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Module-valuations and quasi-valuations

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Abstract

Let F be a field with valuation v and valuation domain O_v. In [Sa1] we showed that for any algebra R over O_v there exists a quasi-valuation on R that is induced from R and v. Let $\mathcal{N}$ be an O_v-module. In this paper we discuss module-valuations on N. We construct the filter module-valuation on N and present some connections between filter module-valuations and filter quasi-valuations.

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1 Introduction and some previous results

The study of valuations and their corresponding valuation rings in the scope of commutative algebra has long been an interesting and productive topic. Studying valuations on division rings has also been fruitful, and has been a key ingredient in the construction of various counterexamples, such as Amitsur's construction of noncrossed products division algebras.

Recall that a valuation on a field F is a function $v : F \to \Gamma \cup \{\infty\}$, where Γ is a totally ordered abelian group and where v satisfies the following conditions:

(A1) $v(x) \neq \infty$ iff $x \neq 0$, for all $x \in F$;

(A2) $v(xy) = v(x) + v(y)$ for all $x, y \in F$;

(A3) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in F$.

Throughout the last few decades researchers have generalized the notion of valuation, resulting in some applicable tools. Some notable such generalizations are: the Manis-valuations studied by Knebusch and Zhang (cf. [KZ]), pseudo-valuations studied by Cohn (cf. [Co]) and Huckaba (cf. [Hu]), the value functions associated with Dubrovin valuation rings that were studied by Morandi (cf. [Mor]), and the gauges that were studied by Tignol and Wadsworth (cf. [TW]). Another approach was initiated in 2012 by the author in developing the notion of a quasi-valuation.

Recall (cf. [Sa1, Introduction]) that a quasi-valuation on a ring R is a function $w : R \to M \cup \{\infty\}$, where M is a totally ordered abelian monoid, to which we adjoin an element ∞, which is greater than all elements of M, and where w satisfies the following properties:

(B1) $w(0) = \infty$;

(B2) $w(xy) \geq w(x) + w(y)$ for all $x, y \in R$;

(B3) $w(x+y) \geq \min\{w(x), w(y)\}$ for all $x, y \in R$.

Let v be a valuation on a field F; the corresponding valuation domain of v is the integral domain whose field of fractions is F, defined by

$$O_v = \{x \in F \mid v(x) \geq 0\}.$$

We denote by Γ_v the value group of the valuation v. Likewise, let w be a quasi-valuation on a ring R; the quasi-valuation ring is the subring of R defined by

$$O_w = \{x \in R \mid w(x) \geq 0\}.$$

We denote by M_w the value monoid of the quasi-valuation w, i.e., the submonoid of M generated by $w(R \setminus \{0\})$.

In [Sa1] the theory of quasi-valuations that extend a given valuation was developed. Namely, for a given valuation v on a field F, the corresponding valuation domain O_v, and a finite field extension E/F, we studied quasi-valuations on E extending v on F. We showed that every such quasi-valuation is dominated by some valuation extending v. The most important result presented in [Sa1] was the construction of the filter quasi-valuation, for any algebra over a valuation domain. We showed that if A is an F-algebra and R is an O_v-subalgebra of A lying over O_v then there exists a quasi-valuation on $R \otimes_{O_v} F$ (called the filter quasi-valuation) extending v on F such that the quasi-valuation ring is equal to R (under the identification of R with $R \otimes_{O_v} 1$). In particular, if R is an O_v-subalgebra of A lying over O_v such that $RF = A$ then there exists a quasi-valuation on A extending v on F. It was also shown that there exists a tight connection between the prime spectra of O_v and R.

In this paper the symbol $\subset$ means proper inclusion and the symbol $\subseteq$ means inclusion or equality.

We recall now some basic definitions and the main steps in constructing the filter quasi-valuation introduced in [Sa1]. For further details and proofs, see [Sa1, Section 9].

The first step is to construct a value monoid, constructed from the value group of the valuation. We call this value monoid the cut monoid. We start by reviewing some of the basic notions of Dedekind cuts of ordered sets. For further information on Dedekind cuts see, for example, [FKK] or [Weh].

Definition 1.1. Let T be a totally ordered set. A subset S of T is called initial if for every $\gamma \in S$ and $\alpha \in T$, if $\alpha \leq \gamma$ then $\alpha \in S$. A cut $\mathcal{A} = (\mathcal{A}^L, \mathcal{A}^R)$ of T is a partition of T into two subsets $\mathcal{A}^L$ and $\mathcal{A}^R$, such that, for every $\alpha \in \mathcal{A}^L$ and $\beta \in \mathcal{A}^R$, $\alpha < \beta$.

The set of all cuts $\mathcal{A} = (\mathcal{A}^L, \mathcal{A}^R)$ of the ordered set T contains the two cuts $(\emptyset, T)$ and $(T, \emptyset)$; these are commonly denoted by $-\infty$ and ∞, respectively. However, we do not use the symbols $-\infty$ and ∞ to denote the above cuts since we define a "different" ∞. So, as usual, we adjoin to an element ∞ greater than all cuts.

Given $\alpha \in T$, we denote

$$(-\infty, \alpha] = \{\gamma \in T \mid \gamma \leq \alpha\}$$

and

$$(\alpha, \infty) = \{\gamma \in T \mid \gamma > \alpha\}.$$

One defines similarly the sets $(-\infty, \alpha)$ and $[\alpha, \infty)$.

To define a cut we often write $\mathcal{A}^L = S$, meaning that $\mathcal{A}$ is defined as $(S, T \setminus S)$ when S is an initial subset of T. The ordering on the set of all cuts of T is defined by $\mathcal{A} \leq \mathcal{B}$ iff $\mathcal{A}^L \subseteq \mathcal{B}^L$ (or equivalently $\mathcal{A}^R \supseteq \mathcal{B}^R$). Given $S \subseteq T$, S^+ is the smallest cut $\mathcal{A}$ such that $S \subseteq \mathcal{A}^L$. In particular, for $\alpha \in T$ we have $\{\alpha\}^+ = ((-\infty, \alpha], (\alpha, \infty))$.

Now, let Γ be a totally ordered abelian group; we denote by $\mathcal{M}(\Gamma)$ the set of all cuts of Γ. For subsets $S, S' \subseteq \Gamma$ and $n \in \mathbb{N}$, we define

$$S + S' = \{\alpha + \beta \mid \alpha \in S, \beta \in S'\};$$

$$nS = \{s_1 + s_2 + \ldots + s_n \mid s_1, s_2, \ldots, s_n \in S\}.$$

For $\mathcal{A}, \mathcal{B} \in \mathcal{M}(\Gamma)$, their (left) sum is the cut defined by

$$(\mathcal{A} + \mathcal{B})^L = \mathcal{A}^L + \mathcal{B}^L.$$

The zero in $\mathcal{M}(\Gamma)$ is the cut $((-\infty, 0], (0, \infty))$.

For $\mathcal{A} \in \mathcal{M}(\Gamma)$ and $n \in \mathbb{N}$, the cut $n\mathcal{A}$ is defined by

$$(n\mathcal{A})^L = n\mathcal{A}^L.$$

It is well known (see for example [FKK] or [Weh]) that $(\mathcal{M}(\Gamma), +, \leq)$ is a totally ordered abelian monoid. $\mathcal{M}(\Gamma)$ is called the cut monoid of Γ.

Note that there is a natural monomorphism of monoids $\varphi : \Gamma \to \mathcal{M}(\Gamma)$ defined in the following way: for every $\alpha \in \Gamma$,

$$\varphi(\alpha) = ((-\infty, \alpha], (\alpha, \infty))$$

For $\alpha \in \Gamma$ and $\mathcal{B} \in \mathcal{M}(\Gamma)$, we denote $\mathcal{B} - \alpha$ for the cut $\mathcal{B} + (-\alpha)$ (viewing $-\alpha$ as an element of $\mathcal{M}(\Gamma)$).

The following theorem is of utmost importance to the study of quasi-valuations.

Theorem 1.2. *(cf. [Sa1, Theorem 9.19]) Let v be a valuation on a field F with value group Γ_v. Let O_v be the valuation domain of v and let R be an algebra over O_v. Let $\mathcal{M}(\Gamma_v)$ denote the cut monoid of Γ_v. Then there exists a quasi-valuation $w : R \to \mathcal{M}(\Gamma_v) \cup \{\infty\}$ induced by (R, v).*

The quasi-valuation discussed in Theorem 1.2 is called the filter quasi-valuation induced by (R, v).

2 Module-Valuations

In this section we present a value function on modules over valuation rings, to which we call a module-valuation. We shall present a construction of a module-valuation - the filter module-valuation - constructed in a similar way as the filter quasi-valuation. Then, we present an interesting and quite surprising connection between filter module-valuations and filter quasi-valuations. All modules are assumed to be left modules.

Definition 2.1. Let F be a field with valuation v and valuation ring O_v. Let $\mathcal{N}$ be an O_v-module. A module-valuation on $\mathcal{N}$ (with respect to v) is a function $w : \mathcal{N} \to M \cup \{\infty\}$ where M is a totally ordered abelian monoid such that:

$$w(x_1 + x_2) \geq \min\{w(x_1), w(x_2)\} \text{ for all } x_1, x_2 \in \mathcal{N} \qquad (1)$$

$$w(ax) = v(a) + w(x) \text{ for all } a \in O_v \text{ and } x \in \mathcal{N}. \qquad (2)$$

For w a module-valuation on $\mathcal{N}$ (with respect to v), we define

$$\mathcal{N}_w = \{k \in \mathcal{N} \mid w(k) \geq 0\}.$$

It is easy to see that $\mathcal{N}_w$ is an O_v-submodule of $\mathcal{N}$. We denote by M_w the submonoid of M generated by $w(\mathcal{N} \setminus \{0\})$.

We shall now present the construction of the filter module-valuation.

Let v be a valuation on a field F with value group Γ_v. Let O_v be the corresponding valuation domain, and let $\mathcal{N}$ be an O_v-module. For every $x \in \mathcal{N}$, the O_v-*support* of x in $\mathcal{N}$ is the set

$$S_x^{\mathcal{N}/O_v} = \{a \in O_v | x \in a\mathcal{N}\}.$$

We suppress $\mathcal{N}/O_v$ when it is understood.

Note that $\mathcal{N}$ need not be a ring; thus $x \in a\mathcal{N}$ is not equivalent to $x\mathcal{N} \subseteq a\mathcal{N}$, as in the construction of the filter quasi-valuation (since $x\mathcal{N}$ may not be defined).

Recall that for every $A \subseteq O_v$ we denote $(v(A))^{\geq 0} = \{v(a) \mid a \in A\}$; in particular,

$$(v(S_x))^{\geq 0} = \{v(a) | a \in S_x\};$$

The same proof (see [Sa, Lemma 9.16]) as in the construction of the filter quasi-valuation holds here to show that $(v(S_x))^{\geq 0}$ is an initial subset of $(\Gamma_v)^{\geq 0}$.

So, we define

$$v(S_x) = (v(S_x))^{\geq 0} \cup (\Gamma_v)^{<0};$$

and note that $v(S_x)$ is an initial subset of Γ_v.

Definition 2.2. Let v be a valuation on a field F with value group Γ_v. Let O_v be the corresponding valuation domain and let $\mathcal{N}$ be an O_v-module. Let $\mathcal{M}(\Gamma_v)$ denote the cut monoid of Γ_v. We say that the function $w : \mathcal{N} \to \mathcal{M}(\Gamma_v) \cup \{\infty\}$ is *induced by* $(\mathcal{N}, v)$ if w satisfies the following:

1. $w(x) = (v(S_x), \Gamma_v \setminus v(S_x))$ for all $0 \neq x \in \mathcal{N}$. I.e., $w(x)^L = v(S_x)$;
2. $w(0) = \infty$.

Remark 2.3. Notation as in the previous Definition and let $0 \neq x \in \mathcal{N}$. We note that it is possible to have $v(S_x) = \Gamma_v$ and thus $w(x) = (\Gamma_v, \emptyset)$; for example, take $\mathcal{N} = F$; in this case for every $0 \neq x \in \mathcal{N}$, we have $w(x) = (\Gamma_v, \emptyset)$. On the other hand, by definition, $v(S_x) \supseteq (\Gamma_v)^{<0}$ and $0 \in v(S_x)$; therefore $v(S_x) \supseteq (-\infty, 0]$; i.e., $w(x) \geq 0$. Thus, for w a function induced by $(\mathcal{N}, v)$ and $x \in \mathcal{N}$, we have $w(x) \geq 0$ (recall that by definition $w(0) = \infty$). In particular, w cannot satisfy $w(x)^L = \emptyset$, i.e., $(\emptyset, \Gamma_v) \notin im(w)$.

Lemma 2.4. *Let $\mathcal{N}$ be an O_v-module, let x be a non-zero element of $\mathcal{N}$, and let $a, b \in O_v$ such that $v(b) = v(a)$. Then $a \in S_x$ iff $b \in S_x$.*

Proof. Assume that $a \in S_x$. By assumption $x \neq 0$ and thus $b \neq 0$; hence, $a = bb^{-1}a$. Since $v(b) = v(a)$, we get $b^{-1}a \in O_v$. Therefore, $x = az$ for some $z \in \mathcal{N}$ implies $x = b(b^{-1}a)z \in b\mathcal{N}$.

□

Lemma 2.5. *Let $\mathcal{N}$ be an O_v-module, and let x, y be non-zero elements of $\mathcal{N}$. Then $S_x \subseteq S_y$ iff $v(S_x) \subseteq v(S_y)$.*

Proof. The right to left implication is obvious. For the other direction, assume to the contrary that there exists $a \in S_x \setminus S_y$. By assumption $x \neq 0$ and thus $a \neq 0$. Thus, By Lemma 2.4, for every $b \in O_v$ satisfying $v(b) = v(a)$, one has $b \notin S_y$. Hence, $v(a) \notin v(S_y)$, a contradiction. □

The following proposition holds for arbitrary O_v-modules.

Proposition 2.6. *Let v be a valuation on a field F with value group Γ_v. Let O_v be the corresponding valuation domain and let $\mathcal{N}$ be an O_v-module. Let $\mathcal{M}(\Gamma_v)$ denote the cut monoid of Γ_v. Then, the function $w : \mathcal{N} \to \mathcal{M}(\Gamma_v) \cup \{\infty\}$ induced by $(\mathcal{N}, v)$ satisfies $w(x + y) \geq \min\{w(x), w(y)\}$ for all $x, y \in \mathcal{N}$.*

Proof. By definition, for every $0 \neq x \in \mathcal{N}$, $w(x)^L = v(S_x)$ and $w(0) = \infty$. We prove that w satisfies $w(x + y) \geq \min\{w(x), w(y)\}$ for all $x, y \in \mathcal{N}$. First note that if at least one of them is zero then it is easily seen that $w(x+y) \geq \min\{w(x), w(y)\}$. Also, if $x + y = 0$ then the required inequality is trivial. So, we may assume that x, y and $x+y$ are non-zero. Assume that $w(x) \leq w(y)$, i.e., $v(S_x) \subseteq v(S_y)$. Thus, by Lemma 2.5, $S_x \subseteq S_y$. Now, let

$a \in S_x$; then $x = az$ for some $z \in \mathcal{N}$, and thus $y = az'$ for some $z' \in \mathcal{N}$. Hence,

$$x + y = az + az' = a(z + z') \in a\mathcal{N};$$

i.e., $a \in S_{x+y}$. Therefore $S_x \subseteq S_{x+y}$ and by Lemma 2.5, $v(S_x) \subseteq v(S_{x+y})$; i.e., $w(x+y)^L \supseteq w(x)^L$. Thus,

$$w(x + y) \geq w(x) = \min\{w(x), w(y)\}.$$

□

Remark 2.7. Let $\mathcal{N}$ be a torsion free module over an integral domain C. Let $0 \neq c \in C$, $b \in C$ satisfying $c^{-1}b \in C$; let $x, y \in \mathcal{N}$ and assume $cx = by$. Since

$$by = c(c^{-1}b)y,$$

we may cancel c and conclude that $x = (c^{-1}b)y$. Of course, if $\mathcal{N}$ is not torsion free, then this fact is not valid.

Theorem 2.8. *Notation as in Proposition 2.6, and assume in addition that $\mathcal{N}$ is torsion free over O_v; then*

$$w(cx) = v(c) + w(x)$$

for every $c \in O_v$, $x \in \mathcal{N}$. In other words, w is a module-valuation.

Proof. First note that if $c = 0$ or $x = 0$ then $w(cx) = v(c) + w(x)$ is clear. Now, let $v(a) \in (-\infty, v(c)]$ and $v(b) \in v(S_x)$ for $a, b \in O_v$. We have $c \in aO_v$ and $x \in b\mathcal{N}$; thus $cx \in ab\mathcal{N}$, i.e., $v(a) + v(b) \in v(S_{cx})$. Namely,

$$w(cx) \geq v(c) + w(x).$$

Note that we may assume above that $a, b \in O_v$ since if one of them (or both) is in $F \setminus O_v$ then one can take $a', b' \in O_v$ satisfying $v(a) \leq v(a') \in (-\infty, v(c)]$ and $v(b) \leq v(b') \in v(S_x)$. Thus, by the proof above, $v(a') + v(b') \in v(S_{cx})$. Now, $v(a) + v(b) \leq v(a') + v(b')$ and $v(S_{cx})$ is an initial subset of Γ_v; therefore $v(a) + v(b) \in v(S_{cx})$.

For the other direction we need to show that if $v(b) \in v(S_{cx})$ for $b \in O_v$, then

$$v(b) \in (-\infty, v(c)] + v(S_x).$$

Note that if $v(b) < v(c)$ then clearly $v(b) \in (-\infty, v(c)] + v(S_x)$. (Indeed, $0 \in v(S_x)$ and $(-\infty, v(c)]$ is an initial subset of Γ_v and thus $v(b) \in (-\infty, v(c)]$). Thus, we may assume that $v(b) \geq v(c)$, i.e., $c^{-1}b \in O_v$. Therefore, by the definition of S_{cx} and Remark 2.7, we have

$$b \in S_{cx} \Rightarrow cx \in b\mathcal{N} \Rightarrow x \in c^{-1}b\mathcal{N}.$$

So we have $c^{-1}b \in S_x$, and writing $b = c(c^{-1}b)$, we conclude that

$$v(b) = v(c) + v(c^{-1}b) \in (-\infty, v(c)] + v(S_x).$$

In light of the previous results, we define the filter module-valuation.

Definition 2.9. Let $\mathcal{N}$ be a torsion free O_v-module. The function $w : \mathcal{N} \to \mathcal{M}(\Gamma_v) \cup \{\infty\}$ induced by $(\mathcal{N}, v)$ is called the *filter module-valuation induced by* $(\mathcal{N}, v)$.

In fact, when considering the function induced by an O_v-module and v, the second property of the module-valuation distinguishes between the torsion free O_v-modules and the non-torsion free O_v-modules; more precisely, we have the following proposition.

Proposition 2.10. *Let $\mathcal{N}$ be an O_v-module and let w denote the function induced by $(\mathcal{N}, v)$; Then the following conditions are equivalent:*
(1) $\mathcal{N}$ is torsion free over O_v;
(2) w is a module-valuation;
(3) w satisfies the second axiom of the module-valuation.

Proof. The implication (1) $\Rightarrow$ (2) is by Theorem 2.8. The implication (2) $\Rightarrow$ (3) is trivial. We prove that (3) implies (1). Assume that $\mathcal{N}$ is not torsion free over O_v; then there exist $0 \neq c \in O_v$ and $0 \neq x \in \mathcal{N}$ such that $cx = 0$. Hence

$$\infty = w(0) = w(cx) > v(c) + w(x),$$

since $v(c), w(x) < \infty$.

We note that even in the case in which xy is defined for $x, y \in \mathcal{N}$ and even in the case in which $xy \in \mathcal{N}$, one does not necessarily have $w(xy) \geq w(x) + w(y)$, where w is the filter module-valuation induced by $(\mathcal{N}, v)$, as opposed to a quasi-valuation. The following example demonstrates this situation.

Example 2.11. Let p be a prime number and let k be a positive integer. Let v denote the p-adic valuation on $\mathbb{Q}$ with corresponding valuation domain O_v, and let $\mathcal{N} = \frac{1}{p^k} O_v$. Let w denote the filter module-valuation induced by $(\mathcal{N}, v)$; then for any $a \in O_v$, we have $w(\frac{1}{p^k} \cdot a) = v(a)$. In particular, $w(\frac{1}{p^k}) = 0$, $w(p^k) = 2k$ and $w(1) = k$. So,

$$k = w(1) = w(\frac{1}{p^k} \cdot p^k) \ngeq w(\frac{1}{p^k}) + w(p^k) = 2k.$$

The following remark can be easily deduced by [Sa1, Remark 9.29] (letting the algebra R be a module).

Remark 2.12. Let C be an integral domain, S a multiplicative closed subset of C with $0 \notin S$, and $\mathcal{N}$ a module over C. Then every $x \in \mathcal{N} \otimes_C CS^{-1}$ is of the form $n \otimes \frac{1}{\beta}$ for $n \in \mathcal{N}$ and $\beta \in S$.

We now consider the tensor product $\mathcal{N} \otimes_{O_v} F$ where $\mathcal{N}$ is a torsion free algebra over O_v. Our goal is to construct a module-valuation on $\mathcal{N} \otimes_{O_v} F$ using the filter module-valuation induced by $(\mathcal{N}, v)$ that was constructed earlier.

Remark 2.13. Note that if $\mathcal{N}$ is a torsion free module over O_v, then there is an embedding $\mathcal{N} \hookrightarrow \mathcal{N} \otimes_{O_v} F$; we shall see that in this case the module-valuation on $\mathcal{N} \otimes_{O_v} F$ extends the module-valuation on $\mathcal{N}$.

Lemma 2.14. *Let v, F, Γ_v and O_v be as in Proposition 2.6. Let $\mathcal{N}$ be a torsion free module over O_v, S a multiplicative closed subset of O_v with $0 \notin S$, and let $w : R \to M \cup \{\infty\}$ be any module-valuation where M is any totally ordered abelian monoid containing Γ_v. Then there exists a module-valuation W on $\mathcal{N} \otimes_{O_v} O_v S^{-1}$, extending w on $\mathcal{N}$ (under the identification of $\mathcal{N}$ with $\mathcal{N} \otimes_{O_v} 1$), with value monoid $M \cup \{\infty\}$.*

Proof. In view of Remark 2.12, let $x \otimes \frac{1}{\beta} \in \mathcal{N} \otimes_{O_v} O_v S^{-1}$ and define

$$W(x \otimes \frac{1}{\beta}) = w(x) - v(\beta) \ (= w(n) + (-v(\beta)) \ ).$$

Note that W is well defined since if $x \otimes \frac{1}{\beta} = y \otimes \frac{1}{\delta}$ then there exists a non-zero $\alpha \in O_v$ such that $\alpha(\delta x - \beta y) = 0$ and thus, since $\mathcal{N}$ is torsion free, $\delta x = \beta y$. Therefore, by our assumption that w is a module-valuation (and thus $w(cz) = v(c) + w(z)$ for every $c \in O_v$ and $z \in \mathcal{N}$), we have

$$v(\delta) + w(x) = v(\beta) + w(y);$$

i.e., $W(x \otimes \frac{1}{\beta}) = W(y \otimes \frac{1}{\delta})$.

We prove now that W satisfies the axioms of a module-valuation. First note that $W(0 \otimes 1) = w(0) - v(1) = \infty$. Next, note that for every two elements $x \otimes \frac{1}{\beta}, y \otimes \frac{1}{\delta} \in \mathcal{N} \otimes_{O_v} O_v S^{-1}$, assuming that $v(\beta) \leq v(\delta)$, we have $\delta = \alpha\beta$ for some $\alpha \in O_v$ and thus

$$x \otimes \frac{1}{\beta} = x \otimes \frac{\alpha}{\alpha\beta} = \alpha x \otimes \frac{1}{\delta}.$$

Therefore, we may assume that we have elements $x \otimes \frac{1}{\delta}, y \otimes \frac{1}{\delta} \in \mathcal{N} \otimes_{O_v} O_v S^{-1}$; then

$$W(x \otimes \frac{1}{\delta} + y \otimes \frac{1}{\delta}) = W((x+y) \otimes \frac{1}{\delta})$$
$$= w(x+y) - v(\delta) \geq \min\{w(x), w(y)\} - v(\delta)$$
$$= \min\{W(x \otimes \frac{1}{\delta}), W(y \otimes \frac{1}{\delta})\}.$$

Finally, note that $\mathcal{N}$ embeds in $\mathcal{N} \otimes_{O_v} O_v S^{-1}$ and for all $x \in \mathcal{N}$, we have

$$W(x \otimes 1) = w(x) - v(1) = w(x).$$

Theorem 2.15. *Let v, F, Γ_v, O_v and $\mathcal{M}(\Gamma_v)$ be as in Proposition 2.6. Let $\mathcal{N}$ be a torsion free O_v-module and let w denote the filter module-valuation induced by $(\mathcal{N}, v)$; then there exists a module-valuation W on $\mathcal{N} \otimes_{O_v} F$, extending w on $\mathcal{N}$, with value monoid $\mathcal{M}(\Gamma_v) \cup \{\infty\}$ and $N_W = \mathcal{N} \otimes_{O_v} 1$.*

Proof. Apply the previous Lemma by taking $S = O_v \setminus \{0\}$, and thus $F = O_v S^{-1}$, to get a module-valuation W on $\mathcal{N} \otimes_{O_v} F$, extending w on $\mathcal{N}$, with value monoid $\mathcal{M}(\Gamma_v) \cup \{\infty\}$. Note that by Remark 2.12, by taking $C = O_v$ and $S = O_v \setminus \{0\}$, every element of $\mathcal{N} \otimes_{O_v} F$ is of the form $x \otimes \frac{1}{\beta}$ for $x \in \mathcal{N}$ and $\beta \in O_v$. So, W is given by

$$W(x \otimes \frac{1}{\beta}) = w(x) - v(\beta)$$

for every $x \otimes \frac{1}{\beta} \in \mathcal{N} \otimes_{O_v} F$.

Finally, note that for every element $x \in \mathcal{N}$, we have, by Remark 2.3, that $w(x) \geq 0$ and thus $W(x \otimes 1) = w(x) \geq 0$. On the other hand, let $x \otimes \frac{1}{\beta} \in \mathcal{N} \otimes_{O_v} F$ with $W(x \otimes \frac{1}{\beta}) \geq 0$; then $w(x) \geq v(\beta)$; i.e., $\beta \in S_x$ and thus one can write $x = \beta y$ for some $y \in \mathcal{N}$. Hence,

$$x \otimes \frac{1}{\beta} = \beta y \otimes \frac{1}{\beta} = y \otimes 1.$$

Consequently, $\mathcal{N}_W = \mathcal{N} \otimes 1$. $\square$

Note: W as described in the previous Theorem will also be called the filter module-valuation induced by $(\mathcal{N}, v)$

Remark 2.16. Let $\mathcal{N}$ be an O_v-module and let $0 \neq x \in \mathcal{N}$. By Remark 2.3, $v(S_x) \supseteq (-\infty, 0]$. Thus, for every $x \otimes \frac{1}{\beta} \in \mathcal{N} \otimes_{O_v} F$ where $x \neq 0$,

$$(W(x \otimes \frac{1}{\beta}))^L = v(S_x) + (-\infty, -v(\beta)] \supseteq (-\infty, -v(\beta)],$$

i.e., $(W(x \otimes \frac{1}{\beta}))^L \neq \emptyset$. Note that $(W(0 \otimes \frac{1}{\beta})) = \infty - v(\beta) = \infty$. Hence,

$$(\emptyset, \Gamma_v) \notin im(W).$$

Lemma 2.17. *Let $\mathcal{N}_1 \subseteq \mathcal{N}_2$ be two O_v-modules. Let $w_{\mathcal{N}_1}$ and $w_{\mathcal{N}_2}$ denote the filter module-valuations induced by $(\mathcal{N}_1, v)$ and $(\mathcal{N}_2, v)$, respectively. Then for every $x \in \mathcal{N}_1$, $w_{\mathcal{N}_1}(x) \leq w_{\mathcal{N}_2}(x)$. Consequently, for any $y \otimes \frac{1}{\beta} \in \mathcal{N}_1 \otimes_{O_v} F$, one has $W_{\mathcal{N}_1}(y \otimes \frac{1}{\beta}) \leq W_{\mathcal{N}_2}(y \otimes \frac{1}{\beta})$.*

Proof. Let $a \in S_x^{\mathcal{N}_1}$; then $x \in a\mathcal{N}_1 \subseteq a\mathcal{N}_2$. Thus, $a \in S_x^{\mathcal{N}_2}$. So, $S_x^{\mathcal{N}_1} \subseteq S_x^{\mathcal{N}_2}$ and hence $v(S_x^{\mathcal{N}_1}) \subseteq v(S_x^{\mathcal{N}_2})$; i.e., $w_{\mathcal{N}_1}(x) \leq w_{\mathcal{N}_2}(x)$.

Example 2.18. Notation as in Example 2.11. Note that for any $a \in O_v$, we have $w(\frac{1}{p^k} \cdot a) = v(a)$. It is also easy to see that $\mathcal{N} \otimes_{O_v} \mathbb{Q} \approx \mathbb{Q}$ and by the definition of the filter module-valuation we have $W(x) = k + v(x)$ for all $x \in \mathbb{Q}$.

From now on we let A denote an F-central simple algebra, where F is a field with valuation v and a corresponding valuation domain O_v. By Wedderburn's Theorem $A \cong M_n(D)$ where D is a division ring finite dimensional over F. Let R denote an O_v-subalgebra of A such that $FR = A$ and R is lying over O_v. Such R is called an O_v-nice subalgebra of A; see [Sa2], [Sa3] and [sa4] for more information on such algebras. By [Sa1, Theorem 9.34], there exists the filter quasi-valuation on RF extending v on F; we denote it by w. We also denote by e_{ij} the matrix units and for $x \in A$ we denote by $(x)_{ij}$ the element in the i'th row and j'th column of x.

We define, for every $1 \leq i, j \leq n$,

$T_{ij} = \{(x)_{ij} \mid x \in R\}$; and

$R_{ij} = \{(x)_{ij} \mid x \in R \text{ and } (x)_{kl} = 0 \text{ whenever } k \neq i \text{ or } l \neq j\}$.

It is not difficult to see that, for every $1 \leq i, j \leq n$, T_{ij} and R_{ij} are O_v-submodules of D; Moreover, $R_{ij} \subseteq T_{ij}$. Thus there exist filter module-valuations $w_{R_{ij}}$ on $R_{ij} \otimes_{O_v} F$ and $w_{T_{ij}}$ on $T_{ij} \otimes_{O_v} F$ induced by (R_{ij}, v) and (T_{ij}, v), respectively. By Lemma 2.17, $w_{R_{ij}} \leq w_{T_{ij}}$.

Note that $M_w, M_{w_{T_{ij}}}$ and $M_{w_{R_{ij}}}$ are contained in $\mathcal{M}(\Gamma_v)$.

Lemma 2.19. *Notation as above and let $x \in R$. Then for all $1 \leq i, j \leq n$,*

$$w((x)_{ij}e_{ij}) = w_{R_{ij}}((x)_{ij}).$$

Proof. We prove that $a \in S^R_{(x)_{ij}e_{ij}}$ iff $a \in S^{R_{ij}}_{(x)_{ij}}$. Now, $a^{-1}(x)_{ij}e_{ij} \in R$ iff $a^{-1}(x)_{ij} \in R_{ij}$ which is equivalent to $a \in S^{R_{ij}}_{(x)_{ij}}$. So, $w((x)_{ij}e_{ij})^L = w_{R_{ij}}((x)_{ij})^L$ and the lemma is proved.

□

Lemma 2.20. *Notation as above and let $x \in R$. Then*

$$w(x) \geq \min_{1 \leq i,j \leq n} \{w_{R_{ij}}((x)_{ij})\}.$$

Proof. We write $x = \sum_{1 \leq i,j \leq n} (x)_{ij}e_{ij}$ and recall that w is a quasi-valuation. Thus,

$$w(x) = w(\sum_{1 \leq i,j \leq n} (x)_{ij}e_{ij}) \geq \min_{1 \leq i,j \leq n} \{w((x)_{ij}e_{ij})\}.$$

By the previous Lemma, $w((x)_{ij}e_{ij}) = w_{R_{ij}}((x)_{ij})$ and we are done.

□

Lemma 2.21. *Let $x \in R$. Then, for all $1 \leq i, j \leq n$,*

$$w(x) \leq w_{T_{ij}}((x)_{ij}).$$

Proof. By the definition of the T_{ij}'s, $a^{-1}x \in R$ implies $a^{-1}(x)_{ij} \in T_{ij}$ for all $1 \leq i, j \leq n$. Thus, $a \in S_x^R$ implies $a \in S_{(x)_{ij}}^{T_{ij}}$ and $w(x)^L \leq w_{T_{ij}}((x)_{ij})^L$ for all $1 \leq i, j \leq n$.

□

Lemma 2.22. *Let $x \in R$. Then*

$$w(x) \leq \min_{1 \leq i,j \leq n} \{w_{T_{ij}}((x)_{ij})\}.$$

Proof. Follows at once from the previous Lemma.

□

Proposition 2.23. *Let $x \in R$. Then*

$$\min_{1 \leq i,j \leq n} \{w_{ij}^{R_{ij}}((x)_{ij})\} \leq w(x) \leq \min_{1 \leq i,j \leq n} \{w_{ij}^{T_{ij}}((x)_{ij})\}.$$

Proof. By Lemma 2.20 and Lemma 2.22.

□

We shall now see that if R is of a certain type then one can obtain the filter quasi-valuation w induced by (R, v) in terms of appropriate filter module-valuations.

Theorem 2.24. *If R is of the form $R = \oplus_{i,j} S_{ij} e_{ij}$ where each S_{ij} is an O_v-submodule of D. Then*

$$w(x) = \min_{1 \leq i,j \leq n} \{w_{ij}^{S_{ij}}((x)_{ij})\}.$$

Proof. By assumption, for every $1 \leq i, j \leq n$, $R_{ij} = S_{ij} = T_{ij}$. Therefore, the theorem is easily deduced by Proposition 2.23.

□

The surprising thing is that although the w_{ij}'s need not be quasi-valuations (namely, the multiplication property is not necessarily satisfied) their minimum is actually a quasi-valuation.

References

[Co] P. M. Cohn, *An Invariant Characterization of Pseudo-Valuations*, Proc. Camb. Phil. Soc. 50 (1954), 159-177.

[FKK] A. Fornasiero, F.V. Kuhlmann and S. Kuhlmann, *Towers of complements to valuation rings and truncation closed embeddings of valued fields*, J. Algebra 323 (2010), no. 3, 574-600.

[Hu] J. A. Huckaba, *Extensions of Pseudo-Valuations*, Pacific J. Math. Volume 29, Number 2 (1969), 295-302.

[KZ] M. Knebusch and D. Zhang, *Manis Valuations and Prüfer Extensions*, Springer-Verlag, Berlin, 2002.

[MH] M. Mahadavi-Hezavehi, *Matrix Pseudo-Valuations on Rings and their Associated Skew Fields*, Int. Math. J. 2 (2002), no. 1, 7–30.

[Mo] P. J. Morandi, *Value functions on central simple algebras*, Trans. Amer. Math. Soc., 315 (1989), 605-622.

[Sa1] S. Sarussi, *Quasi-valuations extending a valuation*, J. Algebra 372 (2012), 318-364.

[Sa2] S. Sarussi, *Quasi-valuations and algebras over valuation domains*, Comm. Algebra, (2019), DOI: 10.1080/00927872.2018.1522322.

[Sa3] S. Sarussi, *Extensions of integral domains and quasi-valuations*, Comm. Algebra, (2019), DOI: 10.1080/00927872.2019.1677695

[Sa4] S. Sarussi, *Alexandroff Topology of Algebras Over an Integral Domain*, Mediterr. J. Math., (2020), https://doi.org/10.1007/s00009-020-1502-z 1660-5446/20/020001-17

[TW] J.-P. Tignol and A.R. Wadsworth, *Value functions and associated graded rimgs for semisimple algebras*, Trans. Amer. Math. Soc. 362 (2010), 687–726.

[Weh] F. Wehrung, *Monoids of intervals of ordered abelian groups*, J. Algebra 182 (1996), no. 1, 287-328.

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