of an anti-3x+1 sequence are the k successive exponents without (or with) zero, then the sequence is a sequence of k successive exponents without (or with) zero. Therefore, an anti-3x+1 sequence is either a sequence of k successive exponents without zero or a sequence of k successive exponents with zero.
Conclusion 2: (From Definition 6 we know that) no term in the anti-3x+1 sequences can be divided exactly by 3.
Conclusion 3: Suppose x' is the successor of the term x *in* an anti-3x+1 sequence. (1). If $1<x\equiv1 \pmod 3$ then $x'<x$, and the 1 successive exponent of x is $i_1\in N$. (2). If $x\equiv2 \pmod 3$ then $x<x'\equiv2 \pmod 3$, and the k successive exponents of x *are* $i_1=i_2=\ldots=i_k=0$.
We only prove (1) of Conclusion 3: From $x=3n+1(0<n)$ we obtain $2x+1=6n+3$. From $x'= (2x+1)/3^{i1}= (6n+3)/3^{i1}$ we know : $1\le i_1\in N$. Also because $x'= (6n+3)/3^{i1}\le 2n+1<3n+1$, $x'<x$. Hence (1) of Conclusion 3 holds. Q.E.D.
From (2) of Conclusion 3 we know that if an anti-3x+1 sequence E has a term $x\equiv2 \pmod 3$ then E is a sequence of k successive exponents with zero. Therefore, if x is a term of a sequence of k successive exponents without zero, then $x\not\equiv 2 \pmod 3$. From Conclusion 2 we can obtain:
Conclusion 4: For any term x of the sequence of the k successive exponents without zero of the anti-3x+1 sequence we have $x\equiv1 \pmod 3$.
The following is the generation process of the "counterexample".
Suppose any anti-3x+1 sequence has an equal term x, and the k successive exponents of x are $i_1, i_2,\ldots, i_k$. Then, similar to the obtaining of formula (6) we can obtain the following equation with equal terms of the anti-3x+1 sequences:

$$x=(2^{k-1}+2^{k-2}\cdot3^{i_1}+\ldots+2\cdot3^{i_1+i_2+\cdots+i_{k-2}}+3^{i_1+i_2+\cdots+i_{k-1}})/(3^{i_1+i_2+\cdots+i_k}-2^k). \qquad (6')$$

We call the solution with $x\in N(3\nmid x)$, and $i_1, i_2, \ldots, i_k$ being the k successive exponents of x the characteristic solution of the anti-3x+1 sequences.
Similar to the obtaining of Corollary 2 we can obtain the following Corollary:
Corollary 2': The sufficient and necessary condition for any anti-3x+1 sequence to have an equal term is formula (6') to have a characteristic solution of the anti-3x+1 sequence.
Similar to the proof of Theorem 3 we can obtain
Theorem 3': The equation with equal terms of the anti-3x+1 sequence

$$x=(2^{n-1}+2^{n-2}\cdot3^{i_1}+\ldots+2\cdot3^{i_1+i_2+\cdots+i_{n-2}}+3^{i_1+i_2+\cdots+i_{n-1}})/(3^{i_1+i_2+\cdots+i_n}-2^n)$$

only has the following characteristic solution
$x=1$, and $i_1=i_2=\ldots=i_n=1$.
Thus, from Corollary 2' and Theorem 3', similar to the proof of Theorem 4, we can obtain:

Theorem 4': Any anti-3x+1 sequence has an equal term 1.
Obviously, the proof process of Theorem 4' is the same as that of Theorem 4. Yet, sequence 2 of the anti-3x+1 sequence do not have equal term 1. Therefore, sequence 2 is the "counterexample" of Theorem 4'. It also means that Theorem 4 "doesn't hold"
The reason for Theorem 4' to have an "counterexample" lies in that "suppose any anti-3x+1 sequence E to have an equal term x" violates the principle of supposition, for before we make the above "supposition" we have already known the existence of sequence 2. Once we discover such sequence as sequence 2 that do not have an equal term, the proposition "any anti-3x+1 sequence has an equal term" is "necessarily false". From the principles of supposition we know that the supposition is not allowed. We cannot "suppose any anti-3x+1 sequence E to have an equal term x", so we cannot obtain formula (6'), so we cannot obtain Theorem 4', so there is no "counterexample".
Contrary to the above supposition, "Suppose that any all-odd 3x+1 sequence E has an equal term x" do not violate the principles of supposition, for up to now we have not proved that the proposition "any all-odd 3x+1 sequence has an equal term" is necessarily false. Therefore, "Suppose that any all-odd 3x+1 sequence E has an equal term x" is rational. And after the supposition the whole proving process conforms to the inference rules, Therefore, Theorem 4 necessarily holds. The following is an evidence of the correctness of the proof of Theorem 4.
Similar to the case of the all-odd 3x+1 sequences, up to now we cannot prove the proposition "any sequence of k successive exponents without zero of the anti-3x+1 sequences has an equal term" to be necessarily false. From the principles of supposition we know that "suppose that any sequence E of k successive exponents without zero of the anti-3x+1 sequences has an equal term x" is rational. And similar to the proving process of Theorem 4, we can obtain:
Theorem 4": Any sequence of the k successive exponents without zero of the anti-3x+1 sequences has the equal term 1.
It is fortunate that Theorem 4" has a more simple proof..
Proof 2 of Theorem 4": From Conclusion 4 we know that if x is a term of the sequence of the k successive exponents without zero of the anti-3x+1 sequences then $x \equiv 1 \pmod 3$. From (1) of Conclusion 3 we know that when $x > 1$, the successor of x is $x' < x$ From this we can obtain that any sequence of k successive exponents without zero necessarily has the minimal term 1. When $x=1$, the successor of x is $x'=1$. Therefore, Theorem 4" holds. Q.E.D.
That Theorem 4" (and Theorem 4) hold and Theorem 4' does not hold fully demonstrate that the principles of supposition is a very important logical rule. Now our discussion is over.
It is worth mentioning that up to now we cannot find a proof similar to the Proof 2 of Theorem 4" to prove that "any all-odd 3x+1 sequence has the equal term 1" hold. (Up to now, the proof given in the main body of the text is the only proof method). This fact not only demonstrates that 3x+1 problem is a very special mathematical problem, but also demonstrates that our proof of 3x+1 problem is unique and novel. Although "unique" and "novel" is one of the main reasons for this paper to be doubted, it is the value of this paper.

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