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Reflexivity of a Banach Space with a Countable Vector Space Basis

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Abstract:
Background: *All most all the function spaces over real or complex domains and spaces of sequences, which arise in practice as examples of normed complete linear spaces (Banach spaces), are reflexive. These Banach spaces are dual to their respective spaces of continuous linear functionals over the corresponding Banach spaces, meaning that the double dual space coincides with the Banach space. For each of these Banach spaces, a countable vector space basis exists, which is responsible for their reflexivity.*
Materials and Methods: *A vector space basis for a Banach space can be easily formulated, taking the analogy from a Riesz basis for a Hilbert space. However, these Banach spaces are not plainly L^p- or ℓ^p-spaces. For a Banach space with a countable vector space basis, the projection maps onto finite dimensional component subspaces are continuous and surjective, and hence open maps. The coefficient sequence of any vector in the Banach space, with respect to a basis, for which the dual basis functionals are normalized, can be shown to be bounded with respect to sup-norm, and the linear transformation mapping a vector to its coefficient sequence becomes continuous. The dual basis linear functionals become continuous and form a basis for the dual space. Extending the same observation to the double dual, the reflexivity becomes obvious.*
Results: *The projection maps and dual basis linear functionals a Banach space with a countable vector space basis are shown to be continuous open maps, and that the dual space is generated by the linear combinations of the dual basis linear functionals.*
Conclusion: *The double dual space is generated by the linear combinations of the double dual basis linear functionals, and becomes isometrically isomorphic to the given Banach space with a countable vector space basis.*
Key Word: *Normed Linear Spaces; Banach Spaces; Vector Space Basis; Weak Topology.*

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I. Introduction

All most all the function spaces over real or complex domains and spaces of sequences, that arise in practice as examples of normed complete linear spaces (Banach spaces), are reflexive. These Banach spaces are dual to their respective spaces of continuous linear functionals over the corresponding Banach spaces, meaning that the double dual space and the Banach space are duals of each other. For each of these Banach spaces, a countable vector space basis exists, which is responsible for their reflexivity. A vector space basis for a Banach space can be easily formulated, taking the analogy from a Riesz basis for a Hilbert space. However, these Banach spaces are not plainly L^p- or ℓ^p-spaces. For a Banach space with a countable vector space basis, the projection maps onto finite dimensional component subspaces are continuous and surjective, and hence open maps. The coefficient sequence of any vector in the Banach space, with respect to a basis, for which the dual basis functionals are normalized, can be shown to be bounded with respect to sup-norm, and the linear transformation mapping a vector to its coefficient sequence becomes continuous. The dual basis linear functionals become continuous and form a basis for the dual space. Extending the same observation to the double dual, the reflexivity becomes obvious.

II. Material And Methods

Let $\mathbb{N}$ be the set of positive integers, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, equipped with absolute value norm $|x|$, for $x \in \mathbb{F}$. A Banach space is a topologically complete normed linear (vector) space over $\mathbb{F}$. Let $\ell^p(\mathbb{N}, \mathbb{F})$ be the Banach space of $\mathbb{F}$-valued sequences, equipped with the p-norm, for $1 \leq p \leq \infty$. Let $\mathbf{e}_i$, for $i \in \mathbb{N}$, be the standard Euclidian vectors, with j-th component of $\mathbf{e}_i$ equal to $\delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta function, for $i, j \in \mathbb{N}$. Then, $\mathbf{e}_i \in \ell^p(\mathbb{N}, \mathbb{F})$, for every $i \in \mathbb{N}$ and $1 \leq p \leq \infty$, and the set $\{\mathbf{e}_i : i \in \mathbb{N}\}$ forms a basis for $\ell^p(\mathbb{N}, \mathbb{F})$, for $1 \leq p \leq \infty$.

Let $\mathcal{B}$ be a Banach space, and $\{\xi_i : i \in \mathbb{N}\} \subset \mathcal{B}$. If every element $\boldsymbol{x} \in \mathcal{B}$ can be expressed as a unique linear combination $\boldsymbol{x} = \sum_{i=1}^{\infty} c_i \xi_i$, for some scalars $c_i \in \mathbb{F}$, for $i \in \mathbb{N}$, then the Banach space $\mathcal{B}$ is said to admit a countable basis, which is $\{\xi_i : i \in \mathbb{N}\}$. By the uniqueness of the linear combination, it is assumed that if $\sum_{i=1}^{\infty} d_i \xi_i$, for some scalars $d_i \in \mathbb{F}$, then $d_i = 0$, for $i \in \mathbb{N}$. The basis $\{\xi_i : i \in \mathbb{N}\}$ is said to be normalized, if $\|\xi_i\|_{\mathcal{B}} = 1$, for $i \in \mathbb{N}$, where $\| \quad \|_{\mathcal{B}}$ is the norm of $\mathcal{B}$. The space continuous linear functionals, called the *dual*, is denoted by $\widehat{\mathcal{B}}$, with its dual norm $\| \quad \|_{\widehat{\mathcal{B}}}$. The double dual, or *bidual*, is denoted by $\widehat{\widehat{\mathcal{B}}}$, with its bidual norm $\| \quad \|_{\widehat{\widehat{\mathcal{B}}}}$. For a Banach space $\mathcal{B}$, with a countable basis $\{\xi_i : i \in \mathbb{N}\}$, the dual basis is the set of linear functionals $\{\hat{\xi}_i : i \in \mathbb{N}\}$, where $\hat{\xi}_i(\xi_j) = \delta_{i,\ j}$, for $i, j \in \mathbb{N}$. That $\{\hat{\xi}_i : i \in \mathbb{N}\} \subset \widehat{\mathcal{B}}$ and forms a basis for it is the contention of Proposition 1.

Let $\mathbb{I} \subset \mathbb{N}$ be a set of indexes, $\mathbb{J} = \mathbb{N} \setminus \mathbb{I}$, and $M_{\mathbb{I}}$ and $M_{\mathbb{J}}$ be the subspaces of vectors in $\mathcal{B}$ spanned by $\{\xi_i : i \in \mathbb{I}\}$ and $\{\xi_j : j \in \mathbb{J}\}$, respectively, including infinite sums, whenever the corresponding infinite sums represent elements in $\mathcal{B}$.

Proposition 1 *For any subset $\mathbb{I} \subset \mathbb{N}$ of indexes, the subspaces $M_{\mathbb{I}}$ and $M_{\mathbb{J}}$ are closed linear subspaces of $\mathcal{B}$, such that every vector $\boldsymbol{x} \in \mathcal{B}$ can be expressed as $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$, for some $\boldsymbol{y} \in M_{\mathbb{I}}$ and $\boldsymbol{z} \in M_{\mathbb{J}}$ uniquely, i.e., $\mathcal{B} = M_{\mathbb{I}} \bigoplus M_{\mathbb{J}}$.*

Proof Let $\mathbb{I}' = \{i\}$, for some $i \in \mathbb{N}$, and $\mathbb{J}' = \mathbb{N} \setminus \mathbb{I}'$, so that $M_{\mathbb{I}'} = \{\, c\xi_i : c \in \mathbb{F} \,\}$. For each $c \in \mathbb{F}$, it holds that $\|c\xi_i\|_{\mathcal{B}} = |c|.\|\xi_i\|_{\mathcal{B}}$, and $M_{\mathbb{I}'}$ is a closed linear subspace of $\mathcal{B}$. By the linear independence of the series, $M_{\mathbb{I}'} \cap M_{\mathbb{J}'} = \{0\}$, and $M_{\mathbb{J}}$ is a closed linear subspace of $\mathcal{B}$. Now, $M_{\mathbb{I}} = \bigcap_{j \in \mathbb{J}} M_{\mathbb{N} \setminus \{j\}}$ and $M_{\mathbb{J}} = \bigcap_{i \in \mathbb{I}} M_{\mathbb{N} \setminus \{i\}}$, with both $M_{\mathbb{N} \setminus \{j\}}$ and $M_{\mathbb{N} \setminus \{i\}}$ closed in $\mathcal{B}$.
□

Proposition 1 implies that the dual basis linear functional $\hat{\xi}_i$ is continuous, for each $i \in \mathbb{N}$. The continuity of $\hat{\xi}_i$ implies that $1 \le \|\hat{\xi}_i\|_{\widehat{\mathcal{B}}} < \infty$, for each $i \in \mathbb{N}$, and $\{\hat{\xi}_i : i \in \mathbb{N}\}$ becomes a basis for $\widehat{\mathcal{B}}$. Let $\hat{\eta}_i = \frac{\hat{\xi}_i}{\|\hat{\xi}_i\|_{\widehat{\mathcal{B}}}}$, so that $\|\hat{\eta}_i\|_{\widehat{\mathcal{B}}} = 1$ and $\hat{\eta}_i\left(\|\hat{\xi}_j\|_{\widehat{\mathcal{B}}}\, \xi_j\right) = \delta_{i,\ j}$, for $i, j \in \mathbb{N}$. The set $\{\hat{\eta}_i : i \in \mathbb{N}\}$ is a normalized dual basis for $\widehat{\mathcal{B}}$. Let $\eta_i = \|\hat{\xi}_i\|_{\widehat{\mathcal{B}}} \xi_i$, for $i \in \mathbb{N}$. Now, $\{\eta_i : i \in \mathbb{N}\}$ is a countable basis for $\mathcal{B}$, and $\{\hat{\eta}_i : i \in$ $\mathbb{N}$ is a normalized dual basis for $\mathcal{B}$.

Proposition 2 *Let $\mathcal{B}$ be a Banach space, with a countable basis $\{\eta_i : i \in \mathbb{N}\}$, such that the dual basis $\{\hat{\eta}_i :$ $:i\in\mathbb{N}$ is normalized. For any $\boldsymbol{x}\in\mathcal{B}$ and $ci\in\mathbb{F}$, if $\boldsymbol{x}=i=1\infty ci\eta i$, then $ci\le\boldsymbol{x}\mathcal{B}$, for every $i\in\mathbb{N}$.*

Proof Let $\hat{T} : \ell^1(\mathbb{N},\ \mathbb{F}) \to \widehat{\mathcal{B}}$ be the formal mapping, defined by $\hat{T}(\boldsymbol{c}) = \sum_{i=1}^{\infty} c_i \hat{\eta}_i$, for $c = (c_1, c_2, c_3, \ldots) \in \ell^1(\mathbb{N},\ \mathbb{F})$. Let $\boldsymbol{y}_n = \sum_{i=1}^{n} c_i \hat{\eta}_i$, for $n \in \mathbb{N}$. Then $\|\boldsymbol{y}_{m+n} - \boldsymbol{y}_n\|_{\widehat{\mathcal{B}}} = \left\|\sum_{i=n+1}^{m+n} c_i \hat{\eta}_i\right\|_{\widehat{\mathcal{B}}} \le \sup_i \|\hat{\eta}_i\|_{\widehat{\mathcal{B}}} \sum_{i=n+1}^{m+n} |c_i| \le \sum_{i=n+1}^{\infty} |c_i| \to 0$, as $n \to \infty$. Thus, $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\widehat{\mathcal{B}}$, and $\hat{T}(\boldsymbol{c}) \in \widehat{\mathcal{B}}$. Since $\|\hat{T}(\boldsymbol{c})\|_{\widehat{\mathcal{B}}} \le \sum_{i=1}^{\infty} |c_i|$, the linear transformation norm of $\hat{T}$ is at most 1. Let $\hat{\hat{T}} : \widehat{\widehat{\mathcal{B}}} \to \ell^{\infty}(\mathbb{N},\ \mathbb{F})$ be the adjoint linear transformation of $\hat{T}$, and $\hat{\hat{\eta}}_i \in \widehat{\widehat{\mathcal{B}}}$ be the natural imbedding of $\eta_i \in \mathcal{B}$, for $i \in \mathbb{N}$. For $\boldsymbol{x} = \sum_{i=1}^{\infty} c_i \eta_i \in \mathcal{B}$, for any scalars $c_i \in \mathbb{F}$, for $i \in \mathbb{N}$, the vector $\hat{\hat{\boldsymbol{x}}} = \sum_{i=1}^{\infty} c_i \hat{\hat{\eta}}_i \in \widehat{\widehat{\mathcal{B}}}$ is the natural imbedding of $\boldsymbol{x}$ in $\mathcal{B}$, and the natural imbedding is an isometric isomorphism. The linear transformation norm of $\hat{\hat{T}}$ coincides with that of $\hat{T}$, and is at most 1. For a vector $\hat{\hat{\boldsymbol{x}}} = \sum_{i=1}^{\infty} c_i \hat{\hat{\eta}}_i \in \widehat{\widehat{\mathcal{B}}}$, for any scalars $c_i \in \mathbb{F}$, for $i \in \mathbb{N}$, if $\|\hat{\hat{\boldsymbol{x}}}\|_{\widehat{\widehat{\mathcal{B}}}} = 1$, then $\left|\hat{\hat{\boldsymbol{x}}}\left(\hat{T}(\mathbf{e}_i)\right)\right| = |\hat{\hat{\boldsymbol{x}}}(\hat{\eta}_i)| = |c_i| \le 1$, for $i \in \mathbb{N}$. The contention follows by the isometry of the natural imbedding and the linearity of $\hat{\hat{T}}$.
□

Another proof of Proposition 1 is given in this paragraph. The linear transformation $T : \mathcal{B} \to \ell^{\infty}(\mathbb{N},\ \mathbb{F})$, defined by $T(\boldsymbol{x}) = (c_1,\ c_2,\ c_3,\ \ldots)$, for $\boldsymbol{x} = \sum_{i=1}^{\infty} c_i \eta_i = \sum_{i=1}^{\infty} c_i \|\hat{\xi}_i\|_{\widehat{\mathcal{B}}} \xi_i$, is one-to-one and continuous. By Proposition 2, the coefficient sequence $T(\boldsymbol{x}) = \boldsymbol{c} = (c_1,\ c_2,\ c_3,\ \ldots) \in \ell^{\infty}(\mathbb{N},\ \mathbb{F})$ and $\|T(\boldsymbol{x})\|_{\ell^{\infty}(\mathbb{N},\mathbb{F})} = \|\boldsymbol{c}\|_{\ell^{\infty}(\mathbb{N},\mathbb{F})} \le \|\ (\sum_{i=1}^{\infty} c_i \eta_i\ )\|_{\mathcal{B}} =$

$\left\| \left(\sum_{i=1}^{\infty} c_i \|\hat{\xi}_i\|_{\hat{\mathcal{B}}} \xi_i \right) \right\|_{\mathcal{B}}$. The inverse image of the closed linear subspaces of $\ell^{\infty}(\mathbb{N}, \mathbb{F})$, spanned by $\{\mathbf{e}_i : i \in \mathbb{I}\}$ and by $\{\mathbf{e}_j : j \in \mathbb{J}\}$, are the subspaces of $\mathcal{B}$ spanned by $\{\eta_i : i \in \mathbb{I}\}$ and by $\{\eta_j : j \in \mathbb{J}\}$, respectively, which must be closed, since T is continuous. Now, the subspaces of $\mathcal{B}$ spanned by $\{\eta_i : i \in \mathbb{I}\}$ and by $\{\eta_j : j \in \mathbb{J}\}$ are exactly the subspaces of $\mathcal{B}$ spanned by $\{\xi_i : i \in \mathbb{I}\}$ and by $\{\xi_j : j \in \mathbb{J}\}$, respectively, and the subspaces $M_{\mathbb{I}}$ and $M_{\mathbb{J}}$ are closed linear subspaces of $\mathcal{B}$. It is clear that $M_{\mathbb{I}} \cap M_{\mathbb{J}} = \{0\}$, since T is one-to-one. The contention that $\mathcal{B} = M_{\mathbb{I}} \oplus M_{\mathbb{J}}$ follows from the fact that $\ell^{\infty}(\mathbb{N}, \mathbb{F})$ is the direct sum of the closed linear subspaces spanned by $\{\mathbf{e}_i : i \in \mathbb{I}\}$ and by $\{\mathbf{e}_j : j \in \mathbb{J}\}$, respectively.

III. Result

The following is the main result.

Theorem 1 (Rabin) *For a Banach space $\mathcal{B}$, with a countable basis $\{\xi_i : i \in \mathbb{N}\}$, the double dual $\hat{\hat{\mathcal{B}}}$ is isometrically isomorphic to $\mathcal{B}$.*

Proof The dual space $\hat{\mathcal{B}}$ is the closed linear subspace generated by the dual basis continuous linear funtionals $\{\hat{\xi}_i : i \in \mathbb{N}\}$, and likewise, the double dual space $\hat{\hat{\mathcal{B}}}$ is the closed linear subspace generated by the double dual basis continuous linear funtionals $\{\hat{\hat{\xi}}_i : i \in \mathbb{N}\}$, which is isometrically isomorphic to the closed linear subspace spanned by $\{\xi_i : i \in \mathbb{N}\}$. Thus $\hat{\hat{\mathcal{B}}} = \mathcal{B}$.

□

IV. Discussion

A vector space basis is very important for a vector space. For infinite dimensional topological vector space, a precise formulation of a vector space basis can be expressed, extending the definition of linear independence for linear combinations of finitely many vectors to series.

V. Conclusion

The projection maps and dual basis linear functionals a Banach space with a countable vector space basis are shown to be continuous open maps, and that the dual space is generated by the linear combinations of the dual basis linear functionals. The double dual space is generated by arbitrary linear combinations of the double dual basis linear functionals, hence becomes isometrically isomorphic to the given Banach space with a countable vector space basis.

References

[1]. Johm B. Conway. A Course in Functional Analysis. Springer-Verlag Inc. New York, 1990
[2]. Serge Lang. Real and Functional Analysis. Springer-Verlag Inc. New York, 1993
[3]. Robert E. Megginson. An Introduction to Banach Space Theory. Springer-Verlag Inc. New York, 1998
[4]. George F. Simmons. Introduction to Topology and Modern Analysis. McGraw-Hill Inc. Printed by Robert E. Krieger Publishing Company Inc. 1963
[5]. Walter Rudin. Principles of Mathematical Analysis. McGraw-Hill Inc. 1976
[6]. Wilson A. Sutherland. Introduction to Metric and Topological Spaces. Oxford University Press. 2009

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