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Block Extended Trapezoidal Rule of the Second Kind (Etr_2) For the Direct Soltion of Second Order Initial Value Problems of Ordinary Differential Equations

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Abstract
An extended trapezoidal rule of second kind (ETR_2) block method has been developed in this paper. Using the multi-step collocation approach, a single block method for step-number, $k = 3$ has been derived. The error constant of the new block method has been computed and was found to be of fourth order. Some second order initial value problems of ordinary differential equations were solved directly, using the new block method. The absolute errors obtained from the new block method competes favorably to some extend with other comparable block methods. The construction and implementation of the new block method was of few hitches.

Key words: *Block method, Second order Ordinary Differential Equations, Initial Value Problems, Extended Trapezoidal Rule of Second kind.*

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I. Introduction

Ordinary differential equations (ODEs) are found to be useful for mathematical modeling problems in diverse fields, such as engineering, operation research, industrial mathematics, behavioral sciences, artificial intelligence, management and sociology. These types of problems can be formulated either in terms of first-order or higher-order ODEs [1].

In this paper, we consider the development of numerical method for the direct solution of second order initial value problems of ordinary differential equations (ODEs) of the form

$$y'' = f(x, y, y'),\ y(a) = y_0,\ y'(a) = y_1,\ x \in [a, b] \quad (1)$$

A typical approach for solving (1) is by converting the problem to a system of first order ODE and then solving it using numerical method like the *R–K* method or Linear Multistep Methods. The major disadvantages associated with these approach and methods include: Complicated computational work and lengthy execution time [2].

It has been observed that the solutions of second order Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs) have received much attention by researchers. Many of such problems may not be easily solved analytically, hence numerical schemes are developed to approximate the solution [3]. In order alleviate the aforementioned disadvanges associated with other methods of solving second order Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs), block method has been developed based on the nature and type of the differential equation to be solved. Zurni Omar and John Olusola Kuboye [5], Zanariah Abdul Majid *et al*. [6], Kuboye JO. *et al*. [7], are found to be involved in the use of block method for solving second order IVPs of ODEs. In a recent work, L. A. Ukpebor [4] adopted a 4-Point Block Method for Solving Second Order Initial Value Problems in Ordinary Differential Equations.

These diverse methods have their very desirable qualities [8]. However, there is a need for an easier and improve method to some of the existing methods, this work present an Extended Trapezoidal Rules of second kind (ETR_2) type block method for the direct solution of (1) without reduction to first order ODEs. In the next section, the methodology of the work is presented and the derived methods are specified. This paper is presented in the follows order: Section
1 contains the introduction, Section 2 present the methodology, Section 3 shows the basic properties of the method while Section 4 will display the results to the numerical problems considered, and Section 5 concludes this paper.

II. Derivation the method

The constructed block scheme of paper is derived from the Extended Trapezoidal Rule of second kind of the following form:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^2\left(\beta_{k-2} f_{n+k-2} + \beta_{k-1} f_{n+k-1}\right) \tag{2}$$

With h being the step size, $\alpha_j(x)$ and $\beta_j(x)$ being the continuous coefficients of the method.
Using the procedure adopted in G. M. Kumleng *et al*. [10], we obtain the D matrix as:

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_n + h & (x_n+h)^2 & (x_n+h)^3 & (x_n+h)^4 & (x_n+h)^5 \\ 1 & x_n + 2h & (x_n+2h)^2 & (x_n+2h)^3 & (x_n+2h)^4 & (x_n+2h)^5 \\ 1 & x_n + 3h & (x_n+3h)^2 & (x_n+3h)^3 & (x_n+3h)^4 & (x_n+3h)^5 \\ 0 & 0 & 2 & 6x_n + 6h & 12(x_n+h)^2 & 20(x_n+h)^3 \\ 0 & 0 & 2 & 6x_n + 12h & 12(x_n+2h)^2 & 20(x_n+2h)^3 \end{bmatrix}$$

The elements of the inverse of D yield the continuous coeficients $\alpha_j(x), j = 0, \dots, 3$, $\beta_1(x)$ and $\beta_2(x)$ of the method as:

$$\left.\begin{aligned}
\alpha_0 &= \frac{1}{30}\,\frac{(h-x+x_n)(3h-x+x_n)(2h-x+x_n)(5h^2-12hx+12hx_n+3x^2-6xx_n+3x_n^2)}{h^5} \\
\alpha_1 &= \frac{1}{10}\,\frac{(x-x_n)(3h-x+x_n)(2h-x+x_n)(12h^2-10hx+10hx_n+3x^2-6xx_n+3x_n^2)}{h^5} \\
\alpha_2 &= -\frac{1}{10}\,\frac{(x-x_n)(3h-x+x_n)(h-x+x_n)(9h^2-8hx+8hx_n+3x^2-6xx_n+3x_n^2)}{h^5} \\
\alpha_3 &= -\frac{1}{30}\,\frac{(x-x_n)(2h-x+x_n)(h-x+x_n)(4h^2+6hx-6hx_n-3x^2+6xx_n-3x_n^2)}{h^5} \\
\beta_1 &= \frac{1}{10}\,\frac{(x-x_n)(-x+x_n+4h)(3h-x+x_n)(2h-x+x_n)(h-x+x_n)}{h^3} \\
\beta_2 &= \frac{1}{10}\,\frac{(x-x_n)(x+h-x_n)(3h-x+x_n)(2h-x+x_n)(h-x+x_n)}{h^3}
\end{aligned}\right\} \tag{3}$$

Substituting the continuous coefficients (3) into (2) yield the continuous form of the new method

$$
\left.\begin{aligned}
y(x) = {} & \frac{1}{30}\,\frac{\left(h-x+x_n\right)\left(3h-x+x_n\right)\left(2h-x+x_n\right)\left(5h^2-12hx+12hx_n+3x^2-6xx_n+3x_n^2\right)}{h^5}y_n \\
& +\frac{1}{10}\,\frac{\left(x-x_n\right)\left(3h-x+x_n\right)\left(2h-x+x_n\right)\left(12h^2-10hx+10hx_n+3x^2-6xx_n+3x_n^2\right)}{h^5}y_{n+1} \\
& -\frac{1}{10}\,\frac{\left(x-x_n\right)\left(3h-x+x_n\right)\left(h-x+x_n\right)\left(9h^2-8hx+8hx_n+3x^2-6xx_n+3x_n^2\right)}{h^5}y_{n+2} \\
& -\frac{1}{30}\,\frac{\left(x-x_n\right)\left(2h-x+x_n\right)\left(h-x+x_n\right)\left(4h^2+6hx-6hx_n-3x^2+6xx_n-3x_n^2\right)}{h^5}y_{n+3} \\
& +\frac{1}{10}\,\frac{\left(x-x_n\right)\left(-x+x_n+4h\right)\left(3h-x+x_n\right)\left(2h-x+x_n\right)\left(h-x+x_n\right)}{h^3}f_{n+1} \\
& +\frac{1}{10}\,\frac{\left(x-x_n\right)\left(x+h-x_n\right)\left(3h-x+x_n\right)\left(2h-x+x_n\right)\left(h-x+x_n\right)}{h^3}f_{n+2}
\end{aligned}\right\}
$$

(4)

Differentiating the continuous scheme (4) and evaluating at $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}$ and evaluating it's second derivative at $x = x_n, x_{n+3}$, yield the following discrete schemes:

$$
\left.\begin{aligned}
127y_n-216y_{n+1}+81y_{n+2}+8y_{n+3} &= -30hg_n+h^2\left(72f_{n+1}+18f_{n+2}\right) \\
8y_n-39y_{n+1}+24y_{n+2}+7y_{n+3} &= 30hg_{n+1}+h^2\left(18f_{n+1}+12f_{n+2}\right) \\
7y_n+24y_{n+1}-39y_{n+2}+8y_{n+3} &= -30hg_{n+2}+h^2\left(12f_{n+1}+18f_{n+2}\right) \\
8y_n+81y_{n+1}-216y_{n+2}+127y_{n+3} &= 30hg_{n+3}+h^2\left(18f_{n+1}+72f_{n+2}\right) \\
y_{n+1}-2y_{n+2}+y_{n+3} &= \frac{1}{12}h^2\left(f_{n+3}+10f_{n+2}+f_{n+1}\right) \\
y_{n+2}-2y_{n+1}+y_n &= \frac{1}{12}h^2\left(f_{n+2}+10f_{n+1}+f_n\right)
\end{aligned}\right\} \quad (3)
$$

III. Basic properties of the method

3.1 Order of the new block method

To obtain the order of our new method, the method proposed by [18] is adopted.
Hence, the new block method displays the following order:

$[4, 4, 4, 4, 4, 4]^T$ with error constant, $\left[\begin{matrix} -\frac{1}{20} & -\frac{1}{20} & 0 & \frac{1}{6} & \frac{1}{6} & 0 \end{matrix}\right]^T$.

3.2 Consistency of the new block method

By definition, a block method is said to be consistent if it has an order greater or equal to one. Therefore, our method is consistent.

3.3 Zero stability of the new block method

Solving for z as in [8], gives $z_1 = 1, z_2 = 0, z_3 = 0, z_4 = 0, z_5 = 0, z_6 = 0$. This shows that the new block method is zero stable.

3.4 Convergence of the new block method

As it has been expressed in [11], our new block method is convergent.

IV. Implementation of the method

Problem 1

$y'' = -1001\,y' - 1000\,y,\ y(0) = 1, y'(0) = -1, h = 0, 0 \leq x \leq 1$

Exact Solution: $y(x) = e^{-x}$

Problem 2 (see [8])

$$y''=y', \qquad y(0) = 0, y'(0) = -1, h = 0.1$$

Exact Solution: $y(x) = 1 - e^{x}$

Problem 3 (see [13])

$$y'' = -y + 2cosx, \qquad y(0) = 1, y'(0) = 0, 0 \leq x \leq 1$$

Exact solution: $y(x) = cosx + xsinx.$

Problem 4 (see [19])

$$y'' = -100y, \qquad y(0) = 1, y'(0) = 10$$

Exact Solution: $y(x) = sin10x + cos10x$

The results and comparison of error of the problems considered are given in the tables below:

Table 1: Exact solution and computed solution of the new method for Problem 1.

x	Exact Solution	Numerical Solution	Error (New method)
0.1	0.90483741803595957316	0.90483742013240569196	2.09644611880e-9
0.2	0.81873075307798185867	0.81873075103812209200	2.03985976667e-9
0.3	0.74081822068171786607	0.74081823038614197314	9.70442410707e-9
0.4	0.67032004603563930074	0.67032004388459718054	2.15104212020e-9
0.5	0.60653065971263342360	0.60653066227365603140	2.56102260780e-9
0.6	0.54881163609402643263	0.54881163380838467387	2.28564175876e-9
0.7	0.49658530379140951470	0.49658530594448613561	2.15307662091e-9
0.8	0.44932896411722159143	0.44932896207897903236	2.03824255907e-9
0.9	0.40656965974059911188	0.40656966747172670319	7.73112759131e-9
1.0	0.36787944117144232160	0.36787943913724834902	2.03419397258e-9

Table 2: Exact solution and computed solution of the new method for Problem 2, and the errors comparison with [8].

X	Exact Solution	Numerical Solution	Error (New method)	Error ([8])
0.1	-0.105170918	-01051709356	1.76e-8	7.281999e-08
0.2	-0.221402758	-0.2214028008	4.28e-8	3.221988e-07
0.3	-0.349858808	-0.3498588825	7.45e-8	7.844950e-07
0.4	-0.491824698	-0.4918248507	1.527e-7	1.502293e-06
0.5	-0.648721271	-0.6487215175	2.465e-7	2.523574e-06
0.6	-0.822118800	-0.8221191570	3.570e-7	3.904267e-06
0.7	-1.013752707	-1.013753242	5.35e-7	5.709429e-06
0.8	-1.225540928	-1.225541669	7.41e-7	8.012196e-06
0.9	-1.459603111	-1.459604088	9.77e-7	1.089732e-05
1.0	-1.718281828	-1.718283141	1.313e-6	1.446283e-05

Table 3: Exact solution and computed solution of the new method for Problem 3, and the errors comparison with [13].

h	Exact Solution	Numerical Solution	Error (New method) Order 4	Error ([16]) Order **7**
0.01	1.0000499987500069444	1.0000499945834326352	4.16657430e-9	1.428607e-11
0.001	1.0000004999998750000	1.0000004999994583332	4.166668e-13	1.687539e-13
0.0001	1.0000000049999999875	1.0000000049999999458	4.17e-17	5.336842e-12
0.00001	1.0000000000500000000	1.0000000000500000000	0.	5.746437e-11

Table 4: Exact solution and computed solution of the new method for Problem 4, and the errors comparison with [8].

h	x	Exact Solution	Numerical Solution	Error (New method)	Error ([19])
0	0	1	1	0	0
0.001	0.001	1.0099498337508319	1.0099498337508175	1.44e-14	1.00000e-9
	0.002	1.0197986733599109	1.0197986733598778	3.31e-14	3.00000e -9
0.0025	0.0025	1.0246849121904149	1.0246849121867439	3.6710e-12	1.00000e-9
	0.0050	1.0487294296656446	1.0487294296572626	8.3820e-12	3.00000e-9
0.005	0.005	1.0487294296656446	1.0487294294239129	2.417317e-10	0.00000e+00
	0.01	1.0948375819248539	1.0948375813737398	5.511141e-10	2.00000e-9

V. Conclusions

An Extended Trapezoidal Rule of second kind has been used to derive a direct method for solving second order ordinary differential equations. Error constant of the new method is on the lower side when compare with other methods. Consequently, the accuracy of the new method is better than other methods. This work is on a step-number, $k = 3$ only. Therefore, in future work, the value of the step-number, k will be increased in order to observe the accuracy of the results.

References

[1]. Ra'ft Abdelrahim, and Zurni Omar: Direct Solution of Second-Order Ordinary Differential Equation Using a Single-Step Hybrid Block Method of Order Five. Mathematical And Computational Applications, 21(12), (2016).

[2]. Awari Y. S.: 6343 Some Generalized Two-Step Block Hybrid Numerov Method for Solving General Second Order Ordinary Differential Equations without Predictors. Science World Journal. 12(4), (2017).

[3]. E.O. Adeyefa, F.L. Joseph and O.D. Ogwumu: Three-Step Implicit Block Method for Second Order Odes. International Journal of Engineering Science Invention ISSN (Online), 3(2), (2014) 34-38.

[4]. L. A. Ukpebor: A 4-Point Block Method for Solving Second Order Initial Value Problems in Ordinary Differential Equations. American Journal of Computational and Applied Mathematics, 9(3), (2019) 51-56.

[5]. Zurni OMAR and John Olusola KUBOYE: A New Implicit Block Method for Solving Second Order Ordinary Differential Equations Directly. Gazi University Journal of Science (GUJ Sci). 28(4), (2015) 689-694

[6]. Zanariah Abdul Majid, Nur Zahidah Mokhtar and Mohamed Suleiman: Direct Two-Point Block One-Step Method for Solving General Second-Order Ordinary Differential Equations. Hindawi Publishing Corporation Mathematical Problems in Engineering. (2012).

[7]. Kuboye JO, Omar Z., Abolarin OE and Abdelrahim R.: Generalized Hybrid Block Method for Solving Second Order Ordinary Differential Equations Directly. International Publisher of Science, Technology and Medicine, 2018.

[8]. Abhulimen1 C. E., Aigbiremhon A.: Three-Step Block Method for Solving Second Order Differential Equations. International Journal Of Mathematical Analysis And Optimization, (2018) 364–381.

[9]. Kumleng G.M., Chollom J. P, and Omagwu S.: A Class of New Block Generalized Adams Implicit Runge-Kutta Collocation Methods. International Journal of Scientific & Engineering Research. 6 (12), (2015).

[10]. G.M. Kumleng, J.P. Chollom and S. Longwap: A Modified Block Adam Moulton (MOBAM) Method for the Solution of Stiff Initial Value Problems of Ordinary Differential Equations. Research Journal of Mathematics and Statistics. 5(4), (2013) 32-42.

[11]. Kaze Atsi and GM Kumleng: A family of modified backward differentiation formula (BDF) type block methods for the solution of stiff ordinary differential equations. International Journal of Statistics and Applied Mathematics, 5(2), (2020) 09-16.

[12]. J. O. Ehigie, S. A. Okunuga, and A. B. Sofoluwe: 3-Point Block Methods for Direct Integration of General Second-Order Ordinary Differential Equations**.** Hindawi Publishing Corporation Advances in Numerical Analysis, 2011.

[13]. J.O. Kuboye and Zurni Omar: Application of Order Nine Block Method for Solving Second Order Ordinary Differential Equations Directly. Research Journal of Applied Sciences, Engineering and Technology, 11(1), (2015) 19-26.

[14]. D.O. Awoyemi, E. A. Adebile, A.O. Adesanya, T.A. Anake: Modified block method for the direct solution of second order ordinary differential equations. International Journal of Applied Mathematics and Computation, 3(3), (2011) 181-188.

[15]. Zarina Bibi Ibrahim, Nooraini Zainuddin, Khairil Iskandar Othman, Mohamed Suleiman & Iskandar Shah Mohd Zawawi: Variable Order Block Method for Solving Second Order Ordinary Differential Equations. Sains Malaysiana, 48(8), (2019) 1761–1769.

[16]. J. O. Kuboye and Z. Omar: Derivation of a Six-Step Block Method for Direct Solution Of Second Order Ordinary Differential Equations. Mathematical and Computational Applications, 20(3), (2015) 151-159.

[17]. Jimevwo G. Oghonyon, Solomon Adebola Okunuga, Sheila A. Bishop: A 5-Step Block Predictor and 4 Step Corrector Methods for Solving General Second Order Ordinary Differential Equations. Global Journal of Pure and Applied Mathematics, 11(5), (2015) 3847-3862.

Hoo Yann Seong and Zanariah Abdul Majid: Solving second order delay differential equations using direct two-point block method. Ain Shams Engineering Journal, 8 (2017) 59–66.

[18]. Abdu Masanawa Sagir: 2 – Block 3 - Point Modified Numerov Block Methods for Solving Ordinary Differential Equations. World Academy of Science, Engineering and Technology International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering. 7(1), (2013).

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