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POHOZAEV-Type Identity for a Kind Of Fourth Order Elliptic Problem

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Abstract:
In this paper,we establish the Pohozaev-type identity for a kind of fourth order elliptic problem,which has the biharmonic operator.We discuss the problem in a class of domains that are more general than star-shaped ones.
***Key Word**: Pohozaev-type identity;biharmonic operator.*

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I. Introduction

In this paper,we consider the following fourth order elliptic problem:

$$\begin{cases} \Delta^2 u - c\Delta u = f(x,u), in\Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, on\partial\Omega, \end{cases} \quad (1.1)$$

Where $\Delta^2 = \Delta(\Delta)$ is the biharmonic operator,c is a constant, $\Omega \subset R^n$ is a domain with smooth boundary $\partial\Omega$,$\nu(x)$ denotes the outward normal to $\partial\Omega$ at x and $f(x,u): R^n \times R \to R$ is continuous.

The Pohozaev identity was first introduced by S.I.Pohozaev in paper [1]. Many authors generalized the identity to more general equations under the conditions that Ω is star-shaped. Others considered the case of domains more general than star-shaped ones in paper [2-6].In this paper,we discuss a kind of fourth order elliptic problem,which has the biharmonic operator in a class of domains that are more general than star-shaped ones. We establish the Pohozaev-type identity of (1.1) ,which can play important role in considering the existence of the solution.

II. Important results And the Pohozaev identity

We need the following lemma,which is similar but also has some differences with paper [8].

Lemma 2.1

Assume that $V(x) = (V_1(x),\cdots,V_n(x))$ is a linear vector filed on R^n and $f(x,u): R^n \times R \to R$ is continuous and satisfies $F(x,t) = \int_0^t f(x,s)ds$ and $F_1(x,t) = \left\langle V(x), F(x,t)\right\rangle = \sum_{i=1}^{n} V_i \dfrac{\partial F(x,t)}{\partial x_i}$ .If $u \in W_0^{2,2}(\Omega) \cap C^5(\overline{\Omega})$ is a solution of (1.1),then

$$\int_\Omega F(x,u) divV(x)dx + \int_\Omega F_1(u)dx = -\int_\Omega f(x,u)\left\langle V(x), \nabla u\right\rangle dx \qquad (2.1)$$

Theorem2.2

Suppose that $V(x)$ is a linear vector filed on R^n with the form

$$V(x) = \begin{pmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1n} \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ a_{n1} & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

Where $f(x,u): R^n \times R \to R$ is continuous and satisfies $F(x,t) = \int_0^t f(x,s)ds$

and $F_1(x,t) = \left\langle V(x), F(x,t) \right\rangle = \sum_{i=1}^n V_i \frac{\partial F(x,t)}{\partial x_i}$ .If $u \in W_0^{2,2}(\Omega) \cap C^5(\overline{\Omega})$ is a solution of (1.1),then

$$\int_{\partial\Omega} |\Delta u|^2 \left\langle V(x), \nu(x) \right\rangle ds = 4\int_\Omega u\Delta^2 u dx - n\int_\Omega uf(x,u)dx + 2c\int_\Omega |\nabla u|^2 dx + 2n\int_\Omega F(x,u)dx + 2\int_\Omega F_1(x,u)dx,$$

where $x = (x_1, x_2, \cdots, x_n) \in R^n$, $divV(x) = n$ and $\left\langle V(x), x \right\rangle = x_1^2 + x_2^2 + \cdots + x_n^2$.

Proof

$$\int_\Omega \Delta u \left\langle V(x), \nabla u \right\rangle dx = \int_\Omega u\Delta \left\langle V(x), \nabla u \right\rangle dx$$

$$= \int_\Omega u \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \sum_{i=1}^n V_i(x) \frac{\partial u}{\partial x_i} dx$$

$$= \int_\Omega u \sum_{i=1}^n V_i(x) \frac{\partial \Delta u}{\partial x_i} dx + 2\int_\Omega u \sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} dx$$

$$= -\int_\Omega \Delta u \sum_{i=1}^n \frac{\partial}{\partial x_i}(uV_i(x))dx - 2\int_\Omega u \sum_{i=1}^n (\sum_{j=1}^n a_{ij} \frac{\partial u}{\partial x_j}) \frac{\partial u}{\partial x_i} dx$$

$$= -\int_\Omega \Delta u \left\langle V(x), \nabla u \right\rangle dx - n\int_\Omega u\Delta u dx - 2\int_\Omega |\nabla u|^2 dx$$

Since

$$\int_\Omega \left\langle V(x), \nabla(\Delta u) \right\rangle \Delta u dx = \int_\Omega \Delta \left\langle V(x), \nabla(\Delta u) \right\rangle u dx$$

$$= \int_\Omega u \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} (\sum_{i=1}^n V_i(x) \frac{\partial \Delta u}{\partial x_i}) dx$$

$$= 2\int_\Omega u \sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial^2 \Delta u}{\partial x_i \partial x_j} dx + \int_\Omega u \sum_{j=1}^n V_i(x) \frac{\partial \Delta^2 u}{\partial x_i}) dx$$

$$= 2\int_\Omega u\Delta^2 u dx - n\int_\Omega u\Delta^2 u dx - \int_\Omega \Delta^2 u \left\langle V(x), \nabla u \right\rangle dx$$

Moreover,

$$\int_\Omega \left\langle V(x), \nabla(\Delta u) \right\rangle \Delta u dx = \int_\Omega \Delta u \sum_{i=1}^n V_i(x) \frac{\partial \Delta u}{\partial x_i}) dx$$

$$= \int_{\partial\Omega} |\Delta u|^2 \left\langle V(x), \nu(x) \right\rangle ds - n\int_\Omega u\Delta^2 u dx - \int_\Omega \left\langle V(x), \nabla(\Delta u) \right\rangle \Delta u dx$$

We have

$$2\int_\Omega \Delta^2 u \left\langle V(x), \nabla u \right\rangle dx = 4\int_\Omega u\Delta^2 u dx - n\int_\Omega u\Delta^2 u dx - \int_{\partial\Omega} |\Delta u|^2 \left\langle V(x), \nu(x) \right\rangle ds.$$

Then by(2.1),

$$\int_\Omega f(x,u) \left\langle V(x), \nabla u \right\rangle dx = -\int_\Omega F(x,u) divV(x)dx - \int_\Omega F_1(x,u)dx$$

$$= -n\int_\Omega F(x,u)dx - \int_\Omega F_1(x,u)dx,$$

Then

$$2\int_\Omega f(x,u) \left\langle V(x), \nabla u \right\rangle dx = 2\int_\Omega \Delta^2 u \left\langle V(x), \nabla u \right\rangle dx - 2c\int_\Omega \Delta u \left\langle V(x), \nabla u \right\rangle dx$$

$$= 4\int_\Omega u\Delta^2 u dx - n\int_\Omega uf(x,u)dx + 2c\int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} |\Delta u|^2 \left\langle V(x), \nu(x) \right\rangle ds$$

$$= -2n\int_\Omega F(x,u)dx - 2\int_\Omega F_1(x,u)dx$$

Thus,

$$\int_{\partial\Omega} |\Delta u|^2 \langle V(x), \nu(x) \rangle ds = 4\int_\Omega u\Delta^2 u\, dx - n\int_\Omega uf(x,u)dx + 2c\int_\Omega |\nabla u|^2 dx$$

$$+2n\int_\Omega F(x,u)dx + 2\int_\Omega F_1(x,u)dx$$

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