IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 18, Issue 1 Ser. I (Jan. – Feb. 2022), PP 47-54
www.iosrjournals.org

Series of Sobolev Inequalities with Remainder Terms

Sulima Ahmed Mohammed [(1)], Mohand M. Abdelrahim Mahgob[(2)] ,Shawgy Hussein [(3)]

1. *Department of Mathematics,College of Sciences and Arts, ArRass,Qassim University, Buraydah, Saudi Arabia*
2. *Mathematics Department, Faculty of Sciences and Arts-Almikwah-Albaha University- Saudi Arabia*
   Mathematics Department, Faculty of Sciences - Omderman Islamic University-Sudan
3. *Sudan University of Science and Technology, College of Science, Department of Math, Sudan*

Abstract

The Series of Sobolev inequality in $\mathbb{R}^{3+\epsilon}, \epsilon \geq 0$, asserts that $\left\|\sum \nabla f_j\right\|_2^2 \geq \mathbb{S}_{3+\epsilon}\left(\sum\left\|f_j\right\|_{\frac{2(3+\epsilon)}{1+\epsilon}}^2\right)$, with $\mathbb{S}_{3+\epsilon}$ being the sharp constant. This paper is concerned, with functions restricted to bounded domains $\Omega \subset \mathbb{R}^{3+\epsilon}$. Following H. Brezis, E. Lieb [13] two kinds of inequalities are established: (i) If $f_j = 0$ on $\partial\Omega$, then $\left\|\sum \nabla f_j\right\|_2^2 \geq \mathbb{S}_{3+\epsilon}\left(\sum\left\|f_j\right\|_{\frac{2(3+\epsilon)}{1+\epsilon}}^2\right) + C(\Omega)\left(\sum\left\|f_j\right\|_{\frac{(3+\epsilon)}{1+\epsilon},w}^2\right)$ and $\sum\left\|\nabla f_j\right\|_2^2 \geq \mathbb{S}_{3+\epsilon}\left(\sum\left\|f_j\right\|_{2^*}^2\right) + D(\Omega)\left(\sum\left\|f_j\right\|_{\frac{3+\epsilon}{2+\epsilon},w}^2\right)$. (ii) If $f_j \neq 0$ on $\partial\Omega$, then $\sum\left\|\nabla f_j\right\|_2 + C(\Omega)\left(\sum\left\|f_j\right\|_{\frac{3+\epsilon}{2+\epsilon},\partial\Omega}\right) \geq S_{3+\epsilon}^{1/2}\left(\sum\left\|f_j\right\|_{\frac{2(3+\epsilon)}{1+\epsilon}}\right)$ with $\epsilon^2 + a\epsilon + 5 = 0$. Some further results and open problems in this area are also presented.

Date of Submission: 03-01-2022 Date of Acceptance: 15-01-2022

I. Introduction

The usual Series of Sobolev Sobolev inequality in $\mathbb{R}^{3+\epsilon}$, $\epsilon \geq 0$, for the L^2 norm of the gradient is

$$\left\|\sum \nabla f_j\right\|_2^2 \geq S_{3+\epsilon}\left(\sum\left\|f_j\right\|_{\frac{2(3+\epsilon)}{1+\epsilon}}^2\right), \tag{1.1}$$

for all functions f_j with $\sum \nabla f_j \in L^2$ and with f_j vanishing at infinity in the weak sense that means$\{x||f_j(x)| > a<\infty$ for all $a>0$ (see [12]). The sharp constant S3+ϵ, is known to be

$$s_{3+\epsilon} = \pi(3+\epsilon)(1-2)[\Gamma((3+\epsilon)/2)/\Gamma(3+\epsilon)]^{2/3+\epsilon}. \tag{1.2}$$

The constant $S_{3+\epsilon}$, is achieved in (1.1) if and only if

$$f_j(x) = a[\varepsilon^2 + |\epsilon|^2]^{-(1+\epsilon)/2} \tag{1.3}$$

for some $a \in \mathbb{C}$, $\varepsilon \neq 0$ and $(x+\epsilon) \in \mathbb{R}^{3+\epsilon}$[1,2,6,7,9,11].

We consider appropriate modifications of (1.1) when $\mathbb{R}^{3+\epsilon}$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^{3+\epsilon}$. There are two main problems (See [13]):

Problem A. If $\sum f_j = 0$ on $\partial\Omega$, then (1.1) still holds (with $L^{\frac{(3+\epsilon)}{1+\epsilon}}$ norms in Ω, of course), since f_j can be extended to be zero outside of Ω. In this case (1.1) becomes a strict inequality when $\sum f_j \neq 0$ (in view of (1.3). However, $S_{3+\epsilon}$, is still the sharp constant in (1.1) (since $\sum\left\|\nabla f_j\right\|_2/\left\|f_j\right\|_{\frac{2(3+\epsilon)}{1+\epsilon}}$ is scale invariant). Our goal, in this case, is to give a lower bound to the difference of the two sides in (1.1) for $f_j \in H_0^1(\Omega)$. In Section II we shall prove the following inequalities (1.4) and (1.6):

$$\sum\left\|\nabla f_j\right\|_2^2 \geq S_{3+\epsilon}\left(\sum\left\|f_j\right\|_2^2\right) + C(\Omega)\left(\sum\left\|f_j\right\|_{\frac{3+\epsilon}{1+\epsilon},w}^2\right), \tag{1.4}$$

Where $C(\Omega)$ depends on Ω and $3+\epsilon$, $\frac{3+\epsilon}{1+\epsilon}$, and w denotes the weak $L^{\frac{3+\epsilon}{1+\epsilon}}$ norm defined by

$$\sum\left\|f_j\right\|_{\frac{3+\epsilon}{1+\epsilon},w} = \sup_A |A|^{-1/\left(\frac{3+\epsilon}{1+\epsilon}\right)'} \int_A \sum\left|f_j(x)\right| dx,$$

With A being a set of finite measure $|A|$.

The inequality (1.4) was motivated by the weaker inequality in [3],

$$\sum\left\|\nabla f_j\right\|_2^2 \geq S_{3+\epsilon}\left(\sum\left\|f_j\right\|_{\frac{2(3+\epsilon)}{1+\epsilon}}^2\right) + C_{\frac{3+\epsilon}{1+\epsilon}}(\Omega)\left(\sum\left\|f_j\right\|_{\frac{3+\epsilon}{1+\epsilon}}^2\right), \tag{1.5}$$

which holds for all $\frac{3+\epsilon}{1+\epsilon}$ (with $C_{\frac{3+\epsilon}{1+\epsilon}}(\Omega) \to 0$ as $\frac{2(3+\epsilon)}{1+\epsilon}$). The proof of (1.5) in [3] was very indirect compared to the proof of (1.4) given here. Inequality (1.4) is best possible in the sense that (1.5) cannot hold with $\frac{3+\epsilon}{1+\epsilon}$; this can be shown by taking the f_j in (1.3), applying a cutoff function to make f_j vanish on the boundary, and then expanding the integrals (as in [3]) near $\varepsilon = 0$.

An inequality stronger than (1.4), and involving the gradient norm is

$$\left\|\sum \nabla f_j\right\|_2^2 \geq S_{3+\epsilon}\left(\sum \|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right) + D(\Omega)\left(\sum \|\nabla f_j\|^2_{\frac{3+\epsilon}{2+\epsilon},w}\right), \tag{1.6}$$

with $\frac{3+\epsilon}{2+\epsilon}$. (The reason that (1.6) is stronger than (1.4) is that the Sobolev inequality has an extension to the weak norms, by Young's inequalities in weak $L^{\frac{3+\epsilon}{1+\epsilon}}$ spaces).

Among the open questions concerning (1.4)-(1.6) are the following:

(a) What are the sharp constants in (1.4)-(1.6)? Are they achieved? Except in one case, they are not known, even for a ball. If $\epsilon = 0$, Ω is a ball of radius R and $\epsilon = 2$ in (1.6), then $C_2(\Omega) = \pi^2/(4R^2)$; however, this constant is not achieved [3].

(b) What can replace the right side of (1.4)-(1.6) when Ω is unbounded, e.g., a half-space?

(c) Is there a natural way to bound $\sum\|\nabla f_j\|_2^2 - S_{3+\epsilon}\left(\sum\|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right)$ from below in terms of the "distance" of f_j from the set of optimal functions (1.3)?

Problem B. If $\sum f_j \neq 0$ on $\partial\Omega$, then (1.1) does not hold in Ω (simply take $\sum f_j = 1$ in Ω). Let us assume now that Ω is not only bounded but that $\partial\Omega$ (the boundary of Ω) has enough smoothness. Then (1.1) might be expected to hold if suitable boundary integrals are added to the left side. In Section III we shall prove that for $\sum f_j = \text{constant} \equiv \sum f_j(\partial\Omega)$ on $\partial\Omega$

$$\left\|\sum \nabla f_j\right\|_2^2 + E(\Omega)\left|\sum f_j(\partial\Omega)\right|^2 \geq S_{3+\epsilon}\left(\sum\|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right). \tag{1.7}$$

On the other hand, if f_j is not constant on $\partial\Omega$, then the following two inequalities hold.

$$\left\|\sum \nabla f_j\right\|_2^2 + F(\Omega)\left(\left\|\sum f_j\right\|^2_{H^{1/2}(\partial\Omega)}\right) \geq S_{3+\epsilon}\left(\sum\|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right), \tag{1.8}$$

$$\left\|\sum \nabla f_j\right\|_2 + G(\Omega)\left(\left\|\sum f_j\right\|_{\frac{3+\epsilon}{2+\epsilon},\partial\Omega}\right) \geq S^{1/2}_{3+\epsilon}\left(\sum\|f_j\|_{\frac{2(3+\epsilon)}{1+\epsilon}}\right), \tag{1.9}$$

with $\epsilon^2 + 4\epsilon + 5 = 0$, which is sharp. (Note the absence of the exponent 2 in (1.9)).

In addition to the obvious analogues of questions (a)-(c) for Problem B, one can also ask whether (1.9) can be improved to

$$\left\|\sum \nabla f_j\right\|_2^2 + H(\Omega)\left(\left\|\sum f_j\right\|^2_{\frac{3+\epsilon}{2+\epsilon},\partial\Omega}\right) \geq S_{3+\epsilon}\left(\sum\|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right). \tag{1.10}$$

We do not know.

If Ω is a ball of radius R, we shall establish that the sharp constant in (1.7) is $E(\Omega) = \sigma_{3+\epsilon}R^{1+\epsilon}/(1+\epsilon)$, where $\sigma_{3+\epsilon}$ is the surface area of the ball of unit radius in $\mathbb{R}^{3+\epsilon}$. With this $E(\Omega)$, (1.7) is a strict inequality. Given this fact, one suspects (in view of the solution to Problem A) that some term could be added to the right side of (1.5). However, such a term cannot be any $L^{\frac{3+\epsilon}{1+\epsilon}}(\Omega)$ norm of f_j, as will be shown.

To conclude this Introduction, let us mention two' related inequalities. First, if one is willing to replace $S_{3+\epsilon}$, on the right side of (1.10) by the smaller constant $2^{-2/3+\epsilon}S_{3+\epsilon}$, then for a ball one can obtain the inequality

$$\int \sum|\nabla f_j|^2 + I(\Omega)\left(\sum\|f_j\|^2_{2,\partial\Omega}\right) \geq S^{-2/3+\epsilon}S_{3+\epsilon}\left(\sum\|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right). \tag{1.11}$$

This is proved in Section (1.1). Inequalities related to (1.11) were derived by Cherrier [4] for general manifolds.

Second, one can consider the doubly weighted Hardy-Littlewood-Sobolev inequality [7,10] which in some sense is the dual of (1.1), namely,

$$\left|\iint \sum f_j(x) f_j(x+\epsilon)|\epsilon|^{-\lambda}|x|^{-\alpha}|x+\epsilon|^{-\alpha}dxd(x+\epsilon)\right| \leq P_{\alpha,\lambda,3+\epsilon}\left(\sum\|f_j\|^2_{\frac{3+\epsilon}{1+\epsilon}}\right), \tag{1.12}$$

with $\left(\frac{3+\epsilon}{1+\epsilon}\right)' = 23 + \epsilon/(\lambda + 2\alpha)$, $0 < \lambda < 3+\epsilon$, $0 \leq \alpha < 3 + \epsilon/\left(\frac{3+\epsilon}{1+\epsilon}\right)'$. If f_j is restricted to have support in a bounded domain Ω and if P is (by definition) the sharp constant in $\mathbb{R}^{3+\epsilon}$, one should expect to be able to add some additional term to the left side of (1.12). When $\epsilon = 2$ this is indeed possible, and the additional term is:

$$J_n|\Omega|^{-\lambda/3+\epsilon}\left\{\int\sum f_j(x)|x|^{-\alpha}dx\right\}^2. \quad (1.13)$$

This was proved in [5] for $n=3, \lambda=2, \alpha=\frac{1}{2}$, and Ω being a ball, but the method easily extends (for a ball) to other $3+\epsilon, \lambda$. The result (1.4) further extends to general Ω (with the same constant $J_{3+\epsilon}$) by using the Riesz rearrangement inequality. On the other hand, when $\epsilon \neq 2$, it does not seem to be easy to find the additional term on the left side of (1.12): at least we have not succeeded in doing so. This is an open problem. In particular, in Section III we prove that when $\epsilon=9, \epsilon=0, \lambda=1, \alpha=0$, one cannot even add $\|f_j\|_1^2$ to the left side of (1.12).

II. Proof of Inequalities (1.4) and (1.6):

Proof of Inequalities (1.4)(See [13]): By the rearrangement inequality for the L^2 norm of the gradient we have

$$\left\|\sum \nabla f_j^*\right\|_2 \le \sum\|\nabla f_j\|_2 \quad (2.1)$$

(see, e.g., [8]); in addition we have

$$\sum\|f_j^*\|_{2^*} = \sum\|f_j\|_{2^*}, \qquad \sum\|f_j^*\|_{\frac{3+\epsilon}{1+\epsilon},w} = \sum\|f_j\|_{\frac{3+\epsilon}{1+\epsilon},w}, \quad (2.2)$$

Here, f_j^* denotes the symmetric decreasing rearrangement of the function f_j extended to be zero outside Ω. Therefore, it suffices to consider the case in which Ω is a ball of radius R (chosen to have the same volume as the original domain) and f_j is symmetric decreasing.

Let $g_j \in (\Omega)$ and define u_j to be the solution of

$$\Delta u_j = g_j \quad in \quad \Omega, \qquad u_j = 0 \quad on \quad \partial\Omega \quad (2.3)$$

Let

$$\phi_j(x) = \begin{cases} f_j(x)+u_j(x)+\|u_j\|_\infty & in \quad \Omega, \\ \|u_j\|_\infty (R/|x|)^{n-2} & in \quad \Omega^c. \end{cases} \quad (2.4)$$

The Sobolev inequality in all of $\mathbb{R}^n$ applied to ϕ_j yields

$$\int_\Omega \sum|\nabla f_j+u_j|^2 + \|u_j\|_\infty^2 R^{1+\epsilon}(1+\epsilon)\sigma_{3+\epsilon} \ge S_{3+\epsilon}\left(\sum\|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right) \quad (2.5)$$

Since $\sum f_j \ge 0$ and $u_j + \|u_j\|_\infty \ge 0$. Here

$$\sigma_{3+\epsilon} = 2\,(\pi)^{3+\epsilon/2}/\Gamma(3+\epsilon/2)$$

is the surface area of the unit ball in $\mathbb{R}^{3+\epsilon}$. Therefore, we find

$$\int\sum|\nabla f_j|^2 - 2\int\sum f_j g_j + \int\sum|\nabla u_j|^2 + k\sum\|u_j\|_\infty^2 \ge \sum\|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}, \quad (2.6)$$

where $k = R^{1+\epsilon}(1+\epsilon)\sigma_{3+\epsilon}$. Replacing g_j by λg_j and u_j by λu_j and optimizing with respect to λ we obtain

$$\int\sum|\nabla f_j|^2 \ge S_{3+\epsilon}\left(\sum\|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right) + \sum\left(\int f_j g_j\right)^2 \Big/ \left[\int|\nabla u_j|^2 + k\|u_j\|_\infty^2\right]. \quad (2.7)$$

In inequality (2.7) we can obviously maximize the right side with respect to g_j. In view of the definition of the weak norm we shall in fact restrict our attention to $g_j = 1_A$, namely, the characteristic function of some set A in Ω. We shall now establish some simple estimates for all the quantities in (2.7) in which $C_{3+\epsilon}$, generically denotes constants depending only on $3+\epsilon$,

$$\int\sum f_j g_j = \int_A\sum f_j, \quad (2.8)$$

$$\int\sum|\nabla u_j|^2 \le C_{3+\epsilon}|A|^{1+2/3+\epsilon}, \quad (2.9)$$

$$\|u_j\|_\infty \le C_{3+\epsilon}|A|^{2/3+\epsilon}, \quad (2.10)$$

Indeed we have, by multiplying (2.3) by u_j and using Hölder's inequality,

$$\int\sum|\nabla u_j|^2 = -\int_A\sum u_j \le \sum\|u_j\|_{\frac{2(3+\epsilon)}{1+\epsilon}}|A|^{\frac{5}{2(3+\epsilon)}} \le S_{3+\epsilon}^{-1/2}\left(\sum\|\nabla u_j\|_2\,|A|^{\frac{5}{2(3+\epsilon)}}\right) \quad (2.11)$$

which implies (2.9). Next we have, by comparison with the solution in $\mathbb{R}^{3+\epsilon}$,

$$\begin{aligned} |u_j| &\le C_{3+\epsilon}|x|^{-(1+\epsilon)} * (1_A) \\ &\le C'_{3+\epsilon}|A|^{2/3+\epsilon} \end{aligned} \tag{2.12}$$

since the function $|x|^{-(1+\epsilon)}$ belongs to $L_w^{-\frac{3+\epsilon}{1+\epsilon}}$. Since $|A| \le |\Omega| = \sigma_{3+\epsilon} R^{3+\epsilon}/3+\epsilon$ we obtain

$$\int \sum |\nabla u_j|^2 + k\sum \|u_j\|_\infty^2 \le C_{3+\epsilon}|A|^{4/3+\epsilon}R^{1+\epsilon}. \tag{2.13}$$

Hence (1.4) has been proved (for all Ω) with a constant

$$C(\Omega) = C_{3+\epsilon}|\Omega|^{-\frac{1+\epsilon}{3+\epsilon}}. \tag{2.14}$$

Proof of Inequality (1.6)(See [13]): To a certain extent the previous proof can be imitated except for one important ingredient, namely, the rearrangement technique cannot be used since it is not true that $\|\sum \nabla f_j\|_{\frac{3+\epsilon}{2+\epsilon},w} \le \sum\|\nabla f_j^*\|_{\frac{3+\epsilon}{2+\epsilon},w}$. (However, it is still true that we can replace f_j by $|f_j|$ without changing any of the norms in (1.6), and thus we may and still assume that $\sum f_j \ge 0$). Consequently we have to use a direct approach and the constant $D(\Omega)$ in (1.6) will not depend only on $|\Omega|$; it will in fact depend on the capacity of Ω. It is an open question whether (1.6) holds with $D(\Omega)$ depending only on $|\Omega|$. Our result is that:

$$D(\Omega) = C_{3+\epsilon}/cap(\Omega). \tag{2.15}$$

We begin as before with (2.3), but (2.4) is replaced by:

$$\phi_j = \begin{cases} f_j + u_j + \|u_j\|_\infty & in \quad \Omega, \\ \|u_j\|_\infty v_j & in \quad \Omega^c, \end{cases} \tag{2.16}$$

Where v_j is the solution of

$$\begin{aligned} \Delta v_j &= 0 \qquad in \quad \Omega^c, \\ v_j &= 1 \qquad on \quad \partial\Omega, \end{aligned} \tag{2.17}$$

With $v_j \to 0$ at infinity. By definition,

$$cap(\Omega) = \int \sum |\nabla v_j|^2. \tag{2.18}$$

Inequality (2.7) still holds but with the constant k replaced by $k = cap(\Omega)$. Also we note that (2.7) can be written as

$$\int \sum |\nabla f_j|^2 \ge S_{3+\epsilon}\left(\sum \|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right) + \sum\left(\int \nabla f_j \cdot \nabla u_j\right)^2 \Big/ \left[\int |\nabla u_j|^2 + k\|u_j\|_\infty^2\right], \tag{2.19}$$

which holds for any $u_j \in C_0^\infty(\Omega)$. By density, (2.19) still holds for every u_j in $H_0^1 \cap L^\infty$ (the reason is that for every such u_j there is a sequence $(u_j)_{j_0} \in C_0^\infty(\Omega)$ with $(u_j)_{j_0} \to u_j$ in H_0^1 and $\|(u_j)_{j_0}\|_\infty \to \|u_j\|_\infty$).

We now choose u_j to be the solution of (2.3) with

$$\sum g_j = \frac{\partial}{\partial x_i}\left[\sum\left(sgn\frac{\partial f_j}{\partial x_i}\right)1_A\right] \tag{2.20}$$

This function u_j is in L^∞ as we now verify. We can write

$$u_j = w_j + h_j,$$

where w_j satisfies $\Delta w_j = g_j$ in all of $\mathbb{R}^{3+\epsilon}$, namely,

$$w_j = C_{3+\epsilon}|x|^{-(1+\epsilon)} * g_j. \tag{2.21}$$

Clearly h_j is harmonic and $h_j = -w_j$ on $\partial\Omega$ therefore $\|\sum h_j\|_\infty \le \|\sum w_j\|_{\infty,\partial\Omega} \le \|\sum w_j\|_\infty$ and hence $\sum\|u_j\|_\infty \le 2\sum\|w_j\|_\infty$. On the other hand, and thus

$$w_j = C_{3+\epsilon}\sum\left(\frac{\partial}{\partial x_i}|x|^{-(1+\epsilon)}\right) * \left[\left(sgn\frac{\partial f_j}{\partial x_i}\right)1_A\right],$$

and thus

$$|w_j| \le C_{3+\epsilon}(1+\epsilon)|x|^{-(2+\epsilon)} * 1_A. \tag{2.22}$$

Since $|x|^{-(2+\epsilon)} \in L_{w_j}^{3+\epsilon/2+\epsilon}$ we obtain

$$\left\|\sum u_j\right\|_\infty \le 2\sum\|w_j\|_\infty \le C'_{3+\epsilon}|A|^{1/3+\epsilon}. \tag{2.23}$$

Next, let us estimate $\int \sum |\nabla u_j|^2$. Multiplying (2.3) by u_j we have

$$\int \sum |\nabla u_j|^2 = \int \sum (sgn\,\partial f_j/\partial x_i)1_A(\partial u_j/\partial x_i) \le \left[\int \sum |\nabla u_j|^2\right]^{1/2} |A|^{1/2}$$

and thus

$$\int \sum |\nabla u_j|^2 \le |A|. \tag{2.24}$$

Finally, since $\sum f_j = 0$ on $\partial\Omega$,

$$\int \sum \nabla f_j \cdot \nabla u_j = -\int \sum f_j \Delta u_j = \int \sum |\partial f_j/\partial x_i| 1_A. \tag{2.25}$$

Using these estimates (2.19) we find

$$\int \sum |\nabla f_j|^2 \geq S_{3+\epsilon}\left(\sum \|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right) + C_{3+\epsilon}\left(\sum\left(\int_A |\partial f_j/\partial x_i|\right)^2\right)\Big/\left(cap(\Omega)|A|^{2/3+\epsilon}\right),$$

Since $|A|^{1-(2/3+\epsilon)} \leq |\Omega|^{1-(2/3+\epsilon)} \leq S_{3+\epsilon}^{-1} cap(\Omega)$ by Sobolev's inequality applied to the function $\widetilde{v_j} = v_j$ in Ω^c and $\widetilde{v_j} = 1$ in Ω. This completes the proof of (1.6) with the constants given in (2.15).

III. Proofs of (1.7)-(1.9) and Related Matters

Proof of (1.8)(See [13]): Let us define:

$$\phi_j = \begin{cases} f_j & in \quad \Omega, \\ w_j & in \quad \Omega^c, \end{cases} \tag{3.1}$$

Where w_j is the harmonic function that vanishes at infinity and agrees with f_j on $\partial\Omega$. Using ϕ_j in (1.1) we find:

$$\int_\Omega \sum |\nabla f_j|^2 + \int_{\Omega^c} \sum |\nabla w_j|^2 \geq S_{3+\epsilon}\left(\sum \|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right). \tag{3.2}$$

On the other hand, we have

$$\int_{\Omega^c} \sum |\nabla w_j|^2 \sim \sum \|f_j\|^2_{H^{1/2}(\partial\Omega)}. \tag{3.3}$$

This concludes the proof of (1.8).

Proof of (1.7)(See [13]): Now suppose that f_j is a constant on $\partial\Omega$. We shall first investigate the case that Ω is a ball of radius R centered at zero. In this case $w_j(x) = f_j(\partial\Omega) R^{(3+\epsilon)-2}|x|^{2-(3+\epsilon)}$. Above Inequality (3.2), then yields (1.7) with:

$$E(\Omega) = cap(\Omega) = \sigma_{3+\epsilon} R^{1+\epsilon}/1+\epsilon = \frac{(3+\epsilon)|\Omega|}{1+\epsilon}\left\{\frac{\sigma_{3+\epsilon}}{(3+\epsilon)|\Omega|}\right\}^{2/3+\epsilon} \tag{3.4}$$

Furthermore, (1.7) is a strict inequality with this $E(\Omega)$ because the function ϕ_j is not of the form (1.3). Also, $E(\Omega)$ given by the sharp constant. To see this we apply (1.9) with $f_j = (f_j)_\varepsilon$, given by (1.3) with $a = 1$ and $x + \epsilon = 0 =$ center of the ball. We have:

$$\int_{\mathbb{R}^{3+\epsilon}} \sum \left|\nabla (f_j)_\varepsilon\right|^2 = S_{3+\epsilon}\left(\sum \left\|(f_j)_\varepsilon\right\|^2_{\frac{2(3+\epsilon)}{1+\epsilon},\mathbb{R}^{3+\epsilon}}\right).$$

On the other hand, as $\varepsilon \to 0$

$$\int_{\mathbb{R}^{3+\epsilon}} \sum \left|\nabla (f_j)_\varepsilon\right|^2 = \int_\Omega \sum \left|\nabla (f_j)_\varepsilon\right|^2 + \int_{\Omega^c} \sum \left|\nabla (f_j)_\varepsilon\right|^2$$
$$= \int_\Omega \sum \left|\nabla (f_j)_\varepsilon\right|^2 + cap(\Omega)\left(\sum \left|(f_j)_\varepsilon(\partial\Omega)\right|^2\right) + o(1). \tag{3.6}$$

Here we have to note that as $\varepsilon \to 0$ for $|x| > R$

$$(f_j)_\varepsilon(x) \to |x|^{-(1+\epsilon)}$$

in the appropriate topologies. On the other hand,

$$\int_{\mathbb{R}^{3+\epsilon}} \sum \left|(f_j)_\varepsilon\right|^{\frac{2(3+\epsilon)}{1+\epsilon}} - \int_\Omega \sum \left|(f_j)_\varepsilon\right|^{\frac{2(3+\epsilon)}{1+\epsilon}} = \int_{\Omega^c} \sum \left|(f_j)_\varepsilon\right|^{\frac{2(3+\epsilon)}{1+\epsilon}} \to C.$$

Thus

$$\sum \left\|(f_j)_\varepsilon\right\|^2_{\frac{2(3+\epsilon)}{1+\epsilon},\mathbb{R}^{3+\epsilon}} = \sum \left\|(f_j)_\varepsilon\right\|^2_{\frac{2(3+\epsilon)}{1+\epsilon},\Omega} + o(1). \tag{3.7}$$

This proves that $E(\Omega)$ in (1.7) is greater than or equal to $cap(\Omega)$ when Ω is a ball, and thus that (3.4) is sharp.

The same calculation with $(f_j)_\varepsilon$, as above shows that if Ω is a ball there is no inequality of the type:

$$\int_\Omega \sum |\nabla f_j|^2 + cap(\Omega)\left(\sum |f_j(\partial\Omega)|^2\right) \geq S_{3+\epsilon}\left(\sum \|f_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right) + d\sum \|f_j\|_1^2 \tag{3.8}$$

with $\epsilon \geq 0$, because the additional term $\sum \left\|(f_j)_\varepsilon\right\|_1 = O(1)$ as $\varepsilon \to 0$.

Now we consider a general domain with $f_j(\partial\Omega) = constant = C$. We can assume $C \geq 0$ and note that we can also assume $f_j \geq C$ in Ω. (This is so because replacing f_j by If $\sum|f_j - C| + C \geq \sum f_j$ does not decrease the $L^{\frac{2(3+\epsilon)}{1+\epsilon}}$ norm and leaves $\|\sum \nabla f_j\|_2$ invariant.) Consider the function $g_j = \sum f_j - C \geq 0$ which vanishes on $\partial\Omega$ and hence can be extended to be zero on Ω^c. Apply to g_j the rearrangement inequality for the L^2 norm of the gradient, as was done in Section II. Finally considers $\tilde{f}_j = g_j^* + C$ in the ball Ω^* whose volume is $|\Omega|$. Since $\tilde{f}_j(\partial\Omega^*) = C = f_j(\partial\Omega)$ we have

$$\int_{\Omega^*} \sum |\nabla \tilde{f}_j|^2 + E(\Omega^*)\left(\sum |f_j(\partial\Omega)|^2\right) \geq S_n\left(\sum \|\tilde{f}_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon},\Omega^*}\right),$$

As we remarked, $\|\sum \nabla f_j\|_2 \geq \|\sum \nabla \tilde{f}_j\|_2$. Also since $f_j \geq C$, it is easy to check that $\sum\|f_j\|_{\frac{2(3+\epsilon)}{1+\epsilon}} = \sum\|\tilde{f}_j\|_{\frac{2(3+\epsilon)}{1+\epsilon}}$.

The conclusion to be drawn from this exercise is that (1.7) holds for general Ω with $E(\Omega)$ given by (3.4), namely, $cap(\Omega^*)$. We also note that (1.7), with this $E(\Omega)$, is strict, since it is strict for a ball.

Question: Is $E(\Omega)$ given by (3.4) the sharp constant in general?

Proof of (1.9)(See [13]): Given f_j in Ω we consider the harmonic function h_j in Ω which equals f_j on $\partial\Omega$ We write

$$f_j = h_j + u_j \tag{3.9}$$

With $u_j = 0$ on $\partial\Omega$ and thus

$$\int \sum |\nabla u_j|^2 \geq S_{3+\epsilon}\left(\sum \|u_j\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right). \tag{3.10}$$

On the one hand

$$\int \sum |\nabla u_j|^2 = \int \sum |\nabla(f_j - h_j)|^2 = \int \sum |\nabla f_j|^2 - \int \sum |\nabla h_j|^2 \tag{3.11}$$

(note that $\int_\Omega \sum |\nabla h_j|^2 = \int_{\partial\Omega} \sum h_j(\partial h_j/\partial 3+\epsilon) = \int_{\partial\Omega} \sum f_j(\partial h_j/\partial 3+\epsilon) = \int_\Omega \sum(\nabla f_j \nabla h_j)$). On the other hand, by the triangle inequality,

$$\sum \|u_j\|_{\frac{2(3+\epsilon)}{1+\epsilon}} \geq \sum \|f_j\|_{\frac{2(3+\epsilon)}{1+\epsilon}} - \sum \|h_j\|_{\frac{2(3+\epsilon)}{1+\epsilon}}. \tag{3.12}$$

Inserting (3.11) and (3.12) in (3.10) we obtain

$$\sum \|\nabla f_j\|_2 + \sum \|h_j\|_{\frac{2(3+\epsilon)}{1+\epsilon}} \geq S^{1/2}_{3+\epsilon}\left(\sum \|f_j\|_{\frac{2(3+\epsilon)}{1+\epsilon}}\right). \tag{3.13}$$

Next we claim that

$$\sum \|h_j\|_{\frac{2(3+\epsilon)}{1+\epsilon}} \leq G(\Omega)\left(\sum \|f_j\|_{\frac{3+\epsilon}{2+\epsilon},\partial\Omega}\right) \tag{3.14}$$

with $\epsilon^2 + 4\epsilon + 5 = 0$, which will complete the proof of (1.9). The proof is a standard duality argument. Indeed, let ψ_j be the solution of

$$\begin{aligned} \Delta\psi_j &= Y \quad in \quad \Omega, \\ \psi_j &= 0 \quad on \quad \partial\Omega, \end{aligned} \tag{3.15}$$

where Y is some arbitrary function in L^t. We have, by multiplying by h_j and integrating by parts,

$$\int_\Omega \sum h_j Y = \int_{\partial\Omega} \sum f_j \frac{\partial\psi_j}{\partial(3+\epsilon)}. \tag{3.16}$$

However, the $L^{\frac{3+\epsilon}{1+\epsilon}}$ regularity theory shows that $\psi_j \in W^{2,t}$with $\|\sum \psi_j\|_{W^{2,t}(\Omega)} \leq C\|Y\|_t$. In particular, $\|\sum \nabla\psi_j\|_{W^{1,t}(\Omega)} \leq C\|Y\|_t$ and, by trace inequalities,

$$\left\|\sum \frac{\partial\psi_j}{\partial 3+\epsilon}\right\|_{\frac{t(2+\epsilon)}{(3+\epsilon)-t},\partial\Omega} \leq C\|Y\|_t, \tag{3.17}$$

Therefore, by (3.16) and Hölder's inequality,

$$\left|\int \sum h_j Y\right| \leq C \sum \|f_j\|_{\frac{3+\epsilon}{2+\epsilon},\partial\Omega} \|Y\|_t, \tag{3.19}$$

Since (3.19) holds for all Y we conclude that

$$\left\|\sum h_j\right\|_{2^*} \leq C \sum \|f_j\|_{\frac{3+\epsilon}{2+\epsilon},\partial\Omega},$$

when $\epsilon^2 + 4\epsilon + 5 = 0$.

Finally, we claim that there is no inequality of the type (1.9) with $\epsilon^2 + 4\epsilon + 5 = 0$. Indeed, suppose (1.9) holds with some such $\frac{3+\epsilon}{2+\epsilon}$. We choose $f_j = (f_j)_\varepsilon$, as in (1.3) with $a = 1$ and $(x+\epsilon) \in \partial\Omega$. It is obvious that as $\varepsilon \to 0$

$$\sum \int_\Omega |\nabla(f_j)_\varepsilon|^2 \Big/ \int_{\mathbb{R}^{3+\epsilon}} |\nabla(f_j)_\varepsilon|^2 = 1/2 + o(1),$$

$$\sum \int_\Omega |(f_j)_\varepsilon|^{\frac{2(3+\epsilon)}{1+\epsilon}} \Big/ \int_{\mathbb{R}^{3+\epsilon}} |(f_j)_\varepsilon|^{\frac{2(3+\epsilon)}{1+\epsilon}} = 1/2 + o(1),$$

while

$$\int_{\mathbb{R}^{3+\epsilon}} \sum \left|\nabla (f_j)_\varepsilon\right|^2 = S_{3+\epsilon}\left(\sum \left\|(f_j)_\varepsilon\right\|^2_{\frac{2(3+\epsilon)}{1+\epsilon},\mathbb{R}^{3+\epsilon}}\right)$$
$$and \sum \left\|(f_j)_\varepsilon\right\|_{\frac{3+\epsilon}{2+\epsilon},\partial\Omega} \Big/ \left\|(f_j)_\varepsilon\right\|_{\frac{2(3+\epsilon)}{1+\epsilon}} = o(1).$$

This contradicts (1.9).

Remark. The last exercise with $(f_j)_\varepsilon$ given above shows that it is not possible to apply rearrangement techniques when f_j is not constant on $\partial\Omega$, even if Ω is a ball. It also shows that there is no inequality for all $f_j \in H^1$ of the type

$$\left\|\sum \nabla f_j\right\|_2^2 + C\sum \|f_j\|^2_{\frac{3+\epsilon}{2+\epsilon},\Omega} \geq S_{3+\epsilon}\left(\sum \|f\|^2_{\frac{2(3+\epsilon)}{1+\epsilon}}\right)$$

with $\epsilon > -3$.

Proof of (1.11)(See [13]): Let Ω be a ball of radius R centered at zero. For simplicity, assume $R = 1$. Define

$$g_j(x) = \begin{cases} f_j(x), & |x| \leq 1, \\ |x|^{-(1+\epsilon)} f_j(x|x|^{-2}) & |x| \geq 1, \end{cases} \tag{3.20}$$

and apply the usual Sobolev inequality (1.1) to g_j. We note (by a change of variables) that

$$\int_\Omega \sum g_j^{\frac{2(3+\epsilon)}{1+\epsilon}} = \int_{\Omega^c} \sum g_j^{\frac{2(3+\epsilon)}{1+\epsilon}}.$$

$$\int_\Omega \sum \left|\nabla g_j\right|^2 = \int_{\Omega^c} \sum \left|\nabla g_j\right|^2 - (1+\epsilon)\left\|f_j\right\|^2_{2,\partial\Omega}. \tag{3.21}$$

Inserting (3.21) into (1.1) yields (1.11) with $I(\Omega) = (1+\epsilon)/2$.

Remark on the Hardy-Littlewood-Sobolev Inequality

Consider the inequality (in $\mathbb{R}^3$)

$$\sum I(f_j) \leq P\left(\sum \|f_j\|^2_{6,5}\right), \tag{3.22}$$

with

$$\sum I(f_j) = \iint \sum \mathrm{f}_j(x) f_j(x+\epsilon)|\epsilon|^{-1} dx d(x+\epsilon) \geq 0. \tag{3.23}$$

The sharp constant P is known to be [7]

$$P = 4^{5/3}/[3\pi^{1/3}]. \tag{3.24}$$

Let Ω be a ball of radius one centered at zero and assume that $\sum f_j = 0$ outside Ω. In this case, (3.22) is strict because the only functions that give equality in (3.22) are of the form [7]

$$\sum (f_j)_\varepsilon(x) = a[\varepsilon^2 + |\epsilon|^2]^{-5/2}. \tag{3.25}$$

For $\sum f_j = 0$ outside Ω, we ask whether (3.22) can be improved to

$$C\left(\sum \|f_j\|_1^2\right) + \sum I(f_j) \leq P\left(\sum \|f_j\|^2_{6/5}\right). \tag{3.26}$$

Our conclusion is that (3.26) fails for any $C > 0$.

Take $f_j = (\widetilde{f_j})_\varepsilon = (f_j)_\varepsilon 1_\Omega$ with $(f_j)_\varepsilon$ given by (3.25) and with $x + \epsilon = 0$ and with $a = a_\varepsilon$ chosen so that $\Sigma \left\|(f_j)_\varepsilon\right\|_{6/5,\mathbb{R}^3} = 1$. The function $(f_j)_\varepsilon$ satisfies the following (Euler) equation on $\mathbb{R}^3$,

$$\sum \frac{1}{|x|} * (f_j)_\varepsilon = P\left(\sum (f_j)_\varepsilon^{1/5}\right) \tag{3.27}$$

However, for $|x| < 1$

$$\sum \left(\frac{1}{|x|} * (\widetilde{f_j})_\varepsilon\right)(x) + K_\varepsilon = \sum \left(\frac{1}{|x|} * (f_j)_\varepsilon\right)(x), \tag{3.28}$$

where K_ε is a constant bounded above by $D_\varepsilon = \int_{|x|>1} \Sigma (f_j)_\varepsilon$. Multiply (3.27) by $(\widetilde{f_j})_\varepsilon$ and integrate over Ω. Then

$$\sum I(\widetilde{f_j})_\varepsilon + T_\varepsilon\left(\sum \left\|(\widetilde{f_j})_\varepsilon\right\|_1^2\right) \geq \sum I(\widetilde{f_j})_\varepsilon + K_\varepsilon \int \sum (\widetilde{f_j})_\varepsilon$$
$$= P\left(\sum \left\|(\widetilde{f_j})_\varepsilon\right\|_{6/5}^{6/5}\right) \geq \mathrm{P}\left(\sum \left\|(\tilde{\mathrm{f}}_\mathrm{j})_\varepsilon\right\|_{6/5}^2\right), \tag{3.29}$$

where $T_\varepsilon = D_\varepsilon / \int \sum (\tilde{f}_j)_\varepsilon$. From (3.29), we see that (3.26) fails if $C > T_\varepsilon$ for any $\varepsilon > 0$. However, it is obvious that $T_\varepsilon \to 0$ as $\varepsilon \to 0$.

References

[1]. H. BREZIS , E H. LIEB , Sobolev Inequalities with Remainder Terms, Journal of Functional Analysis 62, 73-86 (1985)

[2]. A. Cotsiolis and N. K. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, J. Math. Anal. Appl. 295(2004), 225-236..

[3]. G. Liu, Sharp k-order Sobolev inequalities in Euclidean space Rn and the sphere Sn, preprint 2006..

[4]. J. Dolbeault and G. Toscani. Stability results for logarithmic Sobolev and Gagliardo-Nirenberg inequalities. Int. Math. Res. Not. IMRN, (2):473–498, 2016.

[5]. M. Fathi, E. Indrei, and M. Ledoux. Quantitative logarithmic Sobolev inequalities and stability estimates. Discrete Contin. Dyn. Syst., 36(12):6835–6853, 2016..

[6]. N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative Sobolev inequality for functions of bounded variation. J. Funct. Anal., 244(1):315–341, 2007.

[7]. A. Figalli, F. Maggi, and A. Pratelli. Sharp stability theorems for the anisotropic Sobolev and log-Sobolev inequalities on functions of bounded variation. Adv. Math., 242:80–101, 2013.

[8]. A. Figalli and R. Neumayer. Gradient stability for the Sobolev inequality: the case p ≥ 2. J.Eur. Math. Soc. (JEMS), 21(2):319–354, 2019..

[9]. F. Gazzola and T. Weth. Remainder terms in a higher order Sobolev inequality. Arch. Math. (Basel), 95(4):381–388, 2010..

[10]. E. Indrei and D. Marcon. A quantitative log-Sobolev inequality for a two parameter family of functions. Int. Math. Res. Not., (20):5563–5580, 2014..

[11]. V. Ra˘ dulescu, D. Smets, M. Willem, Hardy–Sobolev inequalities with remainder terms, Topol. Meth. Nonlinear Anal. 20 (2002) 145–149..

[12]. A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli. The sharp Sobolev inequality in quantitative form. Journal of the European Mathematical Society, 11(5):1105{1139, 2009.

[13]. H. Brezis, E. Lieb, Sobolev inequalities with remainder terms, J. Funct. Anal. 62 (1985), 73-86.

[14]. Francesco Maggi and Cdric Villani. Balls have the worst best Sobolev inequalities. Journal of Geometric Analysis, 15(1):83{121, 2005.

Sulima Ahmed Mohammed, et. al. "Series of Sobolev Inequalities with Remainder Terms." *IOSR Journal of Mathematics (IOSR-JM)*, 18(1), (2022): pp. 47-54.