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A Short Review on Calkin-Wilf Tree

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Abstract: *This article presents a short review on Calkin-Wilf Tree. This tree is named after the mathematicians Neil Calkin and Herbert S. Wilf. Some interesting properties of Calkin-Wilf tree are discussed in this article. The sequence of Fibonacci numbers &Lucas numbers, diagonals, continued fractions and paths in Calkin-Wilf tree are presented in this article. Trees like q-Calkin-Wilf tree and (u, v)- Calkin-Wilf tree, which are analogous to Calkin-wilf tree, are given in this article.*

Keywords: *Calkin-Wilf tree, Fibonacci numbers, Lucas numbers,q-Calkin-Wilf tree, (u, v)- Calkin-Wilf tree, Diagonals.*

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I. Introduction

The Calkin-Wilf tree was introduced by Dr. Neil Calkin and Dr. Herbert S. Wilf (June 13,1931-January 7,2012) in the year 2000 in their research paper *'Recounting the Rationals'* [1]. The Calkin-Wilf tree (CW- tree) is a binary tree of positive rational numbers. A number which is written in the form p/q, where p and q are integers, and q is not zero is called rational number. The vertices of this tree are labelled by fractions, which are positive rational numbers.

The CW-tree starts with the number 1, represented as the fraction $\frac{1}{1}$, which is known as 'root' of the tree. Each vertex of $\left(\frac{a}{b}\right)$ of this tree from the first level onwards is a parent of two children, a left child $\left(\frac{a}{a+b}\right)$and a right child $\left(\frac{a+b}{b}\right)$ as shown in Fig.1. The left child retains the numerator of its parent and adds the numerator and denominator of its parent to form its denominator. The right child retains the denominator of its parent and forms its numerator from the sum of numerator and denominator of its parent.

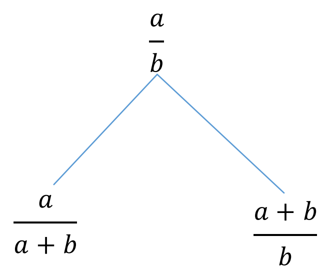

Figure 1: Left and right children of a vertex in CW-tree

C-W tree is formed by an infinite repetition of the criteria given in Fig.1 starting from the root 1/1. The level of 1/1 is taken as the first level and the children are located on a level below the level of their parent. The Fig.2 shows 5 levels of CW-tree.

The rest of the article is organized as follows. Section-II presents the properties of Calkin-Wilf tree. The sequence of Fibonacci numbers in CW-tree is given in Section-III. The sequence of Lucas numbers in CW-tree is presented in Section-IV. The diagonals of this tree are described in Section-V. Continued fraction expansions in the tree are mentioned in Section-VI. Paths in the tree are given in Section-VII. Section-VIII deals with trees analogous to CW-tree. Finally, conclusion is given in Section-IX.

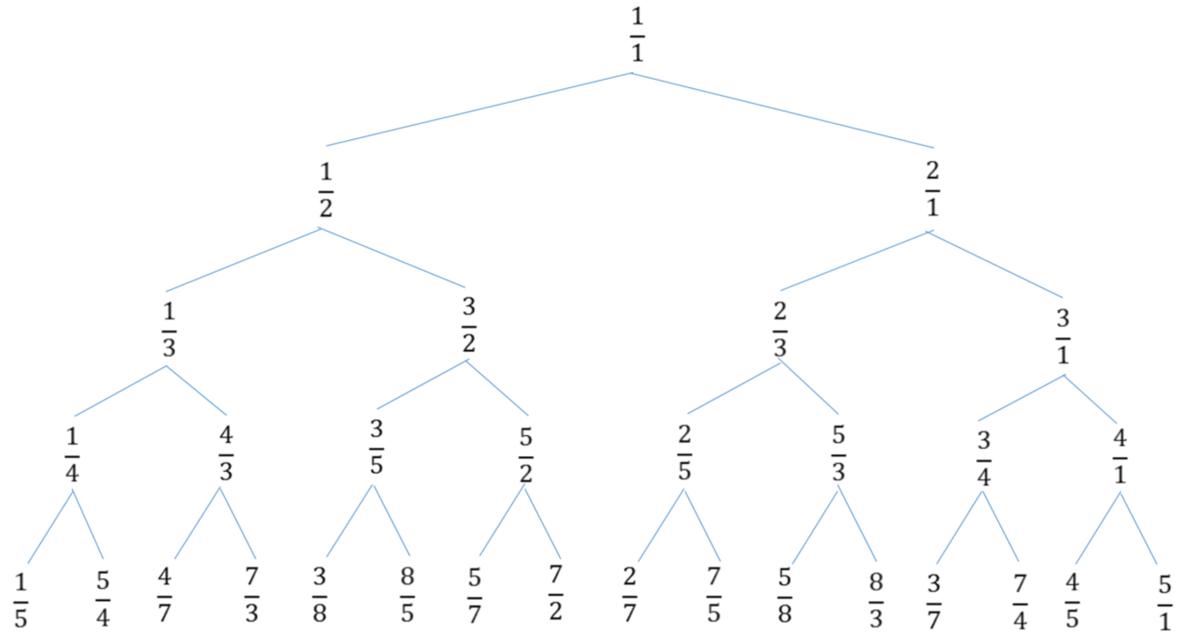

Figure 2: Five levels of Calkin-Wilf tree

II. Properties of Calkin-Wilf tree

(a)*Every positive rational number appears once and only once on the tree.*
(b) *Left child of any vertex is always less than 1 and right child is always greater than 1.*
(c) *At any given level, denominator of any fraction is equal to the numerator of immediate next fraction.*
(d) *In any given level, the ith vertex from left end is the reciprocal of ith vertex from right end of that level.*
For example, on 4th level, the 3rd fraction from left end is 3/5 and the 3rd fraction from right end is 5/3.
(e) *Any vertex is the product of its two children.*
(f) *The product of all fractions in a given level is 1.*
For example, the product of fractions in 3rd level is $\frac{1}{3}\times\frac{3}{2}\times\frac{2}{3}\times\frac{3}{1}=1$
(g) *On any n^{th} level of the tree, $\frac{1}{n}$ is the leftmost fraction and $\frac{n}{1}$ is the rightmost fraction.*
That is, all numerators down the left side of the tree are 1 and all denominators down its right side are 1.
(h) The sequence all fractions in the tree is known as Calkin- Wilf sequence. This sequence is

$$\frac{1}{1},\frac{1}{2},\frac{2}{1},\frac{1}{3},\frac{3}{2}.\frac{2}{3},\frac{3}{1},\frac{1}{4},\frac{4}{3},\frac{3}{5},\frac{5}{2},\frac{2}{5},\frac{5}{3},\frac{3}{4},\frac{4}{1},\ldots \tag{1}$$

If $x_n (n=1,2,3,\ldots)$ denotes n^{th} fraction of Calkin-Wilf sequence, then

$$x_{n+1}=\frac{1}{2\lfloor x_n\rfloor+1-x_n} \tag{2}$$

Where, $\lfloor x_n\rfloor$ is the integer part of x_n. Using (2), any fraction in (1) can be found out knowing its previous fraction. The following two examples verify Eqn. (2).

Example 1: The 6th fraction $(i.e. n=6)$ in (1) is $x_6=\frac{2}{3}=0\frac{2}{3}$ and its integer part of is 0. Using (2), we have

$$x_7=\frac{1}{2\lfloor x_6\rfloor+1-x_6}=\frac{1}{2\times 0+1-\frac{2}{3}}=\frac{3}{1}$$

Since 3/1 is the 7th fraction in sequence (1), Eqn. (2) is verified.

Example 2: The 9th fraction $(i.e. n=9)$ in (1) is $x_9=\frac{4}{3}=1\frac{1}{3}$ and its integer part of is 1. Using (2), we get

$$x_{10}=\frac{1}{2\lfloor x_9\rfloor+1-x_9}=\frac{1}{2\times 1+1-\frac{4}{3}}=\frac{3}{5}$$

Hence, Eqn. (2) is verified as 3/5 is the 10th fraction in (1).

(i)The sequence of numerators of fractions in the tree is

$$1,1,2,1,3,2,3,1,4,3,5,2,5,3,4,1,\ldots \tag{3}$$

The sequence of denominators of fractions in the tree is

$$1,2,1,3,2,3,1,4,3,5,2,5,3,4,1,\ldots \tag{4}$$

It is observed from the above two sequences that if the first term in the sequence of numerators of fractions (3) is removed, then the sequence of denominators (4) is obtained. Let $p(n)$ denotes n^{th} term in sequence of numerators of fractions (3) for $n \geq 0$. That is, $p(0) = 1, p(1) = 1, p(2) = 2, p(3) = 1, p(4) = 3, \dots etc.$The sequence (3) satisfies the following relations for $j \geq 1$.

$$p(2^j - 1) = 1 \qquad (5)$$
$$p(2^j) = j + 1 \qquad (6)$$
$$p(2^j + 1) = j \qquad (7)$$
$$p(2j + 1) = p(j) \qquad (8)$$
$$p(2j) = p(j) + p(j - 1) \qquad (9)$$

for $j \geq 1$

The above relations are verified by the following five examples.

Example 1: Substituting $j = 2$ in (5), we get

$$p(3) = 1$$

As $p(3)$ term in sequence (3) is 1, the relation (5) is verified.

Example 2: Putting $j = 3$ in (6), we obtain

$$p(8) = 4$$

Since $p(8)$ term in sequence (3) is 4, the relation (6) is verified.

Example 3: Substituting $j = 3$ in (7), we have

$$p(9) = 3$$

Since $p(9)$ term in sequence (3) is 3, the relation (7) is verified.

Example 4: Putting $j = 2$ in LHS of (8), we obtain

$p(2j + 1) = p(5) = 2$ [By sequence (3)]

Substituting $j = 2$ in RHS of (8), we have

$p(j) = p(2) = 2$ [By sequence (3)]

Since LHS = RHS, the relation (8) is verified.

Example 5: Putting $j = 4$ in LHS of (9), we get

$p(2j) = p(8) = 4$ [By sequence (3)]

Substituting $j = 4$ in RHS of (9), we obtain

$p(j) + p(j - 1) = p(4) + p(3) = 3 + 1 = 4$ [By sequence (3)]

Since LHS = RHS, the relation (9) is verified.

(j) *Sum of the fractions at a level n of CW-tree is* $3 \times 2^{n-2} - \frac{1}{2}$.

This statement is verified by the following example.

Example: For $n = 4$,we have

$$3 \times 2^{n-2} - \frac{1}{2} = 3 \times 2^2 - \frac{1}{2} = \frac{23}{2}$$

Sum of the fractions at 4th level of the tree $= \frac{1}{4} + \frac{4}{3} + \frac{3}{5} + \frac{5}{2} + \frac{2}{5} + \frac{5}{3} + \frac{3}{4} + \frac{4}{1}$

$$= \frac{15+80+36+150+24+100+45+240}{60} = \frac{690}{60} = \frac{23}{2}$$

Since the sum of the fractions at 4th level of the tree is equal to the value of $3 \times 2^{n-2} - \frac{1}{2}$, the given statement is verified.

(k) *The simplicity of a parent in the tree is equal to sum of the simplicities of children.*

Proof: If the fraction $\frac{a}{b}$ is a parent vertex, then its left child is $\frac{a}{a+b}$ and its right child is $\frac{a+b}{b}$.

Simplicity of left child $= S\left(\frac{a}{a+b}\right) = \frac{1}{a(a+b)}$

Simplicity of Right child $= S\left(\frac{a+b}{b}\right) = \frac{1}{(a+b)b}$

Sum of the simplicities of children of $\frac{a}{b} = \frac{1}{a(a+b)} + \frac{1}{(a+b)b}$

$= \frac{1}{ab} = S\left(\frac{a}{b}\right) =$ Simplicity of the parent $\frac{a}{b}$

Hence, the given statement is proved.

(l) *Sum of the simplicities of all fractions at any level of CW-tree is equal to 1.*

The following example verifies the above statement.

Example: Sum of the simplicities of fractions at 3rd level of the tree

$$= S\left(\frac{1}{3}\right) + S\left(\frac{3}{2}\right) + S\left(\frac{2}{3}\right) + S\left(\frac{3}{1}\right)$$
$$= \frac{1}{1\times3} + \frac{1}{3\times2} + \frac{1}{2\times3} + \frac{1}{3\times1}$$
$$= \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = 1$$

Hence, the given statement is verified.

(m) *The product of complexities of all fractions at any level of the tree is a perfect square.*

This statement is verified by the following example. The complexity of a fraction $\frac{a}{b}$ at any level of the tree is $C\left(\frac{a}{b}\right) = a \times b$.

Example: The product of complexities of all fractions in 3rd level of the tree

$$= C\left(\frac{1}{3}\right) \times C\left(\frac{3}{2}\right) \times C\left(\frac{2}{3}\right) \times C\left(\frac{3}{1}\right) = (1 \times 3) \times (3 \times 2) \times (2 \times 3) \times (3 \times 1)$$

$= 324 = 18^2$, which is a perfect square.

Hence, the given statement is verified.

(n) *The product of complexities of two children of a parent vertex $\left(\frac{a}{b}\right)$ is equal to d^2 times the complexity of the parent, where d^2 is a perfect square.*

Proof: If the fraction $\frac{a}{b}$ is a parent vertex, then its left child is $\frac{a}{a+b}$ and its right child is $\frac{a+b}{b}$.

Complexity of left child $= C\left(\frac{a}{a+b}\right) = a(a+b)$

Complexity of Right child $= C\left(\frac{a+b}{b}\right) = (a+b)b$

Product of complexities of children of $\frac{a}{b} = a(a+b)(a+b)b = (a+b)^2 ab = d^2 C\left(\frac{a}{b}\right)$

Since, $d^2 = (a+b)^2$ is a perfect square and $C\left(\frac{a}{b}\right) = ab$ is the complexity of the parent vertex $\left(\frac{a}{b}\right)$, the given statement is proved.

(o) *The product of complexities of all fractions at any level n of the tree is d^2 times the product of complexities of all fractions at level $(n-1)$, where d^2 is a perfect square.*

This statement is verified by the following example.

Example: The product of complexities of all fractions at 4th level $(i.e. n = 4)$ of the tree

$$= C\left(\frac{1}{4}\right) \times C\left(\frac{4}{3}\right) \times C\left(\frac{3}{5}\right) \times C\left(\frac{5}{2}\right) \times C\left(\frac{2}{5}\right) \times C\left(\frac{5}{3}\right) \times C\left(\frac{3}{4}\right) \times C\left(\frac{4}{1}\right)$$
$$= (1 \times 4) \times (4 \times 3) \times (3 \times 5) \times (5 \times 2) \times (2 \times 5) \times (5 \times 3) \times (3 \times 4) \times (4 \times 1)$$
$$= 51840000 = 160000 \times 324 = (400)^2 \times 324 = d^2 \times 324$$

Here $d^2 = 400^2$ is a perfect square. We know from the case (m) that 324 is the value of product of complexities of all fractions at 3rd level. Hence, the product of complexities of fractions at 4th level of the tree is400^2times the product of complexities of fractions at 3rd level of the tree. This verifies the given statement. If the children are in the level n, then the parent vertex is in the level $(n-1)$. So, this case can be taken as an alternate to previous case (n).

(p) *The sum of traces of all fractions at any level nof the tree is thrice the sum of traces of all fractions at level $(n-1)$.*

Proof: Sum of traces of left child and right child of parent fraction $\left(\frac{a}{b}\right) = t\left(\frac{a}{a+b}\right) + t\left(\frac{a+b}{b}\right)$

$$= \{a + (a+b)\} + \{(a+b) + b\}$$
$= 3(a+b) = 3t\left(\frac{a}{b}\right) = 3 \times$ trace of the parent

If the children are in level n of the tree, then the level of parent is $(n-1)$. Therefore, the given statement is proved.

(q) *Sum of the traces of fractions at level n of the tree is $2 \times 3^{n-1}$.*

Example: Sum of the traces of fractions at 3rd level of the tree

$$= t\left(\frac{1}{3}\right) + t\left(\frac{3}{2}\right) + t\left(\frac{2}{3}\right) + t\left(\frac{3}{1}\right)$$
$$= (1+3) + (3+2) + (2+3) + (3+1) = 18$$

For $n = 3$, the value of $2 \times 3^{n-1} = 2 \times 3^2 = 18$. Thus, the given statement is verified.

(r) *Sum of the complexities of fractions at level n of the tree is equal to the sum of squares of traces of fractions at level $(n-1)$.*

Proof: Sum of complexities of children of fraction $\left(\frac{a}{b}\right) = C\left(\frac{a}{a+b}\right) + C\left(\frac{a+b}{b}\right)$

$$= \{a(a+b)\} + \{(a+b)b\}$$
$= (a+b)^2 = \left[t\left(\frac{a}{b}\right)\right]^2 =$ Square of the trace of parent fraction $\left(\frac{a}{b}\right)$

It is known that if the children are in level n of the tree, then the level of parent is $(n-1)$. Therefore, the given statement is proved.

III. Fibonacci numbers in Calkin-Wilf tree

The sequence of Fibonacci numbers is given by

1,1,2,3,5,8,13,21,34,55,……….. (10)

Each number in this sequence after the first two is the sum of two numbers before it. Let $F_n (n = 0,1,2,3, \ldots)$ denotes n^{th}Fibonacci number. i.e. $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3,$……….etc. Then the Fibonacci sequence (10) satisfies the following relation.

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2 \qquad (11)$$

The Fibonacci sequence (10) appears along a proper zigzag path down the left half of CW-tree. The Fibonacci numbers in the left half of the tree are pointed out by drawing circles around them as shown in the following Fig.3. We notice in Fig.3 that if a child contains the Fibonacci number, then the parent also contains the Fibonacci number.

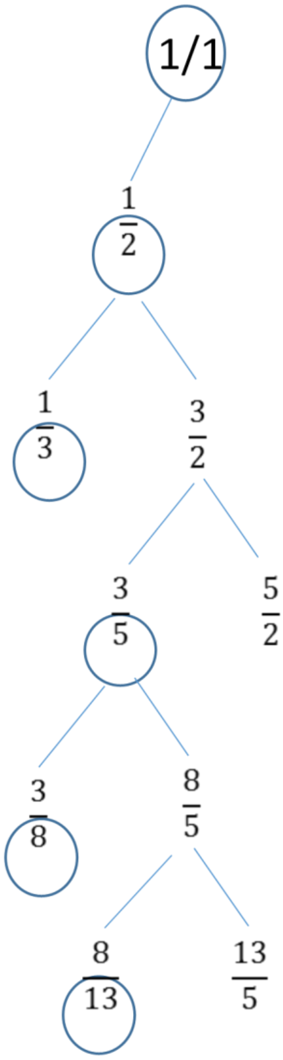

Figure 3: Fibonacci numbers in left half of the tree

The Fibonacci ratio is the fraction of consecutive Fibonacci numbers in the form $\frac{F_n}{F_{n+1}}$ or $\frac{F_{n+1}}{F_n}$. Let us now find out the position of these fractions on CW-tree. If $p\left(\frac{a}{b}\right)$ denotes the position of fraction $\left(\frac{a}{b}\right)$ on CW-tree, where $\frac{a}{b} = \frac{F_n}{F_{n+1}}$ or $\frac{F_{n+1}}{F_n}$, then for $n \geq 1$,

$$p\left(\frac{F_n}{F_{n+1}}\right) = \begin{cases} \frac{5\times 2^n - 2}{3}, if\ n\ is\ even \\ \frac{2^{n+2}-2}{3}, if\ n\ is\ odd \end{cases} \qquad (12)$$

and

$$p\left(\frac{F_{n+1}}{F_n}\right) = \begin{cases} \frac{2^{n+2}-1}{3}, if\ n\ is\ even \\ \frac{5\times 2^n - 1}{3}, if\ n\ is\ odd \end{cases} \qquad (13)$$

The above relations (12) & (13) are verified by the following four examples.

Example 1: Let $n = 2$ (even). Then by Fibonacci sequence (10), we have, $\frac{F_n}{F_{n+1}} = \frac{F_2}{F_3} = \frac{2}{3}$. Using (12), the position of the fraction $\frac{2}{3}$on CW-tree is $p\left(\frac{2}{3}\right) = \frac{5\times 2^n - 2}{3} = \frac{5\times 2^2 - 2}{3} = 6$. It is a fact that the fraction 2/3 is on 3rd level at 6th position in CW-tree (Fig.2).

Example 2: For $n = 1$ (odd), we have $\frac{F_n}{F_{n+1}} = \frac{F_1}{F_2} = \frac{1}{2}$. Using (12), the position of the fraction $\frac{1}{2}$on CW-tree is $p\left(\frac{1}{2}\right) = \frac{2^{n+2} - 2}{3} = \frac{2^3 - 2}{3} = 2$, which is true (Refer Fig.2).

Example 3: For $n = 2$ (even), $\frac{F_{n+1}}{F_n} = \frac{F_3}{F_2} = \frac{3}{2}$. Using (13), the position of the fraction $\frac{3}{2}$on CW-tree is $p\left(\frac{3}{2}\right) = \frac{2^{n+2} - 1}{3} = \frac{2^4 - 1}{3} = 5$, which is correct (Refer Fig.2).

Example 4: For $n = 1$ (odd), $\frac{F_{n+1}}{F_n} = \frac{F_2}{F_1} = \frac{2}{1}$. Using (13), the position of the fraction $\frac{2}{1}$ on CW-tree is $\left(\frac{2}{1}\right) = \frac{5\times 2^n - 1}{3} = \frac{5\times 2 - 1}{3} = 3$, which is true (Refer Fig.2).

Let us now explain 'sum complexity'.

Definition: The sum complexity of a rational number $\left(\frac{a}{b}\right)$ is $(a + b)$.

An interesting fact is that the maximum sum complexity of k^{th} level of CW-tree is F_{k+1}, the $(k+1)th$ Fibonacci number. The following example proves this fact.

Example: In 4th level (i.e. $= 4$) of the tree, there are eight fractions$\frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}$&$\frac{4}{1}$. According to the above definition, the sum complexities of these fractions respectively are 5,7,8,7,7,8,7 & 5. So, the maximum sum complexity of 4th level is 8, which is the Fibonacci number $F_5 = F_{4+1}$.

IV. Lucas numbers in Calkin-Wilf tree

The sequence of Lucas numbers is given by

$$1,3,4,7,11,18,29,\ldots\ldots\ldots.. \tag{14}$$

In the above sequence, each number from third onwards is the sum of two numbers before it. Let $L_n(n = 1,2,3,\ldots)$ denotes n^{th}Lucas number. i.e. $L_1 = 1, L_2 = 3, L_3 = 4,$……….etc. Then the sequence (14) satisfies the following relation.

$$L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 3 \tag{15}$$

The Lucas sequence (14) appears along a proper zigzag path down the left half of CW-tree. The Lucas numbers in the left half of the tree are marked by drawing circles around them as shown in Fig.4. We observe in Fig.4 that if a child contains Lucas number, then the parent also contains Lucas number. The ratio of consecutive Lucas numbers is given by $\frac{L_n}{L_{n+1}}$ or $\frac{L_{n+1}}{L_n}$. Let us now find out the position of these fractions on CW-tree. If $p\left(\frac{X}{Y}\right)$ denotes the position of fraction $\left(\frac{X}{Y}\right)$ on CW-tree, where $\frac{X}{Y} = \frac{L_n}{L_{n+1}}$ or $\frac{L_{n+1}}{L_n}$, then for $n \geq 1$,

$$p\left(\frac{L_n}{L_{n+1}}\right) = \begin{cases} 2^n + \frac{2(1-2^{n+2})}{(-3)}, if\ n\ is\ even \\ 2^n + \frac{2(1-2^{n+1})}{(-3)}, if\ n\ is\ odd \end{cases} \tag{16}$$

and

$$p\left(\frac{L_{n+1}}{L_n}\right) = \begin{cases} 2^n + \frac{(1-2^{n+2})}{(-3)}, if\ n\ is\ even \\ 2^n + \frac{(1-2^{n+3})}{(-3)}, if\ n\ is\ odd \end{cases} \tag{17}$$

The above relations (16) & (17) are verified by the following four examples.

Example 1: Let $n = 2$ (even). Then from Lucas sequence (14), we get, $\frac{L_n}{L_{n+1}} = \frac{L_2}{L_3} = \frac{3}{4}$. Using (16), the position of the fraction $\frac{3}{4}$ on CW-tree is $p\left(\frac{3}{4}\right) = 2^n + \frac{2(1-2^{n+2})}{(-3)} = 2^2 + \frac{2(1-2^4)}{(-3)} = 14$. It is true that the fraction 3/4 is on 4thlevel at 14thposition in CW-tree (Fig.2).

Example 2: For $n = 3$ (odd), we have $\frac{l_n}{L_{n+1}} = \frac{L_3}{L_4} = \frac{4}{7}$. Using (16), the position of the fraction $\frac{4}{7}$ on CW-tree is $\left(\frac{4}{7}\right) = 2^n + \frac{2(1-2^{n+1})}{(-3)} = 2^3 + \frac{2(1-2^4)}{(-3)} = 18$, which is true (Refer Fig.2).

Example 3: For $n = 2$ (even), we get, $\frac{L_{n+1}}{L_n} = \frac{L_3}{L_2} = \frac{4}{3}$. Using (17), the position of the fraction $\frac{4}{3}$ on CW-tree is $p\left(\frac{4}{3}\right) = 2^n + \frac{(1-2^{n+2})}{(-3)} = 2^2 + \frac{(1-2^4)}{(-3)} = 9$. It is a fact that the fraction 4/3 is on 4th level at 9th position in CW-tree (Fig.2).

Example 4: For $n = 3$ (odd), we have $\frac{l_{n+1}}{L_n} = \frac{L_4}{L_3} = \frac{7}{4}$. Using (17), the position of the fraction $\frac{7}{4}$ on CW-tree is $p\left(\frac{7}{4}\right) = 2^n + \frac{(1-2^{n+3})}{(-3)} = 2^3 + \frac{(1-2^6)}{(-3)} = 29$,which is correct (Refer Fig.2).

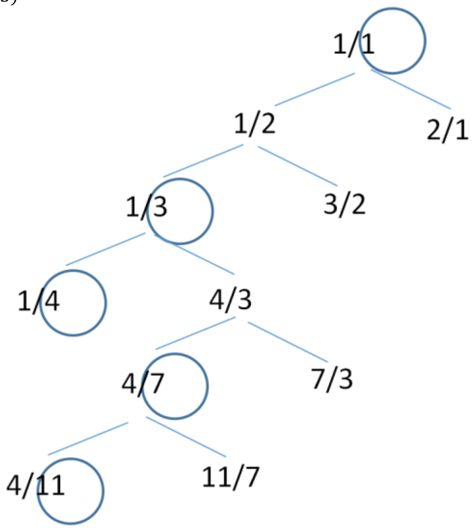

Figure 4: Lucas numbers in left half of CW-tree

V. Diagonals of Calkin-Wilf tree

Definition: Diagonals are sequence of fractions which are at the same relative positions on consecutive levels of the Calkin-Wilf tree.

There are two types of diagonals, left diagonal and right diagonal. The left diagonal $L_j$contains all fractions which lie at the j^{th} position on all levels from left edge of the tree. Similarly, the right diagonal $R_j$contains all fractions which lie at the j^{th} position on all levels from right edge of the tree. The following Table no. 1 gives 10 left diagonals $L_1, L_2, \ldots, L_{10}$ of CW-tree and the corresponding formula for n^{th}fraction ($n = 1,2,3, \ldots etc.$) in left diagonal.

Properties of diagonals:

- *Fractions in left diagonals are the inverse of their corresponding fractions in right diagonal and vice versa.*

For example, the left diagonal L_2 contains all fractions which are at the second position form left edge on all levels of the tree. There is no fraction at 2nd position in 1st level. So, $L_2 = \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$, ...etc. Similarly, the right diagonal R_2 contains all fractions which are at the second position form right edge on all levels of the tree. That is, $R_2 = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, ...etc.

- *Adding 1 to the numerator as well as to the denominator of any fraction in either L_2 or R_2 gives the next fraction.*
- *In L_2 , the numerator of any fraction is more than its denominator by 1 and in R_2 ,the numerator of any fraction is less than its denominator by 1.*
- *The position of a fraction in left diagonals L_1 ,$L_2, L_4, L_{8,}L_{16}, \ldots. etc.$ is given by its denominator.*
- The general formula for n^{th} fraction of left diagonals $L_1, L_2, L_4, L_8, L_{16}, L_{32}, \ldots etc$ is given by $L_{2^j} = \frac{jn+1}{n}$ for $j = 0,1,2,3, \ldots etc.$
- The general formula for n^{th} fraction of left diagonals $L_2, L_3, L_5, L_9, L_{17}, L_{33}, \ldots etc$ is given by $L_{2^j+1} = \frac{n+1}{n(j+1)+j}$ for $j = 0,1,2,3, \ldots etc.$

Table no. 1: 10 left diagonals of CW-tree and their formulas

Left diagonal	Formula for n^{th} fraction in left diagonal ($n = 1,2,3,\ldots$)
$L_1 = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$	$L_1 = \frac{1}{n}$
$L_2 = \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots.$	$L_2 = \frac{n+1}{n}$
$L_3 = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \ldots.$	$L_3 = \frac{n+1}{2n+1}$
$L_4 = \frac{3}{1}, \frac{5}{2}, \frac{7}{3}, \ldots.$	$L_4 = \frac{2n+1}{n}$
$L_5 = \frac{2}{5}, \frac{3}{8}, \ldots.$	$L_5 = \frac{n+1}{3n+2}$
$L_6 = \frac{5}{3}, \frac{8}{5}, \ldots.$	$L_6 = \frac{3n+2}{2n+1}$
$L_7 = \frac{3}{4}, \frac{5}{7}, \ldots.$	$L_7 = \frac{2n+1}{3n+1}$
$L_8 = \frac{4}{1}, \frac{7}{2}, \ldots.$	$L_8 = \frac{3n+1}{n}$
$L_9 = \frac{2}{7}, \frac{3}{11}, \ldots.$	$L_9 = \frac{n+1}{4n+3}$
$L_{10} = \frac{7}{5}, \frac{11}{8}, \ldots.$	$L_{10} = \frac{4n+3}{3n+2}$

VI. Continued Fraction expansions in Calkin-Wilf tree

The continued fraction expansion of a parent $\left(\frac{a}{b}\right)$ at a vertex of CW-tree is given by

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}} \qquad (18)$$

Where, the numbers $a_1, a_2, \ldots, a_n$ are the *partial denominators* of the finite continued fraction and they all are real numbers. The number a_0 may be zero or positive or negative. The continued fraction expansion of $\left(\frac{a}{b}\right)$ is denoted by the symbol $[a_0; a_1, a_2, \ldots., a_n]$ and it is called *simple* if all of the a_i are integers. The value of any finite simple continued fraction will always be a rational number.

That is, $\frac{a}{b} = [a_0; a_1, a_2, \ldots, a_n]$ (19)

The right child $\left(\frac{a+b}{b}\right)$ of $\frac{a}{b}$ is expressed in terms of continued fraction expansion as

$$\frac{a+b}{b} = 1 + \frac{a}{b} = 1 + [a_0; a_1, a_2, \ldots, a_n] = [a_0 + 1; a_1, a_2, \ldots, a_n] \qquad (20)$$

Similarly, the left child $\left(\frac{a}{a+b}\right)$ of $\frac{a}{b}$ is expressed in terms of continued fraction expansion as

$$\frac{a}{a+b} = \frac{1}{(a+b)/_a} = \frac{1}{1+\frac{1}{a/_b}} = [0; 1, a_0, a_1, \ldots, a_n] \qquad (21)$$

Definition: If $\left(\frac{a}{b}\right)$ *is a fraction at the level kof CW-tree and* $[a_0; a_1, a_2, \ldots., a_n]$*is its continued fraction expansion, then* $a_0 + a_1 + a_2 + \cdots + a_n = k$.

This definition is verified by the following example.

Example 1: Consider the fraction $\frac{4}{3}$ at the 4th level (i.e. $k = 4$) of the tree.

$$\frac{4}{3} = 1 + \frac{1}{3} = 1 + \frac{1}{1+2} = 1 + \frac{1}{1+\frac{1}{1/2}} = 1 + \cfrac{1}{1+\cfrac{1}{0+\cfrac{1}{1+\frac{1}{1}}}} = [1; 1,0,1,1]$$

Hence, $a_0 + a_1 + a_2 + a_3 + a_4 = 1 + 1 + 0 + 1 + 1 = 4 = k$

VII. Paths in Calkin-Wilf tree

The following are different types of paths in CW-tree.

(A) Path:

Definition: The path to a fraction $\left(\frac{a}{b}\right)$ *at a vertex in CW-tree is the sequence of left and right turns it takes to reach this fraction stating from the root* $\frac{1}{1}$.

Let us assign 0 to denote a left turn and 1 to denote a right turn. For example, if we want reach the fraction $\frac{2}{3}$ in 3rd level of the tree, we have to take a right turn from $\frac{1}{1}$ and then a left turn $\frac{2}{1}$. Therefore, the path leading to fraction $\frac{2}{3}$ is denoted as '10'.

(B) Reverse path:

Definition: The reverse path of a fraction $\left(\frac{a}{b}\right)$ *in the CW-tree is obtained by reading the sequence of 0s and 1s in the path leading to* $\left(\frac{a}{b}\right)$ *in reverse order.*

The path leading to the fraction $\frac{2}{3}$ is '10'. So, the corresponding reverse path is '01'. In this reverse path, we have to take a left turn from the root$\frac{1}{1}$, then a right turn from $\frac{1}{2}$ to reach $\frac{3}{2}$. So, the reverse path is giving the rational number $\frac{3}{2}$, which is reciprocal of rational number $\frac{2}{3}$ corresponding to original path. However, this is not always true. For example, the path leading to the fraction $\frac{2}{5}$ in 4th level of the tree is represented by 100. Its reverse path 001 leads to the fraction $\frac{4}{3}$. Sometimes, the path and its reverse path lead to the same fraction. For example, the path 111 and its reverse path 111 lead to the same fraction $\frac{4}{1}$. The path 10 and its reverse path 01 are shown in the following Fig.5.

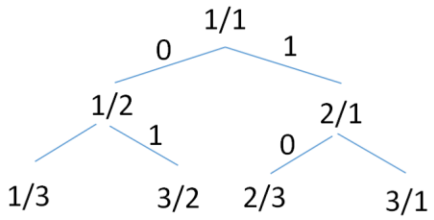

Figure 5: Path 10 and its reverse path 01 in CW-tree

The following Table no. 2 shows few paths and reverse paths and their corresponding fractions.

Table no. 2: Paths and reverse paths in CW-tree

Original fraction	Path	Reverse path	Reverse path fraction
1/1	----	----	1/1
1/2	0	0	1/2
2/1	1	1	2/1
1/3	00	00	1/3
3/2	01	10	2/3
2/3	10	01	3/2
3/1	11	11	3/1
1/4	000	000	1/4
3/5	010	010	3/5
2/5	100	001	4/3
3/4	110	011	5/2
4/7	0010	0100	3/8
3/8	0100	0010	4/7

(C) Inverse path:

Definition: An inverse path of a fraction $\left(\frac{a}{b}\right)$ *in CW-tree is the complement of the original path to* $\frac{a}{b}$.

The inverse path is obtained by replacing 0 by 1 and 1 by 0 in the path leading to the fraction $\frac{a}{b}$. Inverse path will give the reciprocal of original fraction $\frac{a}{b}$. For example, the path 001 leads to the fraction 4/3 and its inverse path 110 reaches the fraction 3/4.

(D) Palindromic path:

Definition: A palindromic path is the path which remains same if it is read forward or backward.

Few palindromic paths in the tree are 00,11,010,111,000,101 &111.

(E) Antipalindromic path:

Definition: The path of a fraction $\frac{a}{b}$ *in CW-tree is known as antipalindromic path when the reverse of path of* $\frac{a}{b}$ *is same as its inverse path.*

For example, the path 1100 leading to the fraction 3/7 in 5th level of the tree is an antipalindromic path. Because, the reverse path of 1100 is 0011, which is also the inverse path of 1100. Other examples of antipalindromic paths are 01 and 10.

It was observed that when the level n of the tree is odd, the number of antipalindromic paths is $2^{(n-1)/2}$. In the level 3, the number of antipalindromic paths is $2^{(3-1)/2} = 2$. These two paths are 01 and 10. They lead to the fractions 3/2 and 2/3 respectively. In case of level 5, there are $2^{(5-1)/2} = 4$ antipalindromic paths. These paths are 0011, 0101, 1010 & 1100. These paths lead to the fractions 7/3, 8/5, 5/8 & 3/7 respectively.

VIII. Trees analogous to Calkin-Wilf tree

The following are three types of trees which are analogous to CW-tree.

(i) q- Calkin- Wilf tree

(ii) (u, v)- Calkin- Wilf tree

(iii) Trees with roots 0/1 and 1/0

8.1 q- Calkin-Wilf tree

The q-Calkin-Wilf tree [2] is a binary tree with root 1/1. As shown in Fig. 6,each vertex $\frac{a}{b}$of this tree from 1st level onwards is a parent of two children, a left child $\frac{a}{qa+b}$and a right child$\frac{qa+b}{b}$, where $q \epsilon N$. The symbol$N$ stands fornatural numbers (1,2,3,…..etc.). The$q$-Calkin-Wilf tree can be formed by an infinite repetition of the criteria given in Fig.6. Three levels of this tree are shown in Fig.7.

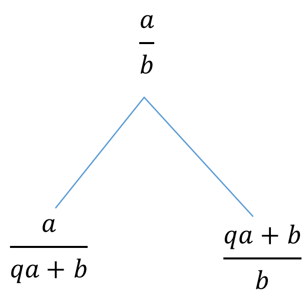

Figure 6: Left child and right child of $\frac{a}{b}$ in q-Calkin-Wilf tree

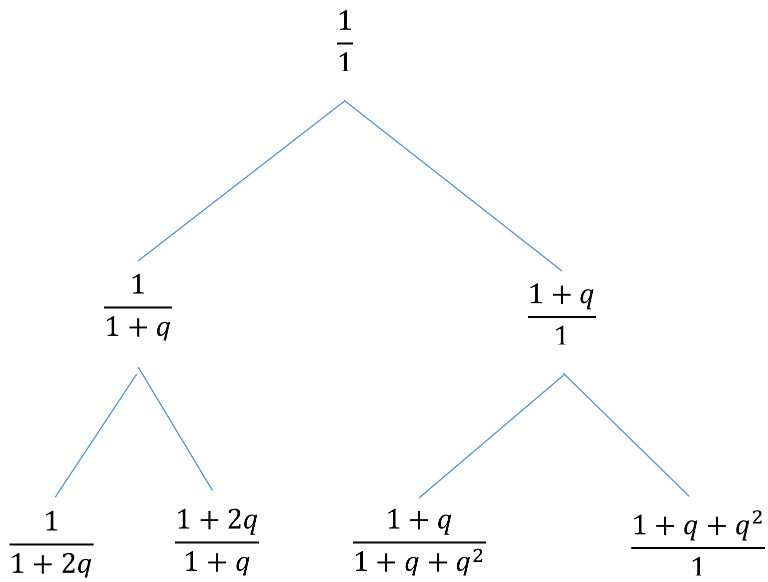

Figure 7: Three levels of q-Calkin-Wilf tree

The following are few observations on q-Calkin-Wilf tree.

(i) The following sequence of fractions in the tree is called q-Calkin-Wilf sequence

$\frac{1}{1}, \frac{1}{1+q}, \frac{1+q}{1}, \frac{1}{1+2q}, \frac{1+2q}{1+q}, \frac{1+q}{1+q+q^2}, \frac{1+q+q^2}{1}, \ldots \ldots \ldots \ldots$

(ii) If x_n be the n^{th}fraction in q-Calkin-Wilf sequence, then

$$x_{2n+1} = \frac{1}{1-qx_{2n}} \qquad (22)$$

Where, $n = 1,2,3, \ldots$

For example, if $n = 2$, then $x_5 = \frac{1}{1-qx_4} = \frac{1}{1-q\left(\frac{1}{1+2q}\right)} = \frac{1+2q}{1+q}$, which is the 5th fraction in q-Calkin-Wilf sequence. Thus, using the above relation (22) one can find out any fraction in this sequence if the previous fraction is known.

(iii) The left most and right most vertices in the k^{th}level of the tree respectively are $\frac{1}{1+(k-1)q}$ and $\frac{1+q+q^2+\cdots+q^{k-1}}{1}$.

8.2 (u, v)- Calkin-Wilf tree

In (u, v)- Calkin-Wilf tree [3], all vertices from the first level onwards are 2×1 column matrices. Any vertex $\binom{a}{b}$ in this tree is a parent of two children, the left child$L_u \binom{a}{b}$ and the right child$R_v \binom{a}{b}$, as shown in Fig.8. L_u and R_v are 2×2 square matrices given by

$$L_u = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \text{ and } R_v = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \qquad (23)$$

Where, $u, v \geq 1$

$$\text{Left child of } \binom{a}{b} = L_u \binom{a}{b} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \binom{a}{b} = \binom{a}{ua+b} \qquad (24)$$

$$\text{Right child of } \binom{a}{b} = R_v \binom{a}{b} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \binom{a}{b} = \binom{a+vb}{b} \qquad (25)$$

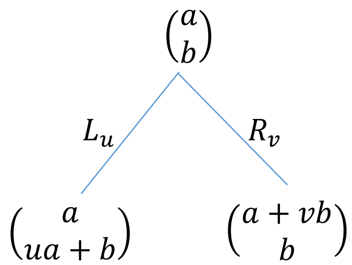

Figure 8: Left and right children of $\binom{a}{b}$ in (u, v)- Calkin-Wilf tree

This tree can be developed by infinite repetition of the criteria given in Fig. (8). The steps for drawing this tree are explained in the following example.

Example: Let the root of the tree is $\binom{2}{3}$. i.e. $a = 2$ & $b = 3$. Suppose, $u = 1$ & $v = 2$. Then, using (24) and (25), we have

$$\text{Left child of } \binom{2}{3} = \binom{a}{ua+b} = \binom{2}{1\times 2+3} = \binom{2}{5} \tag{26}$$

$$\text{Right child of } \binom{2}{3} = \binom{a+vb}{b} = \binom{2+2\times 3}{3} = \binom{8}{3} \tag{27}$$

The above column matrices (26) & (27) form the 2nd level of the tree. Let us now again calculate the children of these matrices using (24) & (25).

$$\text{Left child of } \binom{2}{5} = \binom{a}{ua+b} = \binom{2}{1\times 2+5} = \binom{2}{7} \tag{28}$$

$$\text{Right child of } \binom{2}{5} = \binom{a+vb}{b} = \binom{2+2\times 5}{5} = \binom{12}{5} \tag{29}$$

$$\text{Left child of } \binom{8}{3} = \binom{a}{ua+b} = \binom{8}{1\times 8+3} = \binom{8}{11} \tag{30}$$

$$\text{Right child of } \binom{8}{3} = \binom{a+vb}{b} = \binom{8+2\times 3}{3} = \binom{14}{3} \tag{31}$$

The above column matrices (28),(29),(30) & (31) form the 3rdlevel of the tree. Similarly, fractions of other levels can be calculated. The following Fig. 9 shows 3 levels of (1, 2)-Calkin-Wilf tree.

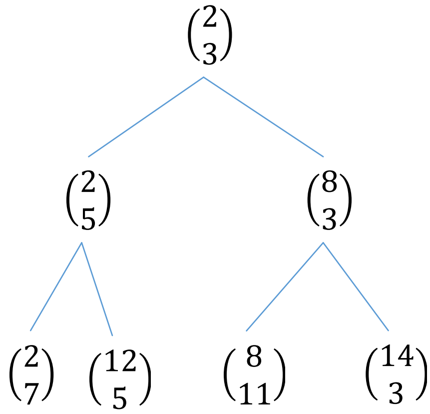

Figure 9: 3 levels of (1, 2)-Calkin-Wilf tree.

8.3 Trees with roots 0/1 and 1/0

Now trees with roots 0/1 and 1/0 are developed following the parent-child relationship given in Fig.1. The tree with root 0/1 is shown in Fig.10 and the right branches of this tree from 2nd level onwards give Calkin-Wilf tree. Fig.11 shows the tree with root 1/0 and its left branches from second level onwards belong to Calkin-Wilf tree.

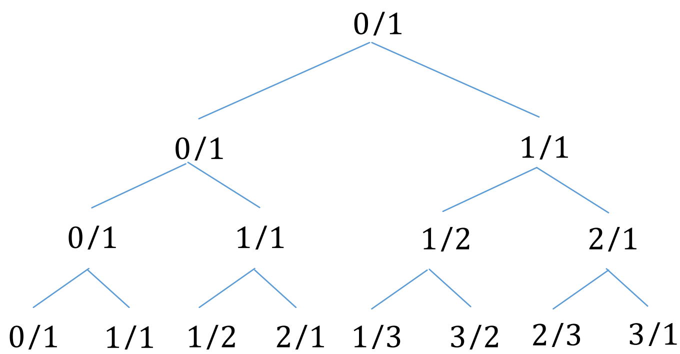

Figure 10: 4 levels of the tree with root 0/1 (Right branches from 2nd level form Cakin-Wilf tree)

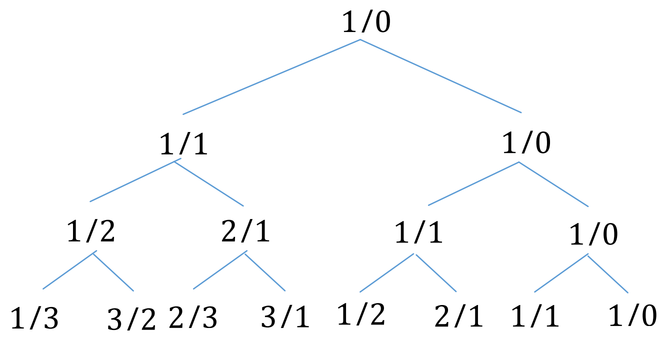

Figure 11: 4 levels of the tree with root 1/0 (Left branches from 2^{nd} level form Cakin-Wilf tree)

IX. Conclusion

The Calkin- Wilf tree is a fascinating mathematical structure. It has very interesting properties. The study of Fibonacci sequence, Lucas sequence, diagonals, continued fractions and paths in this tree will inspire the curious mathematicians to explore it further.

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