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Identity using Ramanujan Sum

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Abstract:
In Number theory , Ramanujan's sum, usually denoted $c_q(n)$, is a function of two positive integer variables q and n defined by the formula. where (a, q) = 1 means that a only takes on values coprime to q. Also a real or complex valued function defined on the set of all positive integers is called an arithmetic function and an arithmetic function is said to be completely multiplicative function if f is not identically zero and f(mn) = f(m)f(n) for all m,n. we know the reduced residue system modulo N is the set of all integers m with gcd (m,N) = 1 and $0 \leq m \leq N$.

In this paper we will use reduced residue system modulo r^k and prove few results.

***Key Word**: Ramanujan sum, Arithmetic function, Multiplicative function, reduced residue system modulo integer*

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I. Introduction

A real or complex valued function defined on the set of all positive integers is called an arithmetic function and an arithmetic function is said to be completely multiplicative function if f is not identically zero and f(mn) = f(m)f(n) for all m, n. The Euler Totient function φ (n) is defined to be the number of positive integers not exceeding n which are relatively prime to n.

(1.1) $\emptyset(n) = \sum_{\substack{d/n \\ (d,n)=1}} 1$

If a and b are integers, not both zero, and k is any integer greater than 1, then $(a,b)_k$ denotes the largest common divisor of a and b which is also a k^{th} power. This will be referred to as k^{th} power greatest common divisor of a and b.

(1.2) If $(a,b)_k = 1$, then a is said to be *relatively K – Prime* to b.

Ecford Cohen [3] introduced a function $\emptyset_k(n)$ which denotes the number of non negative integers less than N^k which are relatively K-Prime to N^k.

(1.3) $\sum_{d/n} \emptyset_k(N/d) = N^k$

(1.4) The Mobius function μ (n) is defined by

$$\mu(n) = \begin{cases} 1 & if\ n = 1 \\ (-1)^k & if\ n = p_1 p_2 \ldots p_k, \\ 0 & otherwise \end{cases}$$

where p_i's are distinct primes.

By the mobius inversion formula, we get that

(1.5) $\emptyset_k(N) = \sum_{d/N} d^k \mu(N/d)$

Any system of $\emptyset(n)$ integers, where $\emptyset(n)$ is the totient function, representing all the residue classes relatively prime to n is called a reduced residue system.

II. Preliminaries

We shall define $C_k(n,r)$ is Ramanujan's sum by

(2.1) $C_k(n,r) = \sum_{(x,r^k)_k=1} e(nx, r^k)$ ($e(a,b) = e^{\frac{2\pi i a}{b}}, b > 0$)

Where summation is taken over a K-reduced residue system $(mod\ r^k)$

That is, over all $x(mod\ r^k)$ Such that $(x, r^k)_k = 1$
The function $C_k(n,r)$ has the following property

(2.2) $C_k(n,r) = \sum_{d|(n,r)} d^k\ \mu\left(\frac{r}{d}\right)$, Where $\mu(d)$ denotes the Mobius function.

A single valued function f(n,r) having values in the field of complex numbers is said to belong to the class E_k if for all n and $r, f(n,r) = f((n,r^k)_k, r)$

In particular, $f \in E_1 \Leftrightarrow f(n,r) = f\big((n,r),r\big)$ for all n, r

Let $g(r)$ be an arithmetical function, then define,

(2.3) $G_s(r) = \sum_{d|r} d^s g\left(\frac{r}{d}\right)$

(2.4) $T_k^s(n,r) = \sum_{d^k|(n,r^k)_k} d^s g\left(\frac{r}{d}\right)$

(2.5) $G_s^*(n,r) = m^s G_s\left(\frac{r}{m}\right)\ where\ m^k = \frac{r^k}{(n,r^k)_k}$

(2.6) $G_{ks}^*(n,r) = r^* \sum_{d|r} \tau_k^{k(s+1)}\ (e^k, r)\ C_k(n,d)$

In Particular, substituting $s = 0, g(r) = \mu(r)$ in (2.3) and (2.5)

(2.7)

$$G_\circ(r) = \sum_{d|r} d^\circ\ \mu\left(\frac{r}{d}\right) = \sum_{d|r} \mu\left(\frac{r}{d}\right) = \begin{cases} 1 & if\ r = 1 \\ 0 & if\ r > 1 \end{cases}$$ and

(2.8)

$$G_\circ^*(n,r) = G_\circ\left(\frac{r}{m}\right) = \begin{cases} 1 & if\ r = m \\ 0 & if\ r \neq m \end{cases} = \begin{cases} 1 & if\ (n,r^k)_k = 1 \\ 0 & if (n,r^k)_k\ \neq 1 \end{cases}$$

Substituting $s = k, n = e^k$ and $g(r) = \mu(r)$ in (2.4)

$$\tau_k^k(e^k, r) = \sum_{d^k|(e^k,r^k)_k} d^k \mu\left(\frac{r}{d}\right) = \sum_{d|(e,r)} d^k \mu\left(\frac{r}{d}\right)$$

$$= \frac{\Phi_k(r)\mu(m)}{\Phi_k(m)}\ where\ m = \frac{r}{(n,r)}$$

Therefore,

(2.9) $\tau_k^k(e^k, r) = \frac{\Phi_k(r)\mu\left(\frac{r}{e}\right)}{\Phi_k\left(\frac{r}{e}\right)}\ if\ e|r$

We shall now prove few Lemmas required for our main result.

2.1 Lemma: $If\ d_1|r, d_2|r, (x, d_1^k)_k = 1, (y, d_2)_k^k = 1\ \ d_1^k \geq x > 0, d_2^k \geq y > 0.\ Then$

$$\sum_{n \equiv a+b(mod\ r^k)} e(ax, d_1^k)\ e(by, d_2^k) = \begin{cases} r^k e(nx, d^k) & if\ x = y, d = d_1 = d_2 \\ 0 & otherwise \end{cases}$$

Proof: If $d_1|r,\ d_2|r$ then $r = d_1e_1 = d_2e_2$

Since $n \equiv a + b(mod\ r^k)$, We have $b \equiv (n-a)(mod\ r^k)$

Consider

$$\sum_{n\equiv a+b(mod\ r^k)} e(ax, d_1^k)\ e(by, d_2^k) = \sum_{a=1}^{r^k} e(ax, d_1^k)\ e\left((n-a)y, d_2^k\right)$$

$$= \sum_{a=1}^{r^k} e(ax, d_1^k)\ e(ny - ay, d_2^k)$$

$$= \sum_{a=1}^{r^k} e^{\frac{2\pi axi}{d_1^k}} e^{\frac{2\pi(ny-ay)i}{d_2^k}}$$

$$= e^{\frac{2\pi nyi}{d_2^k}} \sum_{a=1}^{r^k} e^{\frac{2\pi axi}{d_1^k}} e^{\frac{-2\pi ayi}{d_2^k}}$$

$$= e^{\frac{2\pi nyi}{d_2^k}} \sum_{a=1}^{r^k} e^{\frac{2\pi axe_1^{k}i}{r^k}} e^{\frac{-2\pi aye_2^{k}i}{r^k}}$$

(2.1.1) $$= e(ny, d_2^k) \sum_{a=1}^{r^k} e(a(xe_1^k - ye_2^k), r^k)$$

The hypothesis of the lemma and the definitions of e_1 and e_2 $s$how that the following statements are equivalent.

(2.1.2) $$xe_1^k \equiv ye_2^k(mod\ r^k) \Leftrightarrow xe_1^k = ye_2^k$$

$$\Leftrightarrow xd_2^k = yd_1^k$$

$$\Leftrightarrow d_1 = d_2, x = y$$

Now, define

$$\sum_{a=1}^{r^k} e^{\frac{2n\pi ai}{b^s}} = \begin{cases} b^s & if\ b^s|n \\ 0 & otherwise \end{cases}$$

Then (2.1.1) becomes

$$\sum_{n\equiv a+b(mod\ r^k)} e(ax, d_1^k)\ e(by, d_2^k) = \begin{cases} r^k e(ny, d_2^k) & if\ r^k|xe_1^k - ye_2^k \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} r^k e(ny, d_2^k) & if\ xe_1^k \equiv ye_2^k(mod\ r^k) \\ 0 & otherwise. \end{cases}$$

$$= \begin{cases} r^k e(nx, d^k) & if d = d_1 = d_2, x = y \\ 0 & otherwise \end{cases}, \text{ by (2.1.2)}$$

Hence Lemma follows.

An immediate useful consequences of the above is

2.2 Lemma: If $d|r, c|r$ then

$$\sum_{n\equiv a+b(mod\ r^k)} C_k(a,d)\ C_k(b,e) = \begin{cases} r^k C_k(n,d) & if\ d = e \\ 0 & if\ d \neq e \end{cases}$$

Proof: Suppose $d \geq x \geq 0,\ e \geq y \geq 0$. Then

$$\sum_{n\equiv a+b(mod\ r^k)} C_k(a,d)\ C_k(b,e) = \sum_{\substack{(X,d^k)_k=1 \\ (y,e^k)_k=1}} \sum_{n\equiv a+b(mod\ r^k)} e(ax, d^k)\ e(by, e^k)$$

$$= \begin{cases} \sum_{(X,d^k)_k=1} \quad r^k e(nx, d^k) & if\ x = y, d = e \\ 0 & otherwise \end{cases}, \text{ by lemma 2.1}$$

$$= \begin{cases} r^k C_k(n,d) & if \quad d = e \\ 0 & otherwise. \end{cases}$$

Proving the lemma.

2.3 Lemma:

$$\sum_{n \equiv a+b (mod\ r^k)} f(a,r)\ g(b,r) = r^k \sum_{d|r} \alpha(d,r)\ \beta(d,r)\ C_k(n,d)$$

Proof: Consider

$$\sum_{n \equiv a+b (mod\ r^k)} f(a,r)\ g(b,r) = \sum_{n \equiv a+b (mod\ r^k)} \left(\sum_{d|r} \alpha(d,r)\ c_k(a,d) \right) \left(\sum_{\delta|r} \beta(\delta,r)\ c_k(b,\delta) \right)$$

$$= \sum_{n \equiv a+b (mod\ r^k)} \sum_{\substack{d|r \\ \delta|r}} \alpha(d,r)\ \beta(\delta,r)\ \ c_k(a,b)\ c_k(b,\delta)$$

$$= \sum_{\substack{d|r \\ \delta|r}} \alpha(d,r)\ \beta(\delta,r) \sum_{n \equiv a+b (mod\ r^k)} c_k(a,d)\ c_k(b,\delta)$$

$$= \sum_{d|r} \alpha(d,r)\ \beta(d,r)\ r^k\ c_k(n,d)\ \ by\ lemma\ 2.2$$

$$= r^k \sum_{d|r} \alpha(d,r)\ \beta(d,r)\ c_k(n,d)$$

Thus proving lemma 2.3.

III. Main Result

3.1 Theorem: If $C_k(n,r)$ is Ramanujan's sum then

$$\sum_{(b,r^k)_k=1} c_k(n-b,r) = \mu(r)\ c_k(n,r)$$

Proof: Let us consider

$$\sum_{n \equiv a+b (mod\ r^k)} f(a,r)\ G_{ks}^*(b,r) = r^k \sum_{d|r} \alpha(d,r)\ \tau_k^{k(s+1)}(e^k,r)\ c_k(n,d)\ \ by\ lemma\ 2.3\ and (2.6)$$

$$= \sum_{d|r} \alpha(d,r)\ \tau_k^{k(s+1)}(e^k,r)\ c_k(n,d)$$

In Particular

$$\sum_{n \equiv a+b (mod\ r^k)} f(a,r)\ G_\circ^*(b,r) = \sum_{d|r} \alpha(d,r)\ \tau_k^k(e^k,r)\ c_k(n,d)$$

$$\sum_{\substack{n \equiv a+b (mod\ r^k) \\ (b,r^k)_k=1}} f(a,r) = \sum_{d|r} \frac{\alpha(d,r)\ \Phi_k(r)\ \mu(d)\ C_k(n,d)}{\Phi_k(d)}\ \ by\ (2.8)\ and\ (2.9)$$

$$\sum_{(b,r^k)_k=1} f(n-b,r) = \Phi_k(r)\sum_{d|r}\frac{\alpha(d,r)\,\mu(d)\,C_k(n,d)}{\phi_k(d)}$$

Taking $f(n,r) = C_k(n,r)$, We have

$$\alpha(d,r) = \begin{cases} 1 & if\ d = r \\ 0 & otherwise. \end{cases}$$

Then

$$\sum_{(b,r^k)_k=1} C_k(n-b,r) = \frac{\Phi_k(r)\,\mu(r)\,C_k(n,r)}{\Phi_k(r)}$$

$$= \mu(r)C_k(n,r)$$

Therefore,

$$\sum_{(b,r^k)_k=1} C_k(n-b,r) = \mu(r)\,C_k(n,r)$$

3.2 Theorem:

$$\sum_{(b,r^k)_k=1} C_k(n-b,r) = \Phi_k(r) \sum_{\substack{d|r \\ (n,d^k)_k=1}} \frac{d^k\mu\left(\frac{r}{d}\right)}{\Phi_k(d)}$$

Proof: Consider

$$\sum_{(b,r^k)_k=1} C_k(n-b,r) = \sum_{(b,r^k)_k=1}\ \sum_{d^k|(n-b,r^k)_k} d^k\mu\left(\frac{r}{d}\right)$$

$$= \sum_{(b,r^k)_k=1}\ \sum_{\substack{d|r \\ n\equiv b(mod\ d^k)}} d^k\mu\left(\frac{r}{d}\right)$$

$$= \sum_{d|r} d^k\mu\left(\frac{r}{d}\right) \sum_{\substack{n\equiv b(mod\ d^k) \\ (b,r^k)_k=1}} 1$$

If $(n,d^k)_k = 1$ where d is a divisor of r then there are exactly $\frac{\Phi_k(r)}{\Phi_k(d)}$ K-reduced residue $b(mod\ r^k)$ congruent to $n(mod\ r^k)$.

Therefore,

$$\sum_{(b,r^k)_k=1} C_k(n-b,r) = \sum_{\substack{d|r \\ (n,d^k)_k=1}} d^k\mu\left(\frac{r}{d}\right)\frac{\Phi_k(r)}{\Phi_k(d)}$$

$$= \Phi_k(r) \sum_{\substack{d|r \\ (n,d^k)_k=1}} \frac{d^k\mu\left(\frac{r}{d}\right)}{\Phi_k(d)}$$

Proving the theorem.

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