IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 18, Issue 1 Ser. III (Jan. – Feb. 2022), PP 21-26
www.iosrjournals.org

Secure Dominating Sets of Wheels

K. Lal Gipson[1], Subha T[2]
[1]*Assistant Professor, Department of Mathematics, Scott Christian College (Autonomous), Nagercoil-629003, India.*

[2]*Research Scholar, Reg. No.: 18213112092017, Department of Mathematics, Scott Christian College (Autonomous), Nagercoil-629003, India.*
. Affiliated to ManonmaniamSundaranar University, Abishekapatti, Tirunelveli-627012, India.

Abstract:
Let $G = (V, E)$ be a simple graph. A dominating set S of G is a secure dominating set if for each $u \in V - S$ there exists $v \in N(u) \cap S$ such that $(S - \{v\} \cup \{u\})$ is a dominating set. Let W_n be the wheel and let$\mathcal{D}_s(W_n, i)$ denote the family of all secure dominating sets of W_n with cardinality i. In this paper, we obtain all the secure dominating sets of wheels by recursive method.
***Key Word**: Domination, Secure domination, Secure dominating set, Secure domination number.*

Date of Submission: 29-01-2022 Date of Acceptance: 10-02-2022

I. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2]. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V: uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set of G, if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to at least one vertex in S. A dominating set S of G is a secure dominating set if for each $u \in V - S$ there exists $v \in N(u) \cap S$ such that $(S - \{v\}) \cup \{u\}$ is a dominating set. In this case we say that u is S- defended by v or vS-defends u. The secure domination number $\gamma_s(G)$ is the minimum cardinality of a secure dominating set. The concept secure dominating set is introduced by Cockayne et al [3]. A simple path is a path in which all its internal vertices have degree two, and the end vertices have degree one and is denoted by P_n. A cycle can be defined as a closed path, and is denoted by C_n. A graph G is complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n. For $n \geq 4$, the wheel W_n is defined to be the graph $K_1 + C_{n-1}$.

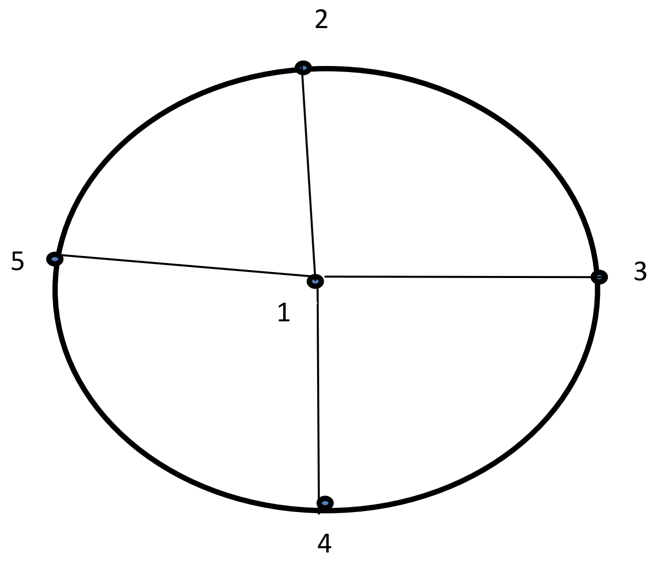
Definition 1.1 [4]

Let X be a dominating set of G. Let $S = \{v \in X: X - \{v\} is a dominating set of G\}$. For $u \in V - X$, let $A(u, X) = \{v \in X: vX - defends u\}$.

Theorem 1.2 [4]

A secure dominating set X is minimal if and only if for each $s \in S$ with $N(s) \cap S \neq \emptyset$, there exists$u_s \in V - X$ such that for each $v \in A(u_s, X) - \{s\}$, one of the following holds:
1. There exists $w \in V - X$such that $N(w) \cap X = \{v, s\}$ and $u_s \notin N(w)$.
2. $N(s) \cap X = \{v\}$ and $u_s \in N(v) - N(s)$.

Example
The graph W_5



$S = \{1,5\}$ is a secure dominating set. For, $V = \{1,2,3,4,5\}$; $V - S = \{2,3,4\}$. The sets $\{2,5\}$, $\{1,3\}$, $\{4,5\}$ are also dominating sets.

In the next section, we construct the families of the secure domination sets of the wheels by recursive method.

As usual we use $\lfloor x \rfloor$for the largest integer less than or equal to x and $\lceil x \rceil$ for the smallest integer greater than or equal to x. Also, we denote the set $\{1,2,\dots,n\}$ by $[n]$, throughout this paper.

II. Secure Dominating Sets of Wheels

Let $\mathcal{D}_s(W_n, i)$ be the family of secure dominating sets of W_n with cardinality i. We need the following lemmas to prove our main results in this section.

Lemma 2.1[4]

Let W_n be the wheel with n vertices. Then $\gamma_s(W_n) = \left\lceil \frac{n}{3} \right\rceil$ for $n \geq 5$.

Lemma 2.2

Let W_n be the wheel with n vertices and $\mathcal{D}_s(W_n, i)$ be the family of secure dominating sets with cardinality i. Then $\mathcal{D}_s(W_n, i) \neq \emptyset$ if and only if $\left\lceil \frac{n}{3} \right\rceil \leq i \leq n$. Also $\mathcal{D}_s(W_n, i) = \emptyset$ if and only if $i < \left\lceil \frac{n}{3} \right\rceil$ or $i > n$.

Proof

By the definition of secure domination number, there is at least one secure dominating set in $\mathcal{D}_s(W_n, i)$ when $i = \gamma_s(W_n) = \left\lceil \frac{n}{3} \right\rceil$. Since all the super set of a secure dominating set is again a secure dominating set. Therefore $\mathcal{D}_s(W_n, i) \neq \emptyset$ if $\left\lceil \frac{n}{3} \right\rceil \leq i \leq n$.

Suppose $i < \left\lceil \frac{n}{3} \right\rceil$. Then by the definition of $\gamma_s(W_n)$, there is no secure dominating sets in $\mathcal{D}_s(W_n, i)$. Therefore $\mathcal{D}_s(W_n, i) = \emptyset$, if $i < \left\lceil \frac{n}{3} \right\rceil$. Clearly $\mathcal{D}_s(W_n, i) = \emptyset$ if $i > n$.

Lemma 2.3

If a graph G contains a wheel of order $3k - 1$, then every secure dominating set of G must contain at least k vertices of the wheel.

Lemma 2.4

If $Y \in \mathcal{D}_s(W_{n-1}, i - 1)$, then $Y \cup \{n\} \in \mathcal{D}_s(W_n, i)$.

Proof

If $Y \in \mathcal{D}_s(W_{n-1}, i - 1)$, then at least one vertex labeled $n - 1$ or $n - 2$ or $n - 3$ or $n - 4$ or $n - 5$ is in Y as an end vertex. If $n - 1 \in Y$, then $Y \cup \{n\} \in X_1$ (say) securely dominate W_n. Thus $X_1 \in \mathcal{D}_s(W_n, i)$. If $n - 2 \in Y$, then $Y \cup \{n\} \in X_2$ (say) securely dominate W_n. Thus $X_2 \in \mathcal{D}_s(W_n, i)$. If $n - 3 \in Y$, then $Y \cup \{n\} \in X_3$ (say) securely dominate W_n. Thus $X_3 \in \mathcal{D}_s(W_n, i)$. If $n - 4 \in Y$, then $Y \cup \{n\} \in X_4$ (say) securely dominate

W_n. Thus $X_4 \in \mathcal{D}_s(W_n, i)$. If $n-5 \in Y$, then $Y \cup \{n\} \in X_5$ (say) securely dominate W_n. Thus $X_5 \in \mathcal{D}_s(W_n, i)$. In all the cases, $Y \cup \{n\}$ securely dominate W_n. Hence $Y \cup \{n\} \in \mathcal{D}_s(W_n, i)$, if $Y \in \mathcal{D}_s(W_{n-1}, i-1)$.

Lemma 2.5

If $Y \in \mathcal{D}_s(W_{n-4}, i-1)$ and there exists $x \in [n]$ such that $Y \cup \{x\} \in \mathcal{D}_s(W_n, i)$, then $Y \in \mathcal{D}_s(W_{n-3}, i-1)$.

Proof

Suppose that $Y \notin \mathcal{D}_s(W_{n-3}, i-1)$. Since $Y \in \mathcal{D}_s(W_{n-4}, i-1)$, Y contains at least one vertex labeled $n-4$ or $n-5$ or $n-6$ or $n-7$ or $n-8$ as an end vertex. If $n-4 \in Y$ and $Y \cup \{x\} \in \mathcal{D}_s(W_n, i)$ for some $x \in [n]$, then $Y \in \mathcal{D}_s(W_{n-3}, i-1)$, a contradiction. If $n-5 \in Y$ and $Y \cup \{x\} \in \mathcal{D}_s(W_n, i)$ for some $x \in [n]$, then $Y \in \mathcal{D}_s(W_{n-3}, i-1)$, a contradiction. If $n-6 \in Y$ and $Y \cup \{x\} \in \mathcal{D}_s(W_n, i)$ for some $x \in [n]$, then $Y \in \mathcal{D}_s(W_{n-3}, i-1)$, a contradiction. If $n-7 \in Y$ and $Y \cup \{x\} \in \mathcal{D}_s(W_n, i)$ for some $x \in [n]$, then $Y \in \mathcal{D}_s(W_{n-3}, i-1)$, a contradiction. If $n-8 \in Y$, but in this case $Y \cup \{x\} \notin \mathcal{D}_s(W_n, i)$ for any $x \in [n]$, a contradiction. Therefore $Y \in \mathcal{D}_s(W_{n-3}, i-1)$.

Lemma 2.6

i) If $\mathcal{D}_s(W_{n-1}, i-1) = \mathcal{D}_s(W_{n-3}, i-1) = \emptyset$, then $\mathcal{D}_s(W_{n-2}, i-1) = \emptyset$.

ii) If $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) \neq \emptyset$, then $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$.

iii) If $\mathcal{D}_s(W_{n-1}, i-1) = \mathcal{D}_s(W_{n-2}, i-1) = \mathcal{D}_s(W_{n-3}, i-1) = \emptyset$, then $\mathcal{D}_s(W_n, i) = \emptyset$.

Proof

i) Since $\mathcal{D}_s(W_{n-1}, i-1) = \emptyset$, by Lemma 2.2, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-1}{3} \right\rceil$.

$$> n-2$$

$\Longrightarrow i-1 > n-2$ (1)

Since $\mathcal{D}_s(W_{n-3}, i-1) = \emptyset$, by Lemma 2.2, $i-1 > n-3$ or $i-1 < \left\lceil \frac{n-3}{3} \right\rceil$

$$< \left\lceil \frac{n-2}{3} \right\rceil$$

$\Longrightarrow i-1 < \left\lceil \frac{n-2}{3} \right\rceil$ (2)

From (1) and (2), we have $i-1 < \left\lceil \frac{n-2}{3} \right\rceil$ or $i-1 > n-2$. By Lemma 2.2, $\mathcal{D}_s(W_{n-2}, i-1) = \emptyset$.

ii) Suppose $\mathcal{D}_s(W_{n-2}, i-1) = \emptyset$, by Lemma 2.2, we have $i-1 > n-2$ or $i-1 < \left\lceil \frac{n-2}{3} \right\rceil$.
If $i-1 > n-2 > n-3$, then $i-1 > n-3$. Therefore$\mathcal{D}_s(W_{n-3}, i-1) = \emptyset$, a contradiction.
If $i-1 < \left\lceil \frac{n-2}{3} \right\rceil < \left\lceil \frac{n-1}{3} \right\rceil$, then $i-1 < \left\lceil \frac{n-1}{3} \right\rceil$. Therefore$\mathcal{D}_s(W_{n-1}, i-1) = \emptyset$, a contradiction.
Thus $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$.

iii) Suppose that $\mathcal{D}_s(W_n, i) \neq \emptyset$. Let $Y \in \mathcal{D}_s(W_n, i)$. Then by Lemma 2.4, $Y - \{n\} \in \mathcal{D}_s(W_{n-1}, i-1)$ for some $Y \in \mathcal{D}_s(W_n, i)$, a contradiction. Therefore $\mathcal{D}_s(W_n, i) = \emptyset$.

Lemma 2.7

If $\mathcal{D}_s(W_n, i) \neq \emptyset$, then

i) $\mathcal{D}_s(W_{n-1}, i-1) = \mathcal{D}_s(W_{n-2}, i-1) = \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) \neq \emptyset$ if and only if $n = 3k$ and $i = k$ for every $k \geq 3$;

ii) $\mathcal{D}_s(W_{n-2}, i-1) = \mathcal{D}_s(W_{n-3}, i-1) = \emptyset$ and $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$ if and only if $i = n$;

iii) $\mathcal{D}_s(W_{n-1}, i-1) = \emptyset$, $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) \neq \emptyset$ if and only if $n = 3k+2$ and $i = \left\lceil \frac{3k+2}{3} \right\rceil$ for some $k \geq 3$;

iv) $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$, $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) = \emptyset$ if and only if $i = n-1$;

v) $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$, $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) \neq \emptyset$ if and only if $\left\lceil \frac{n-1}{3} \right\rceil + 1 \leq i \leq n-2$.

Proof

i) ($\Longrightarrow$) Since $\mathcal{D}_s(W_{n-1}, i-1) = \mathcal{D}_s(W_{n-2}, i-1) = \emptyset$, by Lemma 2.2, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-2}{3} \right\rceil$. If $i-1 > n-1$, then $i > n$. By Lemma 2.2, $\mathcal{D}_s(W_n, i) = \emptyset$, a contradiction. Therefore$i - 1 < \left\lceil \frac{n-2}{3} \right\rceil$.

$\Longrightarrow i < \left\lceil \frac{n-2}{3} \right\rceil + 1$ (3)

Since $\mathcal{D}_s(W_n, i) \neq \emptyset$, by Lemma 2.2, $\left\lceil \frac{n}{3} \right\rceil \leq i \leq n \Longrightarrow \left\lceil \frac{n}{3} \right\rceil \leq i$(4)

From (3) and (4), we have $n = 3k$ and $i = k$ for some $k \geq 3$.
(⟸) Suppose $n = 3k$ and $i = k$ for some $k \geq 3$.
Now $\gamma_s(W_{n-1}) = \left\lceil \frac{n-1}{3} \right\rceil$
$= \left\lceil \frac{3k-1}{3} \right\rceil$
$= \left\lceil k - \frac{1}{3} \right\rceil$
$= \left\lceil i - \frac{1}{3} \right\rceil$
$> i - 1$
By Lemma 2.2, $\mathcal{D}_s(W_{n-1}, i-1) = \emptyset$.
Similarly, we can prove $\mathcal{D}_s(W_{n-2}, i-1) = \emptyset$.
Now $\gamma_s(W_{n-3}) = \left\lceil \frac{n-3}{3} \right\rceil$
$= \left\lceil \frac{3k-3}{3} \right\rceil$
$= \lceil k-1 \rceil$
$= \lceil i-1 \rceil$
$= i - 1$
By Lemma 2.2, $\mathcal{D}_s(W_{n-3}, i-1) \neq \emptyset$.

ii) (⟹) Since $\mathcal{D}_s(W_{n-2}, i-1) = \mathcal{D}_s(W_{n-3}, i-1) = \emptyset$, by Lemma 2.2, $i - 1 < \left\lceil \frac{n-3}{3} \right\rceil$ or $i - 1 > n - 2$.
If $i - 1 < \left\lceil \frac{n-3}{3} \right\rceil$
$< \left\lceil \frac{n-1}{3} \right\rceil$
$\Rightarrow i - 1 < \left\lceil \frac{n-2}{3} \right\rceil$
By Lemma 2.2, $\mathcal{D}_s(W_{n-1}, i-1) = \emptyset$, a contradiction.So $i - 1 > n - 2 \Rightarrow i > n - 1$. (5)
Since $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$, by Lemma 2.2, $\left\lceil \frac{n-1}{3} \right\rceil \leq i - 1 \leq n - 1 \Rightarrow i \leq n$. (6)
From (5) and (6), we have $n - 1 < i \leq n$ which implies $i = n$.
(⟸) Suppose $i = n$.
Then $i - 1 = n - 1 > n - 2$. Therefore $i - 1 > n - 2$. By Lemma 2.2,$\mathcal{D}_s(W_{n-2}, i-1) = \emptyset$.
Similarly, we can prove $\mathcal{D}_s(W_{n-3}, i-1) = \emptyset$.
Since $i - 1 = n - 1$, by Lemma 2.2, $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$.

iii) (⟹) Since $\mathcal{D}_s(W_{n-1}, i-1) = \emptyset$, by Lemma 2.2, $\left\lceil \frac{n-1}{3} \right\rceil > i - 1$ or $i - 1 > n - 1$.
If $i - 1 > n - 1$
$> n - 2$
$\Rightarrow i - 1 > n - 2$
By Lemma 2.2, $\mathcal{D}_s(W_{n-2}, i-1) = \emptyset$, a contradiction.
So $i - 1 < \left\lceil \frac{n-1}{3} \right\rceil$
$\Rightarrow i < \left\lceil \frac{n-1}{3} \right\rceil + 1$ (7)
Since $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$, by Lemma 2.2, $\left\lceil \frac{n-2}{3} \right\rceil \leq i - 1 \leq n - 2$.
$\Rightarrow \left\lceil \frac{n-2}{3} \right\rceil + 1 \leq i$(8)
From (7) and (8), we have $n = 3k + 2$ and $i = k + 1$ for some $k \geq 3$.
(⟸) Suppose $n = 3k + 2$ and $i = k + 1$ for some $k \geq 3$.
Now $\gamma_s(W_{n-1}) = \left\lceil \frac{n-1}{3} \right\rceil$
$= \left\lceil \frac{3k+2-1}{3} \right\rceil$
$= \left\lceil k + \frac{1}{3} \right\rceil$

$> i-1$

By Lemma 2.2, $\mathcal{D}_s(W_{n-1}, i-1) = \emptyset$.

Now $\gamma_s(W_{n-2}) = \left\lceil \frac{n-2}{3} \right\rceil$

$$= \left\lceil \frac{3k+2-2}{3} \right\rceil$$
$$= \lceil k \rceil$$
$$= k$$
$$= i-1$$

By Lemma 2.2, $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$.

Similarly, we can prove $\mathcal{D}_s(W_{n-3}, i-1) \neq \emptyset$.

iv) ($\Longrightarrow$) Since $\mathcal{D}_s(W_{n-3}, i-1) = \emptyset$, by Lemma 2.2, $\left\lceil \frac{n-3}{3} \right\rceil > i-1$ or $i-1 > n-3$.

If $i-1 < \left\lceil \frac{n-3}{3} \right\rceil$.

$$< \left\lceil \frac{n-1}{3} \right\rceil$$

By Lemma 2.2, $\mathcal{D}_s(W_{n-1}, i-1) = \emptyset$, a contradiction.

So $i-1 > n-3 \Longrightarrow i > n-2$. Therefore $i = n-1$ *or* n. Suppose $i = n$.

Since $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$, by Lemma 2.2, $i-1 \leq n-2$ which implies $i \leq n-1$, a contradiction. Hence $i = n-1$.

v) ($\Longleftarrow$) Suppose $i = n-1$. By Lemma 2.2, $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$, $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) = \emptyset$.

($\Longrightarrow$) Since $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$, $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) \neq \emptyset$, by Lemma 2.2, $\left\lceil \frac{n-1}{3} \right\rceil \leq i-1 \leq n-1$, $\left\lceil \frac{n-2}{3} \right\rceil \leq i-1 \leq n-2$ and $\left\lceil \frac{n-3}{3} \right\rceil \leq i-1 \leq n-3$.

Therefore $\left\lceil \frac{n-1}{3} \right\rceil \leq i-1 \leq n-3$ which implies $\left\lceil \frac{n-1}{3} \right\rceil + 1 \leq i \leq n-2$.

($\Longleftarrow$) Suppose $\left\lceil \frac{n-1}{3} \right\rceil + 1 \leq i \leq n-2$.

Now $\left\lceil \frac{n-1}{3} \right\rceil + 1 \leq i \Longrightarrow \left\lceil \frac{n-1}{3} \right\rceil \leq i-1$

$$\Longrightarrow \gamma_s(W_{n-1}) \leq i-1$$

Also $i \leq n-2 \Longrightarrow i-1 \leq n-3 \leq n-1$.

Thus $\gamma_s(W_{n-1}) \leq i-1 \leq n-1$. By Lemma 2.2, $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$.

Similarly, we can prove $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) \neq \emptyset$.

Theorem 2.8

For every $n \geq 9$ and $i \geq \left\lceil \frac{n}{3} \right\rceil$

i) If $\mathcal{D}_s(W_{n-1}, i-1) = \mathcal{D}_s(W_{n-2}, i-1) = \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) \neq \emptyset$, then

$$\mathcal{D}_s(W_n, i) = \left\{X \cup \begin{cases} \{n\} & if\ n-3\ is\ end\ vertex\ of\ X\ or\ 2 \notin X\ and\ n-5\ is\ end\ vertex\ of\ X \\ \{n-1\} & if\ n-4\ or\ n-6\ is\ end\ vertex\ of\ X \\ \{n-2\} & if\ n-10 \in X\ and\ n-5\ is\ the\ end\ vertex\ of\ X\ or\ n-7\ is\ the\ end\ vertex\ of\ X \\ \{n-3\} & if\ n-6\ is\ the\ end\ vertex\ of\ X \\ \{n-4\} & if\ n-7\ is\ the\ end\ vertex\ of\ X \end{cases} / X \in \mathcal{D}_s(W_{n-3}, i-1)\right\}$$

ii) If $\mathcal{D}_s(W_{n-2}, i-1) = \mathcal{D}_s(W_{n-3}, i-1) = \emptyset$ and $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$, then$\mathcal{D}_s(W_n, i) = \{[n]\}$

iii) If $\mathcal{D}_s(W_{n-1}, i-1) = \emptyset$, $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) \neq \emptyset$, then

$$\mathcal{D}_s(W_n, i) = \left\{X_1 \cup \begin{cases} \{n\} & if\ 2 \notin X_1\ and\ n-4\ is\ the\ end\ vertex\ of\ X_1\ or\ n-2\ or\ n-3\ or\ n-5\ is\ the\ end\ vertex\ of X_1 \\ \{n-1\} & if\ n-3\ or\ n-4\ or\ n-5\ or\ n-6\ is\ the\ end\ vertex\ of\ X_1 \\ \{n-2\} & if\ n-4\ or\ n-5\ or\ n-6\ is\ the\ end\ vertex\ of\ X_1 \\ \{n-3\} & if\ n-5\ or\ n-6\ is\ the\ end\ vertex\ of\ X_1 \\ \{n-4\} & if\ n-6\ is\ the\ end\ vertex\ of\ X_1 \end{cases} / X_1 \in \mathcal{D}_s(W_{n-2}, i-1)\right\} \cup \left\{X_2 \cup \begin{cases} \{n\} & if\ 2 \notin X_2\ and\ n-5\ is\ the\ end\ vertex\ of X_2\ or\ n-3\ is\ the\ end\ vertex\ of X_2 \\ \{n-1\} & if\ n-4\ or\ n-6\ is\ the\ end\ vertex\ of\ X_2 \\ \{n-2\} & if\ n-5\ or\ n-7\ is\ the\ end\ vertex\ of\ X_2 \\ \{n-3\} & if\ n-6\ is\ the\ end\ vertex\ of\ X_2 \\ \{n-4\} & if\ n-7\ is\ the\ end\ vertex\ of\ X_2 \end{cases} / X_2 \in \mathcal{D}s(Wn-3,i-1)\right\}.$$

iv) If $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$, $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) = \emptyset$, then $\mathcal{D}_s(W_n, i) = \{[n] - \{x\}/x \in [n]\}$.

v) If $\mathcal{D}_s(W_{n-1}, i-1) \neq \emptyset$, $\mathcal{D}_s(W_{n-2}, i-1) \neq \emptyset$ and $\mathcal{D}_s(W_{n-3}, i-1) \neq \emptyset$, then

$$\mathcal{D}_s(W_n, i) = \{X_1 \cup \begin{cases} \{n\} & if\ n-1\ or\ n-2\ or\ n-3\ or\ n-4\ or\ n-5\ is\ the\ end\ vertex\ of X_1 \\ \{n-1\} & if n-2\ or\ n-3\ or\ n-4\ or\ n-5\ \ is\ the\ end\ vertex\ of\ X_1 \\ \{n-2\} & if\ n-3\ or\ n-4\ or\ n-5\ is\ the\ end\ vertex\ of\ X_1 \\ \{n-3\} & if\ n-4\ or\ n-5\ is\ the\ end\ vertex\ of\ X_1 \\ \{n-4\} & if\ n-5\ is\ the\ end\ vertex\ of\ X_1 \end{cases} /X_1$$

$$\in \mathcal{D}_s(W_{n-1}, i-1)\}\} \cup \{X_2$$

$$\cup \begin{cases} \{n\} & if\ n-2\ is\ the\ end\ vertex\ of X_2 \\ \{n-1\} & if\ n-3\ or\ n-5\ or\ n-6\ is\ the\ end\ vertex\ of X_2 \\ \{n-2\} & if\ n-4\ or\ n-6\ is\ the\ end\ vertex\ of\ X_2 \\ \{n-3\} & if\ n-5\ or\ n-6\ is\ the\ end\ vertex\ of\ X_2 \\ \{n-4\} & if\ n-6\ is\ the\ end\ vertex\ of\ X_2 \end{cases} /X_2$$

$$\in \mathcal{D}_s(W_{n-2}, i-1)\}\} \cup \{X_3 \cup \begin{cases} \{n,n-1\} & if\ n-4\ is\ the\ end\ vertex\ of\ X_3 \\ \{n-1,n-2\} & if\ n-5\ is\ the\ end\ vertex\ of\ X_3 \\ \{n-2,n-3\} & if\ n-6\ is\ the\ end\ vertex\ of\ X_3 \\ \{n-3,n-4\} & if\ \ n-7\ is\ the\ end\ vertex\ of\ X_3 \end{cases} /X_3$$

$$\in \mathcal{D}_s(W_{n-4}, i-2)\}\}$$

III. Conclusion

This paper discusses and analyses the secure dominating sets of wheels. Using recursive method, we constructed the secure dominating sets of wheels.

References

[1]. S. Alikhani and Y-H. Peng, Dominating Sets and Domination Polynomials of Paths, *International Journal of Mathematics and Mathematical Sciences*, Vol.2009, 2009, Article ID 542040.

[2]. G. Chartrand and L. Lesniak (2005), Graphs & Digraphs, Fourth Edition, Champman \& Hall / CRC.

[3]. E.J. Cockayne, P.J.P. Grobler, W.R. Grundlingh J. Munganga and J.H. Van Vuuren, Protection of a graph, *Util., Math.*, 67(2005),19-32.

[4]. S.V. Divya Rashmi, S. Arumugam and Ibrahim Venkat, Secure Domination in Graphs, *Int. J. Advance. Soft Comput. Appl.*, 8(2) (2016), 79-83.

[5]. Frank Harary, Graph Theory, Addison-Wesley Publishing Company, Inc.

[6]. J. Gross and J. Yellen, Handbook of Graph Theory, CRC Press (2004).

[7]. T W Haynes, S T Hedetniemi and P.J Slater, Fundamentals of Domination in Graphs, *Marcel Dekker*, New York, 1998.

[8]. K. Lal Gipson, Subha T, Secure Domination Polynomials Of Paths, *International Journal of Scientific & Technology Research*, Volume 9, Issue 2(Feb 2020), 6517-6521.

[9]. Sahib Sh. Kahat, Abdul Jalil M. Khalaf and RoslanHasni, Dominating Sets and Domination Polynomial of Wheels, *Asian Journal of Applied Sciences*, Volume 02-Issue 03, June 2014, 287-290.

[10]. A. Vijayan and K. Lal Gipson, Dominating Sets and Domination Polynomials of Square of Paths, *Open Journal of Discrete Mathematics*,2013,3,60-69.

[11]. A. Vijayan and K. Lal Gipson, Dominating Sets and Domination Polynomials of Square Of Cycles, *IOSR Journal of Mathematics (IOSR-JM)*, Volume 3, Issue 4. Sep-Oct 2012, PP 04-14.

K. Lal Gipson, et. al. "Secure Dominating Sets of Wheels." *IOSR Journal of Mathematics (IOSR-JM)*, 18(1), (2022): pp. 21-26.