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Null - Controllablility of Non-Linear Control Systems By Brouwer's Fixed Point Theorem.

[1]E.C. Mbah *, [2] S. E. ANIAKU, [3] C.C Asogwa, [4] P.U Nwokoro
[1] *Department of Mathematics, University of Nigeria, Nsukka, Nigeria*
[2] *Department of Mathematics, University of Nigeria,Nsukka, Nigeria.*
[3] *Department of Mathematics, University of Nigeria,Nsukka, Nigeria.*
[4] *Department of Mathematics, University of Nigeria,Nsukka, Nigeria.*

ABSTRACT*: This article is aimed at establishing the null - controllability of a non linear control systems in Euclidean space* $\mathbb{R}$. *Sufficient conditions for a non-linear control system to be steered to a zero target by means of Brouwer's Fixed Point Theorem were developed. Thus, by the application of Brouwer's Fixed Point Theorem, it was proved that the non linear control systems*

$$\dot{x}(t) = A(t)x(t) + f(t, x(t), u(t)), \quad x(0) = x_0.$$

is Euclidean null-Controllable.

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I. Introduction

In this paper,we develope the conditions under which the non - linear control systems .

$$\dot{x}(t) = A(t)x(t) + f(t, x(t), u(t)), \quad x(0) = x_0 \qquad (1.1)$$

is Euclidean null-Controllable. Here if we take $\mathbb{R} = (-\infty, \infty)$, then $\mathbb{R}^n$ is called $n-$dimensional Euclidean space. So, $\mathbb{R}^+ = (0, \infty) = I$, will be an interval of real line $\mathbb{R}$.

In $(1.1), x \in \mathbb{R}^n, A(t)$ is $n \times n$ matrix function on $\mathbb{R}^+$. The control function $u(t)$ is an $m-$ vector valued measurable function $u: (0, \infty) \longrightarrow \Omega$, which is forced to lie in a convex, compact non-empty subset Ω of an $m-$ dimensional Euclidean space $\mathbb{R}^m$. We know that such measurable control function u is said to be admissible. It is assumed that the vector valued function f is $n-$ dimensional and that

$$f: [0, \infty] \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow E^n$$

is assumed to be continuous and non linear in x and also Lipschizian in both x and u.

In this paper, we assumed that the solution of (1.1) exists. It is a routine to denote the absolute continuous solution of (1.1) by $x(t, x_0, u) \equiv x(t)$ say, for convenience.

Now, it is relevant to point out that the problem of forcing the solution of the control systems to zero, which is referred to as null - controllability, has recieved serious attentions from various researchers all over the world. Take for instance, Chukwu [1], Onwuatu [2], Hisato and Etsujiro [3], Aniaku et al [4], [5], Eke[6], Balachandran and Dauer[7], to mention but a few. This problem features because, the solutions to practical problems confronting mankind need to be controlled to zero. Take, for examples, if in problem (1.1) above, x stands for the number of people in community who refuse to pay their taxes, or x can be the course of water shortage in a town, or even x may be the number of cows dying monthly from disease, or x may be the number of building collapsing in a state for one reason or the other. In each of these cases above, x is a problem which need to be controlled to zero in a finite time. A and f are such that the equation of the form (1.1) is satisfied.

This paper, however, adds a new dimension to the solution of the problem, since, as we know, this is the first time that Brouwer's Fixed Point Theorem is used as a crusial instrument for deriving the null- controllability of such systems.

We pursued the objective of this paper by dividing this paper into three major sections. In the first section, which we refer to as the introductory part, we introduce the subject matter of the paper. In the second section, which was referred as the preliminary section, we have the definitions of important terms which we encounter in the paper. We also have lemmas, propositions and theorems, which helped us to achieve the main objective of this paper. In section three, the main result is stated and proved.

II. Preliminaries.

The followings are the definitions of some important terms which we encounter in this paper.

Definitions

Definition 2.1: [5]

The control system (1.1) is said to be Euclidean controllable if for each $x_0 \in \mathbb{R}^n$ and each $x_1 \in \mathbb{R}^n$, there exist a time $t_1 \geq 0$ and an admissible control u such that the solution $x(t, x_0, u) = x(t)$ say, of (1.1) satisfies $x(0) = x_0$ and $x(t_1) = x_1$.

Definition 2.2: [5]

The control system (1.1) is said to be Euclidean null-controllable if in the definition 2.1 above, $x_1 = 0$.

Definition 2.3: (Diffeomorphism)[10]

Diffeomorphism is a map between manifolds which is differentiable and has also a differentiable inverse.

Definition 2.4: (Topological Mapping)

A mapping $f: B \longrightarrow \mathbb{R}^n$ is said to be a topological mapping if it is a diffeomorphism.

Now, if $x(t, x_0, u)$ is a continuous solution of the nonlinear control systems (1.1), then we know, using the method of variation of constants that $x(t, x_0, u)$ is given by

$$x(t, x_0, u) = X(t)x_0 + X(t) \int_0^t X^{-1}(s) f(s, x(s), u(s)) ds \qquad (2.1)$$

where $X(t)$ is the fundamental matrix satisfying the equation

$$\dot{X}(t) = A(t)X(t). \qquad (2.2)$$

If we assume x as a given solution of the non-linear control system (1.1), then null-controllability condition requires us to impose the boundary condition

$$Tx = 0 \qquad (2.3)$$

where T is a bounded linear operator defined on $\mathbb{C}[\mathbb{R}^+, \mathbb{R}^n]$, the space of all bounded and continuous functions in $\mathbb{R}^n$.

We now state the following lemmas which are vital for our work.

Lemma 2.1: (Weiestrass Approximation Theorem)[8]

For every continuous function $f(x)$ on $[a, b]$ and every $\epsilon > 0$, there is a polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon$ for every x in $[a, b]$.

Lemma 2.2: (Brouwer's Fixed Point Theorem)[9]

Let B be a closed, convex unit space of any finite dimensional Euclidean apace and $f: B \longrightarrow B$ is continuous, then f has a fixed point. i.e $f(x) = x$.

Proof:

We consider the case of real Euclidean space. Note that Weiestrass Approximation Theorem for continuos functions of $n-$variables implies that every continous map f of B into itself is the uniform limit of sequence f_k of infinitely differentiable maps for each integer k. Then there is a point $x_k \in B$ such that $f_k(x_k) = x_k$. Since B is compact, some subsequences x_{k_1} converge to a point $x \in B$.

Since $\lim_{i \to \infty} f_{k_i}(x) = f(x)$ uniformly in B,

$$f(x) = \lim_{i \to \infty} f_{k_i}(x_{k_i}) = \lim_{i \to \infty} x_{k_i} = x.$$

This shows that it is sufficient to consider the case that f is infinitely differentiable. This mean that f is an infinitely differentiable map of B into itself and that $f(x) = x \in B$.

Note: In lemma 2.2, we assumed that $\bar{x} = \{(x_1, x_2, \ldots, x_n): (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}} < 1\}$.

We took that $B = f(\bar{x})$, where f is a topological mapping.

We now remark that since the existence of x satisfying (1.1) is assumed here, our work is to prove the existence of an x satisfying both (1.1) and (2.3). When this is done, we have the required null-controllability condition. Here consider the existence of a fixed point which is implied by Brouwer's fixed point theorem. In this, we ensure the existence of a solution to problem (1.1) and (2.3) provided a suitable function f defined on B is continuous.

Let ζ be a Banach space of all bounded continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^n$ and define norm, $||.||$ on ζ as

$$||x|| = \sum_{i=1}^n |x_i|.$$

Let ξ be a positive integer and define S_ξ as

$$S_\xi = \{x \in \mathbb{R}^n : ||x|| \le \xi\}. \qquad (2.4)$$

Suppose $X(t)$ is a fundamental matrix solution of (1.1) and consider T_1X as

$$T_1X = \{Tx(t_1), Tx(t_2), \dots Tx(t_n)\} \qquad (2.5)$$

This means that T_1X is a matrix whose columns are values of T at the corresponding column of X. This means that under this condition, we have

$$[T_1X]\alpha = T(X_\alpha) \qquad (2.6)$$

for any $\alpha \in \mathbb{R}^n$.
It is also assumed that $[TX]$ is non-singular. i.e $[TX]^{-1}$ exists.
We now consider this important lemma which is very vital in this paper.

Lemma 2.3:
Consider the linear control system

$$\dot{x} = A(t)x(t) + B(t)u(t), x(0) = x_0 \qquad (2.7)$$

where A is an $n \times n$ matrix function and B is $n \times m$ matrix of continuous function. Note that the solution $x(t)$ say, of (2.7) exists and its matrix T_1X is invertible. Then the solution of (2.7) satisfying the boundary condition (2.3) is

$$x(t) = p(t) - X(t)[T_1X]^{-1}p \qquad (2.8)$$

where

$$p(t) = \int_0^t X(t)X^{-1}(s)B(s)u(s)ds \qquad (2.9)$$

and the corresponding initial condition x_0 is

$$x_0 = [T_1X]^{-1}Tp \qquad (2.10)$$

Note:
The above formulae (2.8) and (2.10) are for linear systems. A carry over to non-linear systems is also easy as one can see in the main result below.

Proof of Lemma 2.3:
By the variation of constant formula, the solution of (2.7) is

$$x(t) = X(t)x_0 + X(t)\int_0^t X^{-1}(s)B(s)u(s)ds \qquad (2.11)$$

This implies that

$$x(t) = X(t)x_0 + p(t) \qquad (2.12)$$

where $p(t)$ is defined in (2.9) above.
If we apply T to (2.12), we get

$$Tx(t) = TX(t)x_0 + Tp(t).$$

This can be written simply, for convenience, as

$$Tx = TXx_0 + Tp \qquad (2.13)$$

But from (2.3), $Tx = 0$. So for x satisfying (2.7), we have from (2.13) that

$$0 = TXx_0 + Tp \Longrightarrow TXx_0 = -Tp$$

Since $[TX]$ is non-singular, we have

$$x_0 = [TX]^{-1}Tp$$

as in (2.10) above.
If we substitute this value of x_0, which is initial point of (2.7) satisfying the boundry condition (2.3), into (2.11)

and taking into consideration of definition of $p(t)$ in (2.9), we get the required result (2.8) ▪
We hereby state our main result for this paper.

III. Main Result.

Theorem 3
Suppose $x(t, x_0, u)$ is the continuous solution of the non-linear control systems.

$$\dot{x}(t) = A(t)x(t) + f\big(t, x(t), u(t)\big) \qquad (3.1)$$

with the initial condition $x(0) = x_0$ of the form (2.10) of Lemma 2.3 given as

$$x_0 = [T_1X]^{-1}Tp \qquad (3.2)$$

where $p(t)$ is defined as

$$p(t) = \int_0^t X(t)X^{-1}(s)f(s, x(s), u(s))ds \qquad (3.3)$$

We assume the existence and uniqueness of the solution $x(t, x_0, u)$ of (3.1) with initial point $x_0 \in S_\xi$. Let there exist a constant $C > 0$ such that $||x(t, x_0, u)|| \leq C$. Whenever the $||X^{-}(t)f(t, x(t), u(t)|| = q(t)$ say, we have

$$N = \int_0^t q(s)ds.$$

Suppose $L = max||X(t)||$ for $t \geq 0$ then,

$$LN||[T_1X]^{-1}||||T|| \leq \xi.$$

In this case, there exist at least one solution of (3.1) satisfying the boundary condition (2.3). That is the system (3.1) is Euclidean null-controllable.

Proof:
Let $f: S_\xi \longrightarrow S_\xi$ be the function defined by

$$f(t) = [T_1X]^{-1}T\int_0^t X(t)X^{-1}(s)f(s, x(s), u(s))ds \qquad (3.4)$$

From the definition of S_ξ and the hypothesis of the theorem, f is a mapping of S_ξ into itself. We then prove that f is continuous.
Let $\{x_n\} \in S_\xi, x^* \in S_\xi$ with $\lim_{n\to\infty}||x_n - x^*|| = 0$.
Then from (3.4), we have

$$||f(x_n) - f(x^*)|| = ||[T_1X]^{-1}T\int_0^t X(t)X^{-1}(s)[f(s, x_n, u) - f(s, x^*, u)]ds||$$

From this we have

$$||f(x_n) - f(x^*)|| \leq ||[T_1X]^{-1}||||T||||X(t)\int_0^t X^{-1}(s)[f(s, x_n, u) - f(s, x^*, u)]ds|| \qquad (3.5)$$

In (3.5) above, the integrand tends pointwise to zero as $n \longrightarrow \infty$, and is bounded by $2q(t)$, which is itself an integrable function.
Then, from (3.5), we have

$$||f(x_n) - f(x^*)|| \leq ||[T_1X]^{-1}||||T||L\mathbb{N} = L\mathbb{N}||[T_1X]^{-1}||||T|| < \xi$$

from hypothesis.
Hence, from Lebesque Theorem on Dominated convergence, we have

$$\lim_{n\to\infty}||f(x_n) - f(x^*)|| = 0$$

Then as f is a continuous mapping from S_ξ into itself, Brouwer's Fixed point Theorem (Lemma 2.2)implies the existence of at least one vector x_0 such that

$$x_0 = [T_1X]^{-1}T\int_0^t X(t)X^{-1}(s)f(s, x_0, u)ds$$

That is $f(x_0) = x_0$. This means that the given control system (3.1) is Euclidean null-controllable■

IV. Conclusion

We conclude that the null-controllability of linear and non-linear control systems can be estalished using fixed point theorem. In our paper [5], Null-controllability of linear control systems was established by Leray Schauder fixed point theorem. Other fixed point theorems may also be used to establish null-controllability both for linear and non-linear control systems.

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