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A Review on Jacobsthal and Jacobsthal-Lucas Numbers

M. Narayan Murty
(Visiting Professor in Physics, Department of Physics, GIET University, Gunupur-765022, Dist. Rayagada, Odisha, India)

Abstract: *This article presents a review on Jacobsthal and Jacobsthal-Lucas numbers. Some basic properties of Jacobsthal numbers J_n and Jacobsthal-Lucas numbers j_n are given in this article. Except j_0, all J_n and j_n are odd integers. The relations between Jacbsthal and Jacobsthal- Lucas numbers are mentioned in this article. These numbers can be represented by matrices. Fibonacci numbers can be expressed in terms of Jacobsthal numbers and vice versa. Also the identities satisfied by Jacobsthal numbers are discussed in this article.*

Keywords: *Jacobsthal numbers, Jacobsthal- Lucas numbers, Fibonacci numbers, Pell numbers, Pell-Lucas numbers.*

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I. Introduction

The *Jacobsthal numbers* are integers named after the German mathematician Ernst Jacobsthal. The *Jacobsthal sequence* $\{J_n\}$is defined by the recurrence relation

$$J_{n+2} = J_{n+1} + 2J_n, \quad J_0 = 0, J_1 = 1, n \geq 0 \qquad (1)$$

Where, J_n denotes n^{th} Jacobsthal number. Putting $n = 0$ in (1) we get $J_2 = J_1 + 2J_0 = 1$. For $n = 1$, we have $J_3 = J_2 + 2J_1 = 3$. Similarly, other Jacobsthal numbers can be calculated from (1).

The *Jacobsthal-Lucas sequence*$\{j_n\}$is defined by the recurrence relation

$$j_{n+2} = j_{n+1} + 2j_n, \quad j_0 = 2, \ j_1 = 1, n \geq 0 \qquad (2)$$

Where, j_n represents n^{th} Jacobsthal number. Putting $n = 0$ in (2) we have$j_2 = j_1 + 2j_0 = 5$. For $n = 1$, we have $j_3 = j_2 + 2j_1 = 7$. Similarly, other Jacobsthal-Lucas numbers can be found out using (2).

The *Jacobsthal Oblong sequence*$\{J_{on}\}$ is defined by the recurrence relation

$$J_{on} = J_n J_{n+1}, \ n \geq 0 \qquad (3)$$

Where, J_{0n} denotes n^{th}Jacobsthal Oblong number.

From (1), (2) and (3), we have the following Table no.1 for sequences of Jacbsthal numbers J_n, Jacobsthal-Lucas numbers j_n and Jacobsthal Oblong numbers J_{on}. It is observed from this table, all J_n, j_n& J_{on}, except $j_0 = 2$, are odd integers.

Table no.1: Sequences of J_n, j_n& J_{on}.

n	0	1	2	3	4	5	6	7	8	9	10……..
J_n	0	1	1	3	5	11	21	43	85	171	341………
j_n	2	1	5	7	17	31	65	127	257	511	1025……..
J_{on}	0	1	3	15	55	231	903	3655	14535	58311	232903………

The rest of the article is organized as follows. Section-II presents few interesting properties of Jacobsthal and Jacbsthal-Lucas numbers. The interrelationships between Jacobsthal and Jacobsthal-Lucas numbers are mentioned in Section-III. Matrix representations of Jacobsthal and Jacobsthal-Lucas numbers are given in Section-IV. The relation between Jacobsthal numbers and Fibonacci numbers is mentioned in Section-V. The analogy of Jocobsthal and Jacobsthal-Lucas numbers with other number systems is presented in section-VI. The first set, second set and third set of identities satisfied by Jacobsthal numbers are given in Section-VII, Section-VIII and Section-IX respectively. Finally, conclusion is given in Section-X.

II. Properties of Jacobsthal and Jacobsthal-Lucas numbers

The following few important properties of Jacobsthal and Jacobsthal-Lucas numbers are stated by A.F.Horadam in [1].

1. (a) For odd n, Jacobsthal numbers can be known using the expression

$$J_n = \sum_{s=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-s}{s} 2^s \text{ for odd } n \qquad (4)$$

Example: Applying the notation $\binom{p}{x} = \frac{p!}{x!(p-x)!}$ in the above expression (4), we have for $n = 5$,

$$J_5 = \sum_{s=0}^{2} \frac{(4-s)!}{s!(4-2s)!} 2^s$$

$$\Rightarrow J_5 = \frac{4!}{0!4!} 2^0 + \frac{3!}{1!2!} 2^1 + \frac{2!}{2!0!} 2^2 = 1 + 6 + 4 = 11$$

This is true as per the Table no.1.

(b) For even n, Jacobsthal-Lucas numbers can be found out using the expression

$$j_n = \sum_{s=0}^{\left[\frac{n}{2}\right]} \left(\frac{n}{n-s}\right) \binom{n-s}{s} 2^s \text{ for even } n \quad (5)$$

Example: Substituting $n = 6$ in the above expression (5), we get

$$j_6 = \sum_{s=0}^{3} \left(\frac{6}{6-s}\right) \frac{(6-s)!}{s!(6-2s)!} 2^s$$

$$= \frac{6}{6} \times \frac{6!}{0!6!} 2^0 + \frac{6}{5} \times \frac{5!}{1!4!} 2^1 + \frac{6}{4} \times \frac{4!}{2!2!} 2^2 + \frac{6}{3} \times \frac{3!}{3!0!} 2^3$$

$$= 1 + 12 + 36 + 16 = 65$$

This is true as per the Table no.1.

2. (a) Jacobsthal numbers can be found out using the generating function

$$\sum_{i=1}^{\infty} J_i x^{i-1} = (1 - x - 2x^2)^{-1} \quad (6)$$

Using the Binomial expansion in RHS of the above expression (6), we get

$$J_1 + J_2 x + J_3 x^2 + J_4 x^3 + \cdots$$

$$= 1 + (-1)(-x - 2x^2) + \frac{(-1)(-1-1)}{2!}(-x - 2x^2)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(-x - 2x^2)^3 + \cdots$$

$$= 1 + (x + 2x^2) + (x^2 + 4x^4 + 4x^3) + (x^3 + 8x^6 + 3x^2 \times 2x^2 + 3x \times 4x^4) + \cdots$$

$$= 1 + x + 3x^2 + 5x^3 + \cdots$$

Comparing both sides of the above expression, we get

$$J_1 = 1, J_2 = 1, J_3 = 3, \ J_4 = 5, \ldots.$$

Thus the Jacobsthal numbers are calculated from its generating function (6).

(b) Jacobsthal- Lucas numbers can be known using the generating function

$$\sum_{i=1}^{\infty} j_i x^{i-1} = (1 + 4x)(1 - x - 2x^2)^{-1} \quad (7)$$

Using the Binomial expansion in RHS of the above expression (7), we obtain

$$J_1 + J_2 x + J_3 x^2 + J_4 x^3 + \cdots = (1 + 4x)(1 + x + 3x^2 + 5x^3 + \cdots)$$

$$= (1 + 4x) + (1 + 4x)x + (1 + 4x)3x^2 + (1 + 4x)5x^3 + \cdots$$

$$= 1 + 5x + 7x^2 + 17x^3 + \cdots$$

Comparing both sides of the above expression, we have

$$j_1 = 1, \ j_2 = 5, \ j_3 = 7, \ j_4 = 17, \ldots.$$

Thus the Jacobsthal-Lucas numbers are determined from its generating function (7).

3. (a) Jacobsthal numbers satisfy the Binet formula

$$J_n = \frac{\alpha^n - \beta^n}{3} = \frac{2^n - (-1)^n}{3} \quad \text{for } n \geq 0 \quad (8)$$

Where α and β are the roots of quadratic equation

$$x^2 - x - 2 = 0$$

Solving the above quadratic equation, we get $\alpha = 2$ and $\beta = -1$.

(b)The Binet formula for Jacobsthal-Lucas sequence is given by

$$j_n = \alpha^n + \beta^n = 2^n + (-1)^n \quad \text{for } n \geq 0 \quad (9)$$

4. (a) Jacobsthal numbers satisfy the Simson formula

$$J_{n+1} J_{n-1} - J_n^2 = (-1)^n 2^{n-1} \quad (10)$$

<u>Proof:</u> Using the Binet formula (8), we have

$$J_{n+1} J_{n-1} - J_n^2 = \frac{(\alpha^{n+1} - \beta^{n+1})(\alpha^{n-1} - \beta^{n-1}) - (\alpha^n - \beta^n)^2}{9}$$

$$= \frac{(\alpha^{2n} - \alpha^{n+1}\beta^{n-1} - \beta^{n+1}\alpha^{n-1} + \beta^{2n}) - (\alpha^{2n} + \beta^{2n} - 2\alpha^n\beta^n)}{9}$$

$$= \frac{-\alpha^{n+1}\beta^{n-1} - \beta^{n+1}\alpha^{n-1} + 2\alpha^n\beta^n}{9} = \frac{-\alpha^{n-1}\beta^{n-1}(\alpha^2 + \beta^2 - 2\alpha\beta)}{9}$$

$$= \frac{-\alpha^{n-1}\beta^{n-1}(\alpha - \beta)^2}{9}$$

Substituting $\alpha = 2$ and $\beta = -1$ in RHS of above expression, we get

$$J_{n+1} J_{n-1} - J_n^2 = -2^{n-1}(-1)^{n-1} = (-1)^n 2^{n-1}$$

Hence the Simson formula for Jacobsthal numbers is proved.

(b) Jacobsthal-Lucas numbers satisfy the Simson formula

$$j_{n+1} j_{n-1} - j_n^2 = 9(-1)^{n-1} 2^{n-1} \quad (11)$$

<u>Proof:</u> Applying the Binet formula (9), we get

$$j_{n+1} j_{n-1} - j_n^2 = (\alpha^{n+1} + \beta^{n+1})(\alpha^{n-1} + \beta^{n-1}) - (\alpha^n + \beta^n)^2$$

$$= (\alpha^{2n} + \alpha^{n+1}\beta^{n-1} + \beta^{n+1}\alpha^{n-1} + \beta^{2n}) - (\alpha^{2n} + \beta^{2n} + 2\alpha^n\beta^n)$$

$$= \alpha^{n+1}\beta^{n-1} + \beta^{n+1}\alpha^{n-1} - 2\alpha^n\beta^n$$

$= \alpha^{n-1}\beta^{n-1}(\alpha-\beta)^2$

Substituting $\alpha = 2$ and $\beta = -1$ in RHS of above expression, we obtain

$$j_{n+1}j_{n-1} - j_n^2 = 9 \times 2^{n-1}(-1)^{n-1}$$

Hence the Simson formula for Jacobsthal-Lucas numbers is proved.

5. (a) Jacobsthal numbers satisfy the summation formula

$$\sum_{i=2}^{n} J_i = \frac{J_{n+2}-3}{2} \quad \text{for } n \geq 0 \qquad (12)$$

This formula can be verified by taking an example. For $n = 6$,

LHS of (12) $= J_2 + J_3 + J_4 + J_5 + J_6 = 1 + 3 + 5 + 11 + 21 = 41$

RHS of (12) $= \frac{J_8-3}{2} = \frac{85-3}{2} = 41$

Since LHS = RHS, (12) is verified.

(b) Jacobsthal-Lucas numbers satisfy the summation formula

$$\sum_{i=1}^{n} j_i = \frac{j_{n+2}-5}{2} \quad \text{for } n \geq 0 \qquad (13)$$

This formula can be verified by taking an example. For $n = 6$,

LHS of (13) $= j_1 + j_2 + j_3 + j_4 + j_5 + j_6 = 1 + 5 + 7 + 17 + 31 + 65 = 126$

RHS of (13) $= \frac{j_8-5}{2} = \frac{257-5}{2} = 126$

As LHS = RHS, (13) is verified.

6. (a) Jacobsthal numbers satisfy the relation

$$3(J_{n+1} + J_n) = 3 \times 2^n \qquad (14)$$

For $n = 5$, LHS $= 3(J_6 + J_5) = 3(21 + 11) = 96$ and RHS $= 3 \times 2^5 = 96$. Hence (14) is verified.

(b) Jacobsthal-Lucas numbers satisfy the relation

$$j_{n+1} + j_n = 3 \times 2^n \qquad (15)$$

For $n = 5$, LHS = $j_6 + j_5 = 65 + 31 = 96$ and RHS $= 3 \times 2^5 = 96$. Thus (15) is verified.

III. Interrelationships between Jacobsthal and Jacobsthal-Lucas numbers

The following interrelationships between Jacobsthal numbers and Jacobsthal-Lucas numbers are given in [1]. These relations are verified by taking examples and showing that the LHS of a relation is equal to its RHS. Table no.1 is referred for the values of Jacobsthal and Jacobsthal-Lucas numbers.

1. $$j_n J_n = J_{2n} \qquad (16)$$

Proof: Using Binet formulas (8) and (9) in LHS of above expression(16) , we get

$$j_n J_n = \frac{(\alpha^n+\beta^n)(\alpha^n-\beta^n)}{3} = \frac{\alpha^{2n}-\beta^{2n}}{3} = J_{2n}$$ = RHS of (16) [By eqn. (8)]

Hence, (16) is proved. To verify (16) consider the example $n = 3$. Then LHS $= j_3 J_3 = 7 \times 3 = 21$ and RHS $= J_6 = 21$. Since LHS = RHS = 21, the relation (16) is verified.

2. $$j_n = J_{n+1} + 2J_{n-1} \qquad (17)$$

For $n = 5$, LHS = $j_5 = 31$ and RHS= $J_6 + 2J_4 = 21 + 2 \times 5 = 31$.

3. $$9J_n = j_{n+1} + 2j_{n-1} \qquad (18)$$

For $n = 6$, LHS = $9J_6 = 9 \times 21 = 189$ and RHS = $j_7 + 2j_5 = 127 + 2 \times 31 = 189$.

4. $$j_{n+1} + j_n = 3(J_{n+1} + J_n) \qquad (19)$$

For $n = 5$, LHS = $j_6 + j_5 = 65 + 31 = 96$ and RHS= $3(J_6 + J_5) = 3(21 + 11) = 96$.

5. $$j_{n+1} - j_n = 3(J_{n+1} - J_n) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1} \qquad (20)$$

For $n = 5$,

$j_{n+1} - j_n = j_6 - j_5 = 65 - 31 = 34$,

$3(J_{n+1} - J_n) + 4(-1)^{n+1} = 3(J_6 - J_5) + 4(-1)^6 = 3(21 - 11) + 4 = 34$ and

$2^n + 2(-1)^{n+1} = 2^5 + 2(-1)^6 = 32 + 2 = 34$.

Since the above three expressions have same value for $n = 5$, the relation (20) is verified.

6. $$j_{n+1} - 2j_n = 3(2J_n - J_{n+1}) = 3(-1)^{n+1} \qquad (21)$$

For $n = 4$,

$j_{n+1} - 2j_n = j_5 - 2j_4 = 31 - 2 \times 17 = -3$,

$3(2J_n - J_{n+1}) = 3(2 \times J_4 - J_5) = 3(2 \times 5 - 11) = -3$ and

$3(-1)^{n+1} = 3(-1)^5 = -3$.

. As the above three expressions have same value for $n = 4$, the relation (21) is verified

7. $$2j_{n+1} + j_{n-1} = 3(2J_{n+1} + J_{n-1}) + 6(-1)^{n+1} \qquad (22)$$

For $n = 4$, LHS = $2j_5 + j_3 = 2 \times 31 + 7 = 69$ and RHS $= 3(2 \times J_5 + J_3) + 6(-1)^5 = 3(2 \times 11 + 3) - 6 = 69$.

8. $$j_{n+r} + j_{n-r} = 3(J_{n+r} + J_{n-r}) + 4(-1)^{n-r} = 2^{n-r}(2^{2r} + 1) + 2(-1)^{n-r} \qquad (23)$$

For $n = 4$ and $r = 2$,

$j_{n+r} + j_{n-r} = j_6 + j_2 = 65 + 5 = 70,$
$3(J_{n+r} + J_{n-r}) + 4(-1)^{n-r} = 3(J_6 + J_2) + 4(-1)^2 = 3(21 + 1) + 4 = 70$
and $2^{n-r}(2^{2r} + 1) + 2(-1)^{n-r} = 2^2(2^4 + 1) + 2(-1)^2 = 70$.
Since the above three expressions have same value for $n = 4$ and $r = 2$, the relation (23) is verified.

9. $$j_{n+r} - j_{n-r} = 3(J_{n+r} - J_{n-r}) = 2^{n-r}(2^{2r} - 1) \quad (24)$$

For $n = 4$ and $r = 2$,
$j_{n+r} - j_{n-r} = j_6 - j_2 = 65 - 5 = 60,$
$3(J_{n+r} - J_{n-r}) = 3(J_6 - J_2) = 3(21 - 1) = 60$ and
$2^{n-r}(2^{2r} - 1) = 2^2(2^4 - 1) = 60.$
As the above three expressions have same value for $n = 4$ & $r = 2$, the relation (24) is verified.

10. $$j_n = 3J_n + 2(-1)^n \quad (25)$$

For $n = 5$, LHS = $j_5 = 31$ and RHS = $3J_5 + 2(-1)^5 = 3 \times 11 - 2 = 31.$

11. $$3J_n + j_n = 2^{n+1} \quad (26)$$

For $n = 6$, LHS = $3J_6 + j_6 = 3 \times 21 + 65 = 128$ and RHS= $2^7 = 128.$

12. $$J_n + j_n = 2J_{n+1} \quad (27)$$

For $n = 6$, LHS = $J_6 + j_6 = 21 + 65 = 86$ and RHS= $2J_7 = 2 \times 43 = 86.$

13. $$\lim_{n \to \infty} \left(\frac{J_{n+1}}{J_n}\right) = \lim_{n \to \infty} \left(\frac{j_{n+1}}{j_n}\right) = 2 \quad (28)$$

For $n = 9, \frac{J_{n+1}}{J_n} = \frac{J_{10}}{J_9} = \frac{341}{171} \approx 1.994$ and for $= 14, \frac{J_{n+1}}{J_n} = \frac{J_{15}}{J_{14}} = \frac{10923}{5461} \approx 2.001.$
For $n = 8, \frac{j_{n+1}}{j_n} = \frac{j_9}{j_8} = \frac{511}{257} \approx 1.988$ and for $= 9, \frac{j_{n+1}}{j_n} = \frac{j_{10}}{j_9} = \frac{1025}{511} \approx 2.005.$
The above examples show that as n increases, both the ratios $\frac{J_{n+1}}{J_n}$ and $\frac{j_{n+1}}{j_n}$ approach to 2. Hence the relation (28) is verified.

14. $$\lim_{n \to \infty} \frac{j_n}{J_n} = 3 \quad (29)$$

For $n = 9, \frac{j_9}{J_9} = \frac{511}{171} \approx 2.988.$
For $n = 10, \frac{j_{10}}{J_{10}} = \frac{1025}{341} \approx 3.005.$
The above examples show that as n increases $\frac{j_n}{J_n}$ become closer and closer to 3. Hence the relation (29) is verified.

15. $$j_{n+2}j_{n-2} - j_n^2 = -9(J_{n+2}J_{n-2} - J_n^2) = 9(-1)^n 2^{n-2} \quad (30)$$

<u>Proof:</u> Using the Binet formula (9) in left terms of the above expression (30), we get

$$\begin{aligned} j_{n+2}j_{n-2} - j_n^2 &= (\alpha^{n+2} + \beta^{n+2})(\alpha^{n-2} + \beta^{n-2}) - (\alpha^n + \beta^n)^2 \\ &= \alpha^{n+2}\beta^{n-2} + \beta^{n+2}\alpha^{n-2} - 2\alpha^n\beta^n \\ &= \alpha^{n-2}\beta^{n-2}(\alpha^2 - \beta^2)^2 \\ &= 2^{n-2}(-1)^{n-2} \times 9 = 9(-1)^n 2^{n-2} \quad (30\text{ a}) \end{aligned}$$
$$[\because \alpha = 2 \ \& \ \beta = -1]$$

Using the Binet formula (8) in the middle terms of the expression (30), we have

$$\begin{aligned} -9(J_{n+2}J_{n-2} - J_n^2) &= -(\alpha^{n+2} - \beta^{n+2})(\alpha^{n-2} - \beta^{n-2}) - (\alpha^n - \beta^n)^2 \\ &= \alpha^{n+2}\beta^{n-2} + \beta^{n+2}\alpha^{n-2} - 2\alpha^n\beta^n \\ &= \alpha^{n-2}\beta^{n-2}(\alpha^2 - \beta^2)^2 \\ &= 2^{n-2}(-1)^{n-2} \times 9 = 9(-1)^n 2^{n-2} \quad (30\text{ b}) \end{aligned}$$
$$[\because \alpha = 2 \ \& \ \beta = -1]$$

As the values in (30 a) and (30 b) are equal, Eqn.(30) is proved. To verify (30) take an example with $n = 6$. Then
$j_{n+2}j_{n-2} - j_n^2 = j_8j_4 - j_6^2 = 257 \times 17 - 65^2 = 4369 - 4225 = 144$,
$-9(J_{n+2}J_{n-2} - J_n^2) = -9(J_8J_6 - J_6^2) = -9(85 \times 5 - 21^2) = -9(425 - 441) = 144$
and $9(-1)^n 2^{n-2} = 9(-1)^6 2^4 = 144.$
Since the above three expressions have same value for $= 6$, the relation (30) is verified.

16. $$J_m j_n + J_n j_m = 2J_{m+n} \quad (31)$$

This relation same as (16) for $m = n.$ When $m = 4$ & $n = 5$, LHS = $J_4j_5 + J_5j_4 = 5 \times 31 + 11 \times 17 = 155 + 187 = 342$ and RHS = $2J_9 = 2 \times 171 = 342.$

17. $$j_m j_n + 9J_m J_n = 2j_{m+n} \quad (32)$$

For $m = 4$ & $n = 5$, LHS $= j_4j_5 + 9J_4J_5 = 17 \times 31 + 9 \times 5 \times 11 = 527 + 495 = 1022$ and RHS $= 2j_9 = 2 \times 511 = 1022.$

18. $$j_n^2 + 9J_n^2 = 2j_{2n} \quad (33)$$

This relation is same as the previous relation (32) for $m = n$. If $n = 4$, LHS $= j_4^2 + 9J_4^2 = 17^2 + 9 \times 5^2 = 289 + 225 = 514$ and RHS $= 2j_8 = 2 \times 257 = 514$.

19. $$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n} \quad (34)$$

For $m = 5$ & $n = 4$, LHS $= J_5 j_4 - J_4 j_5 = 11 \times 17 - 5 \times 31 = 187 - 155 = 32$ and RHS $= (-1)^4 \times 2^5 J_1 = 1 \times 32 \times 1 = 32$.

20. $$j_m j_n - 9J_m J_n = (-1)^n 2^{n+1} j_{m-n} \quad (35)$$

For $m = 5$ & $n = 4$, LHS $= j_5 j_4 - 9J_5 J_4 = 31 \times 17 - 9 \times 11 \times 5 = 527 - 495 = 32$ and RHS $= (-1)^4 \times 2^5 j_1 = 1 \times 32 \times 1 = 32$.

21. $$j_n^2 - 9J_n^2 = (-1)^n 2^{n+1} j_0 \quad (36)$$

This relation is same as the previous relation (35) for $m = n$. If $n = 4$, LHS $= j_4^2 - 9J_4^2 = 17^2 - 9 \times 5^2 = 289 - 225 = 64$ and RHS $= (-1)^4 2^5 j_0 = 1 \times 32 \times 2 = 64$.

IV. Matrix representations of Jacobsthal and Jacobsthal-Lucas numbers

The matrix description of Jacobsthal and Jacobsthal-Lucas numbers is given by Koke and Bozkurt in [2]. These authors have defined Jacobsthal F-matrix as follows:

$$F = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \quad (37)$$

and proved for any natural number n that

$$F^n = \begin{pmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{pmatrix} \quad (38)$$

The Jacobsthal F-matrix (37) can be generated by taking $n = 1$in (38).Thus we have

$$F = \begin{pmatrix} J_2 & 2J_1 \\ J_1 & 2J_0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

Jacobsthal numbers satisfy the following matrix relation.

$$\begin{pmatrix} J_{n+1} \\ J_n \end{pmatrix} = F \begin{pmatrix} J_n \\ J_{n-1} \end{pmatrix} \quad (39)$$

This relation can be verified by taking $n = 2$. Using (37) in (39), we have

$$\text{RHS of (39)} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} J_2 \\ J_1 \end{pmatrix} = \begin{pmatrix} J_2 + 2J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} J_3 \\ J_2 \end{pmatrix} \; [\because J_3 = J_2 + 2J_1 \text{for } n = 1 \text{in (1)}]$$
$$= \text{LHS of (39)}$$

Jacobsthal-Lucas numbers satisfy the following matrix relation.

$$\begin{pmatrix} j_{n+1} \\ j_n \end{pmatrix} = F \begin{pmatrix} j_n \\ j_{n-1} \end{pmatrix} \quad (40)$$

The above relation can be verified by taking $n = 3$. Using (37) in (40), we get

$$\text{RHS of (40)} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} j_3 \\ j_2 \end{pmatrix} = \begin{pmatrix} j_3 + 2j_2 \\ j_3 \end{pmatrix} = \begin{pmatrix} j_4 \\ j_3 \end{pmatrix} \; [\because j_4 = j_3 + 2j_2 \text{ for } n = 2 \text{ in (2)}]$$
$$= \text{LHS of (40)}$$

The matrix relation (38) can be verified by taking $n = 3$. Then,

LHS of (38) = $F^3 = F \times F \times F = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 3 & 2 \end{pmatrix}$

RHS of (38) = $\begin{pmatrix} J_4 & 2J_3 \\ J_3 & 2J_2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \times 3 \\ 3 & 2 \times 1 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 3 & 2 \end{pmatrix}$

Since LHS= RHS, relation (38) is verified.

V. Relation between Jacobsthal numbers and Fibonacci numbers

If F_n denotes the n^{th} Fibonacci number, the Fibonacci sequence is given the following recurrence relation.

$$F_{n+2} = F_n + F_{n+1}, \; n \geq 0, \quad F_0 = 1 \; \& \; F_1 = 1 \quad (41)$$

The Fibonacci sequence using the above relation (41) is written as

0,1,1,2,3,5,8,13,21,34,55,89,144,233,…….. (42)

Any Fibonacci number is a product of two Jacobsthal numbers. That is,

$$F_k = J_m J_n \quad (43)$$

This relation is satisfied for the following sets of integers (k, m, n).

$(k, m, n) \equiv$ (1,1,1), (2,1,1), (1,1,2), (2,1,2), (1,2,2), (2,2,2), (4,1,3), (4,2,3), (5,1,4), (5,2,4), (10,4,5), (8,1,6), (8,2,6), etc.

For example, if $(k, m, n) \equiv (10,4,5)$, the relation $F_{10} = J_4 J_5$ is satisfied since $F_{10} = 55$, $J_4 = 5$ & $J_5 = 11$.

Similarly, any Jacobsthal number is a product of two Fibonacci numbers.

$$J_k = F_m F_n \quad (44)$$

The above relation is satisfied by the following sets of integers.

$(k, m, n) \equiv$ (1,1,1), (2,1,1), (1,2,1), (2,2,1), (1,2,2), (2,2,2), (3,4,1), (3,4,2), (4,5,1), (4,5,2), (6,8,1), (6,8,2), etc.

For example, if $(k, m, n) \equiv$ (6,8,2), the relation $J_6 = F_8 F_2$ is satisfied since $J_6 = 21, F_8 = 21$ & $F_2 = 1$.

VI. Analogy of Jacobsthal and Jacobsthal-Lucas numbers with other number systems

The sequence of Fibonacci numbers $\{F_n\}$ was defined by the recurrence relation (41). The sequences of Lucas numbers $\{L_n\}$ and Pell numbers $\{P_n\}$ and Pell-Lucas numbers $\{Q_n\}$ are defined by the following recurrence relations.

$$L_n = L_{n-1} + L_{n-2}\ ,\ n \geq 2 \text{ with } L_0 = 2 \text{ and } L_1 = 1 \qquad (45)$$

$$P_n = 2P_{n-1} + P_{n-2}\ ,\ n \geq 2 \text{ with } P_0 = 0 \text{ and } P_1 = 1 \qquad (46)$$

$$Q_n = 2Q_{n-1} + Q_{n-2}\ ,\ n \geq 2 \text{ with } Q_0 = 0 \text{ and } Q_1 = 1 \qquad (47)$$

Some values of sequences $\{L_n\}, \{P_n\}$ & $\{Q_n\}$ are given in following Table no.2.

Table no.2: Values of $\{L_n\}\{P_n\}$ & $\{Q_n\}$.

n	0	1	2	3	4	5	6	7	8	9	10……..
L_n	2	1	3	4	7	11	18	29	47	76	123………
P_n	0	1	2	5	12	29	70	169	408	985	2378……..
Q_n	2	2	6	14	34	82	198	478	1154	2786	6726 ……

The relation between Jacobsthal and Jacobsthal-Lucas numbers is given by

$$J_n j_n = J_{2n} \qquad (48) \quad \text{[Refer eqn.(16)]}$$

The above relation (48) is analogous to the following relation between Fobonacci and Lucas numbers

$$F_n L_n = F_{2n} \qquad (49)$$

This relation can be verified using the Fibonacci sequence (42) and the values of Lucas numbers given in Table no.2.

Similarly the relation (48) is also analogous to the following relation between Pell and Pell-Lucas numbers .

$$P_n Q_n = P_{2n} \qquad (50)$$

The above relation (50) can be verified using the values of Pell and Pell-Lucas numbers given in Table no.2.

VII. First set of Identities satisfied by Jacobsthal numbers

The following set of identities satisfied by Jacobsthal numbers are stated in [3]. These identities are verified by taking examples and showing the LHS of identity is equal to its RHS. The Table no.1 is referred for the values of Jacobsthal numbers. Let

$$\beta_n = \frac{2^{4n+2}-1}{3} \qquad (51)$$

$$\gamma_n = \frac{2^{4n+2}+1}{5} \qquad (52)$$

$$\tau_n = \frac{2^{4n}-1}{15} \qquad (53)$$

$$\eta_n = 2^{4n+4} + 1 \qquad (54)$$

$$\sigma_n = 5J_{4n+3} \qquad (55)$$

$$\delta_n = \frac{2^{4n+1}+1}{3} \qquad (56)$$

$$\nu_n = \frac{5(2^{4n+5}+1)}{3} \qquad (57)$$

$$\mu_n = \frac{2^{4n+3}+1}{3} \qquad (58)$$

Using the above expressions (51)-(58) and Table no.1 for Jacobsthal numbers, the values of $\beta_n, \gamma_n, \tau_n, \eta_n, \sigma_n, \delta_n, \nu_n$ & μ_n for $n = 0\&1$ are calculated. These values are given in the following Table no.3.

Table no.3: Values of $\beta_n, \gamma_n, \tau_n, \eta_n, \sigma_n, \delta_n, \nu_n$ & μ_n for $n = 0\&1$

β_0 β_1	γ_0 γ_1	τ_0 τ_1	η_0 η_1	σ_0 σ_1	δ_0 δ_1	ν_0 ν_1	μ_0 μ_1
1 21	1 13	0 1	17 257	15 215	1 11	55 855	3 43

Jacobsthal numbers satisfy the following identities.

1. For every $n \geq 0$ and $k \geq 0$,

$$(\beta_{n+1}\gamma_{n+1} - \beta_n\gamma_n)J_{2k}^2 + 8(\beta_{n+1}\tau_{n+1} - \beta_n\tau_n)J_{2k} = J_{2k+4n+4}^2 + J_{2k+4n+2}^2 - J_{4n+4}^2 - J_{4n+2}^2 \quad (59)$$

Verification: This identity is verified by taking an example.

For $n = 0$ & $k = 2$, LHS $= (\beta_1\gamma_1 - \beta_0\gamma_0)J_4^2 + 8(\beta_1\tau_1 - \beta_0\tau_0)J_4$
$= (21 \times 13 - 1 \times 1)5^2 + 8(21 \times 1 - 1 \times 0)5$
$= (273 - 1)25 + 8 \times 21 \times 5 = 7640$

For $n = 0$ & $k = 2$, RHS$= J_8^2 + J_6^2 - J_4^2 - J_2^2 = 85^2 + 21^2 - 5^2 - 1^2$
$= 7225 + 441 - 25 - 1 = 7640$

[Refer Tables no.1 & no.3]

This identity is verified since LHS=RHS.

2. For every $n \geq 0$ and $k \geq 0$,

$$(\tau_{n+2}\eta_{n+1} - \tau_{n+1}\eta_n)J_{2k}^2 + 8(\beta_{n+1}\tau_{n+2} - \beta_n\tau_{n+1})J_{2k} = J_{2k+4n+6}^2 + J_{2k+4n+4}^2 - J_{4n+6}^2 - J_{4n+4}^2 \quad (60)$$

Verification:

For $n = 0$ & $k = 1$, LHS $= (\tau_2\eta_1 - \tau_1\eta_0)J_2^2 + 8(\beta_1\tau_2 - \beta_0\tau_1)J_2$

Using (53), $\tau_2 = \frac{2^8-1}{15} = 17$. Then,

LHS $= (17 \times 257 - 1 \times 17)1^2 + 8(21 \times 17 - 1 \times 1)1 = (4369 - 17) + 8(357 - 1)$
$= 4352 + 8 \times 356 = 7200$

For $n = 0$ & $k = 1$, RHS $= J_8^2 + J_6^2 - J_6^2 - J_4^2 = J_8^2 - J_4^2 = 85^2 - 5^2 = 7200$

[Refer Tables no.1 & no.3]

This identity is verified since LHS = RHS.

3. For every $m \geq 0$ & $k \geq 0$,

$$\sum_{i=0}^{m} J_{2k+2i}^2 = \sum_{i=0}^{m} J_{2i}^2 + \beta_n J_{2k}\ [\gamma_n J_{2k} + 8\tau_n] \quad (61)$$

if $m = 2n$ and $n = 0,1,2, \ldots.. etc.$

Verification: when $n = 0$, we have $m = 2n = 0$, then

LHS = J_{2k}^2

RHS = $J_0^2 + \beta_0 J_{2k}[\gamma_0 J_{2k} + 8\tau_0]$

Since, $J_0 = 0, \beta_0 = 1, \gamma_0 = 1$& $\tau_0 = 0$, we get

RHS = J_{2k}^2

This identity is verified since LHS = RHS.

4. For every $m \geq 0$ & $k \geq 0$,

$$\sum_{i=0}^{m} J_{2k+2i}^2 = \sum_{i=0}^{m} J_{2i}^2 + \tau_{n+1} J_{2k}\ [\eta_n J_{2k} + 8\beta_n] \quad (62)$$

if $m = 2n + 1$ and $n = 0,1,2, \ldots.. etc.$

Verification: when $n = 0$, we have $m = 2n + 1 = 1$, then

LHS = $\sum_{i=0}^{1} J_{2k+2i}^2 = J_{2k}^2 + J_{2k+2}^2$

RHS = $\sum_{i=0}^{1} J_{2i}^2 + \tau_1 J_{2k}[\eta_0 J_{2k} + 8\beta_0] = J_0^2 + J_2^2 + \tau_1 J_{2k}[\eta_0 J_{2k} + 8\beta_0]$

Since, $J_0 = 0, J_2 = 1, \tau_1 = 1, \eta_0 = 17$ & $\beta_0 = 1$, we obtain

RHS = $1 + J_{2k}[17J_{2k} + 8] = 1 + 17J_{2k}^2 + 8J_{2k}$

$$= J_{2k}^2 + (4J_{2k} + 1)^2 \quad (63)$$

For $k \geq 0$, Jacobsthal numbers satisfy the relation

$$J_{2k} = 4J_{2k-2} + 1 \text{ [Refer Eqn.(71)]} \quad (64)$$

Replacing k by $(k + 1)$ in (63) we get

$$J_{2k+2} = 4J_{2k} + 1 \quad (65)$$

Using (65) in (63), RHS $= J_{2k}^2 + J_{2k+2}^2$, which is also the LHS of (62). Hence the above identity (62) is verified.

5. For $k \geq 0$,

$$J_{2k+1}^2 = 16J_{2k}J_{2k-2} + 8J_{2k} + 1 \quad (66)$$

Verification: Let $k = 3$, then LHS = $J_7^2 = 43^2 = 1849$ and RHS $= 16J_6J_4 + 8J_6 + 1 = 16 \times 21 \times 5 + 8 \times 21 + 1$= 1680 + 168 +1= 1849. As LHS = RHS, this identity is verified.

6. For $k \geq 0$,

$$J_{2k+1}^2 + J_{2k+3}^2 = 10 + 8J_{2k}(34J_{2k-2} + 15) \quad (67)$$

Verification: When $k = 2$, LHS $= J_5^2 + J_7^2 = 11^2 + 43^2 = 121 + 1849 = 1970$ and RHS $= 10 + 8J_4(34J_2 + 15) = 10 + 8 \times 5(34 \times 1 + 15) = 1970$. Since LHS = RHS, this identity is verified.

7. For every $m \geq 0$ & $k \geq 0$,

$$\sum_{i=0}^{m} J_{2k+2i+1}^2 = \sum_{i=0}^{m} J_{2i+1}^2 + 8\beta_n J_{2k}\ [2\gamma_n J_{2k-2} + \delta_n] \quad (68)$$

if $m = 2n$ and $n = 0,1,2, \ldots.. etc.$

Verification: when $n = 0$, we have $m = 2n = 0$, then
LHS = J_{2k+1}^2
RHS = $J_1^2 + 8\beta_0 J_{2k}[2\gamma_0 J_{2k-2} + \delta_0]$
Now taking $k = 1$, we get
LHS = $J_3^2 = 3^2 = 9$
RHS = $J_1^2 + 8\beta_0 J_2[2\gamma_0 J_0 + \delta_0] = 1^2 + 8 \times 1 \times 1[2 \times 1 \times 0 + 1] = 9$
This identity is verified since LHS = RHS.

8. For every $m \geq 0$ & $k \geq 0$,
$$\sum_{i=0}^{m} J_{2k+2i+1}^2 = \sum_{i=0}^{m} J_{2i+1}^2 + 8\tau_{n+1} J_{2k}\ [2\eta_n J_{2k-2} + \sigma_n] \quad (69)$$
if $m = 2n + 1$ and $n = 0,1,2, \ldots\ldots etc.$
Verification: when $n = 0$, we have $m = 2n + 1 = 1$, then
LHS = $\sum_{i=0}^{1} J_{2k+2i+1}^2 = J_{2k+1}^2 + J_{2k+3}^2$
RHS = $\sum_{i=0}^{1} J_{2i+1}^2 + 8\tau_1 J_{2k}[2\eta_0 J_{2k-2} + \sigma_0]$
Now taking $k = 1$, we get
LHS = $J_3^2 + J_5^2 = 3^2 + 11^2 = 9 + 121 = 130$
RHS = $J_1^2 + J_3^2 + 8\tau_1 J_2[2\eta_0 J_0 + \sigma_0]$
$= 1^2 + 3^2 + 8 \times 1 \times 1[2 \times 17 \times 0 + 15] = 130$
This identity is verified since LHS = RHS.

9. For $k \geq 0$,
$$J_{2k+1} = 8J_{2k-2} + 3 \quad (70)$$
Proof:
Replacing n by $(2k + 1)$ in the Binet formula (8), we obtain
$$J_{2k+1} = \frac{2^{2k+1} - (-1)^{2k+1}}{3}$$
$$= \frac{2^{2k+1} + (-1)^{2k+2}}{3}$$
$$= \frac{2^{2k+1} + 1}{3} \qquad [\because \text{For } k \geq 0, (-1)^{2k+2} = 1]$$
$$= \frac{2^{2k-2} \times 2^3 - 8 + 9}{3}$$
$$= 8\left(\frac{2^{2k-2} - 1}{3}\right) + 3$$
$$= 8\left[\frac{2^{2k-2} - (-1)^{2k-2}}{3}\right] + 3 \qquad [\because \text{For } k \geq 0, (-1)^{2k-2} = 1]$$
$\Rightarrow J_{2k+1} = 8J_{2k-2} + 3$ [Using Binet formula (8)]
Hence (70) is proved.

10. For every $k \geq 0$,
$$J_{2k} = 4J_{2k-2} + 1 \quad (71)$$
Proof: Using the Binet formula (8) the RHS can be written as
RHS = $4J_{2k-2} + 1 = 4\left[\frac{2^{2k-2} - (-1)^{2k-2}}{3}\right] + 1 = \frac{2^{2k} - 4(-1)^{2k-2} + 3}{3}$
$$= \frac{2^{2k} - 4 + 3}{3} \qquad [\because \text{For } k \geq 0, (-1)^{2k-2} = 1]$$
$$= \frac{2^{2k} - 1}{3}$$
$$= \frac{2^{2k} - (-1)^{2k}}{3} \qquad [\because \text{For } k \geq 0, (-1)^{2k} = 1]$$
$= J_{2k}$ [By Binet formula (8)]
= LHS

Hence (71) is proved.

11. For every $k \geq 0$,
$$J_{2k}J_{2k+1} + J_{2k+2}J_{2k+3} = 3 + J_{2k}(136J_{2k-2} + 55) \quad (72)$$
Verification: When $k = 1$, LHS $= J_2J_3 + J_4J_5 = 1 \times 3 + 5 \times 11 = 58$ and RHS $= 3 + J_2(136J_0 + 55) = 3 + 1(136 \times 0 + 55) = 58$. Since LHS = RHS, this identity is verified.

12. For every $k \geq 0$,
$$J_{2k+2}^2 - J_{2k}^2 = 1 + J_{2k}(60J_{2k-2} + 23) \quad (73)$$
Verification: When $k = 2$, LHS $= J_6^2 - J_4^2 = 21^2 - 5^2 = 441 - 25 = 416$ and RHS $= 1 + J_4(60J_2 +$ *23=1+560×1+23=416.* As LHS = RHS, this identity is verified.

13. For every $k \geq 0$,
$$J_{2k+3}J_{2k+2} - J_{2k+1}J_{2k} = 3 + J_{2k}(120J_{2k-2} + 49) \quad (74)$$

<u>Verification:</u> When $k = 2$, LHS $= J_7J_6 - J_5J_4 = 43 \times 21 - 11 \times 5 = 903 - 55 = 848$ and RHS $= 3 + J_4(120J_2 + 49) = 3 + 5(120 \times 1 + 49) = 848$. Since LHS = RHS, this identity is verified.

14. For every $n \geq 0$ & $k \geq 0$,

$$J_{2k}[8(\beta_{n+1}\gamma_{n+1} - \beta_n\gamma_n)J_{2k-2} + \beta_{n+1}\mu_{n+1} - \beta_n\mu_n] = J_{2k+4n+5}J_{2k+4n+4} + J_{2k+4n+3}J_{2k+4n+2} - J_{4n+5}J_{4n+4} - J_{4n+3}J_{4n+2} \quad (75)$$

<u>Verification:</u> For $n = 0$ & $k = 2$,

LHS= $J_4[8(\beta_1\gamma_1 - \beta_0\gamma_0)J_2 + \beta_1\mu_1 - \beta_0\mu_0]$

$= 5[8(21 \times 13 - 1 \times 1)1 + 21 \times 43 - 1 \times 3] = 15380$

RHS= $J_9J_8 + J_7J_6 - J_5J_4 - J_3J_2 = 171 \times 85 + 43 \times 21 - 11 \times 5 - 3 \times 1 = 15380$

As LHS = RHS, this identity is verified.

15. For every $m \geq 0$ & $k \geq 0$,

$$\sum_{i=0}^{m} J_{2k+2i}J_{2k+2i+1} = \sum_{i=0}^{m} J_{2i}J_{2i+1} + \beta_n J_{2k}[8\gamma_n J_{2k-2} + \mu_n] \quad (76)$$

if $m = 2n$ and $n = 0,1,2, \ldots\ldots etc.$

<u>Verification:</u> For $n = 1$ & $k = 2$,

LHS = $\sum_{i=0}^{2} J_{4+2i}J_{5+2i} = J_4J_5 + J_6J_7 + J_8J_9 = 5 \times 11 + 21 \times 43 + 85 \times 171$

$= 55 + 903 + 14535 = 15493$

RHS = $\sum_{i=0}^{2} J_{2i}J_{2i+1} + \beta_1 J_4[8\gamma_1 J_2 + \mu_1]$

$= J_0J_1 + J_2J_3 + J_4J_5 + \beta_1 J_4[8\gamma_1 J_2 + \mu_1]$

$= 0 \times 1 + 1 \times 3 + 5 \times 11 + 21 \times 5[8 \times 13 \times 1 + 43] = 15493$

As LHS = RHS, this identity is verified.

16. For every $m \geq 0$ & $k \geq 0$,

$$\sum_{i=0}^{m} J_{2k+2i}J_{2k+2i+1} = \sum_{i=0}^{m} J_{2i}J_{2i+1} + \tau_{n+1} J_{2k}[8\eta_n J_{2k-2} + \nu_n] \quad (77)$$

if $m = 2n + 1$ and $n = 0,1,2, \ldots\ldots etc.$

<u>Verification:</u> For $n = 1$ & $k = 2$,

LHS = $\sum_{i=0}^{3} J_{4+2i}J_{5+2i} = J_4J_5 + J_6J_7 + J_8J_9 + J_{10}J_{11}$

$= 5 \times 11 + 21 \times 43 + 85 \times 171 + 341 \times 683$

$= 55 + 903 + 14535 + 232903 = 248396$

RHS= $\sum_{i=0}^{3} J_{2i}J_{2i+1} + \tau_2 J_4[8\eta_1 J_2 + \nu_1]$

$= J_0J_1 + J_2J_3 + J_4J_5 + J_6J_7 + \tau_2 J_4[8\eta_1 J_2 + \nu_1]$

$= 0 \times 1 + 1 \times 3 + 5 \times 11 + 21 \times 43 + 17 \times 5[8 \times 257 \times 1 + 855]$

$[\tau_2 = 17 \; using \; (53)]$

$= 3 + 55 + 903 + 85[2056 + 855] = 248396$

Since LHS = RHS, this identity is verified.

17. For every $k \geq 0$,

$$J_{2k+1}^2 = 1 + 8J_{2k}[2J_{2k-2} + 1] \quad (78)$$

<u>Verification:</u> For $k = 2$,

LHS = $J_5^2 = 11^2 = 121$

RHS = $1 + 8J_4[2J_2 + 1] = 1 + 8 \times 5[2 \times 1 + 1] = 121$

As LHS = RHS, this identity is verified.

18. For every $k \geq 0$,

$$J_{2k+3}^2 - J_{2k+1}^2 = 8 + 8J_{2k}[30J_{2k-2} + 13] \quad (79)$$

<u>Verification:</u> For $k = 2$,

LHS = $J_7^2 - J_5^2 = 43^2 - 11^2 = 1849 - 121 = 1728$

RHS = $8 + 8J_4[30J_2 + 13] = 8 + 8 \times 5[30 \times 1 + 13] = 1728$

VIII. Second set of Identities satisfied by Jacobsthal numbers

The following set of identities satisfied by Jacobsthal numbers are mentioned in [3]. These identities are verified by taking examples and showing the LHS of identity is equal to its RHS. The Table no.1 is referred for the values of Jacobsthal numbers.

Define

$\tau_0^* = 1$ and $(\tau_{n+1}^* - \tau_n^*) = 2^{4n+3}(50 \times 2^{4n} - 1)$ (80)

for $n = 0,1,2, \ldots\ldots etc.$

$\gamma_n^* = \frac{2^{8n+4}+1}{17}$ for $n = 0,1,2, \ldots\ldots etc.$ (81)

$\beta_0^* = 23$ and $(\beta_{n+1}^* - \beta_n^*) = 2^{4n+5}(200 \times 2^{4n} - 1)$ (82)

for $n = 0,1,2, \ldots\ldots etc.$

$\eta_n^* = \frac{2^{8n+8}-1}{17}$ for $n = 0,1,2, \ldots\ldots etc.$ (83)

Jacobsthal numbers satisfy the following identities.

1. For every $n \geq 0$ and $k \geq 0$,

$$J_{2k}\{4(\gamma_{n+1}^* - \gamma_n^*)J_{2k-2} + \tau_{n+1}^* - \tau_n^*\} = J_{2k+4n+4}^2 - J_{2k+4n+2}^2 - J_{4n+4}^2 + J_{4n+2}^2 \quad (84)$$

Verification: For $n = 1$ & $k = 2$,

LHS = $J_4\{4(\gamma_2^* - \gamma_1^*)J_2 + \tau_2^* - \tau_1^*\}$ (85)

Substituting $n = 1$ in (80), we get

$(\tau_2^* - \tau_1^*) = 2^7(50 \times 2^4 - 1) = 102272$ (86)

Applying (81) we obtain

$\gamma_1^* = \frac{2^{12}+1}{17} = 241$ and $\gamma_2^* = \frac{2^{20}+1}{17} = 61681$ (87)

Using (86) & (87) in (85) we have

LHS = $5\{4(61681 - 241) \times 1 + 102272\} = 1740160$ (88)

$[J_4 = 5 \,\&\, J_2 = 1]$

Now, for $n = 1$ & $k = 2$, RHS of (84) is given by

RHS = $J_{12}^2 - J_{10}^2 - J_8^2 + J_6^2 = 1365^2 - 341^2 - 85^2 + 21^2$

$= 1863225 - 116281 - 7225 + 441 = 1740160$ (89)

The values of LHS in (88) is equal to the value of RHS in (89). Hence this identity is verified.

2. For every $n \geq 0$ and $k \geq 0$,

$$J_{2k}\{4(\eta_{n+1}^* - \eta_n^*)J_{2k-2} + \beta_{n+1}^* - \beta_n^*\} = J_{2k+4n+6}^2 - J_{2k+4n+4}^2 + J_{4n+4}^2 - J_{4n+6}^2 \quad (90)$$

Verification: For $n = 0$ & $k = 2$,

LHS = $J_4\{4(\eta_1^* - \eta_0^*)J_2 + \beta_1^* - \beta_0^*\}$ (91)

Using (83) we obtain

$\eta_0^* = \frac{2^8-1}{17} = 15$ and $\eta_1^* = \frac{2^{16}-1}{17} = 3855$ (92)

Substituting $n = 0$ in (82), we get

$(\beta_1^* - \beta_0^*) = 2^5(200 - 1) = 6368$ (93)

Using (92) & (93) in (91) we get

LHS = $5\{4(3855 - 15) \times 1 + 6368\} = 108640$ (94)

$[J_4 = 5 \,\&\, J_2 = 1]$

Now, for $n = 0$ & $k = 2$, RHS of (90) is given by

RHS = $J_{10}^2 - J_8^2 + J_4^2 - J_6^2 = 341^2 - 85^2 + 5^2 - 21^2$

$= 116281 - 7225 + 25 - 441 = 108640$ (95)

The values of LHS in (94) and RHS in (95) are equal. Hence this identity is verified

3. For every $m \geq 0$ & $k \geq 0$,

$\sum_{i=0}^{m}(-1)^i J_{2k+2i}^2 = \sum_{i=0}^{m}(-1)^i J_{2i}^2 + J_{2k}\,[4\gamma_n^* J_{2k-2} + \tau_n^*]$ (96)

if $m = 2n$ and $n = 0,1,2, \ldots\ldots etc.$

Verification: when $n = 1$, we have $m = 2n = 2$, then

LHS = $\sum_{i=0}^{2}(-1)^i J_{2k+2i}^2$ and RHS = $\sum_{i=0}^{2}(-1)^i J_{2i}^2 + J_{2k}[4\gamma_1^* J_{2k-2} + \tau_1^*]$

Now taking $k = 2$, we get

LHS = $\sum_{i=0}^{2}(-1)^i J_{4+2i}^2 = J_4^2 - J_6^2 + J_8^2 = 5^2 - 21^2 + 85^2$

$= 25 - 441 + 7225 = 6809$

RHS = $\sum_{i=0}^{2}(-1)^i J_{2i}^2 + J_4[4\gamma_1^* J_2 + \tau_1^*]$

$= J_0^2 - J_2^2 + J_4^2 + J_4[4\gamma_1^* J_2 + \tau_1^*]$ (97)

Substituting $n = 0$ in (80), we have

$\tau_1^* - \tau_0^* = 2^3(50-1) = 392$

$\Rightarrow \tau_1^* = 392 + \tau_0^* = 392 + 1 = 393$ (98)

Using (87),(98) and the values of Jacobsthal numbers in (97), we obtain

RHS $= 0^2 - 1^2 + 5^2 + 5[4 \times 241 \times 1 + 393] = 6809$

This identity is verified since the value of either LHS or RHS is equal to 6809.

4. For every $m \geq 0$ & $k \geq 0$,

$\sum_{i=0}^{m}(-1)^i J_{2k+2i}^2 = \sum_{i=0}^{m}(-1)^i J_{2i}^2 - J_{2k}\ [4\eta_n^* J_{2k-2} + \beta_n^*]$ (99)

if $m = 2n+1$ and $n = 0,1,2, \dots\dots etc.$

Verification: when $n = 1$, we have $m = 2n + 1 = 3$, then

LHS = $\sum_{i=0}^{3}(-1)^i J_{2k+2i}^2$ and RHS = $\sum_{i=0}^{3}(-1)^i J_{2i}^2 - J_{2k}[4\eta_1^* J_{2k-2} + \beta_1^*]$

Now taking $k = 2$, we get

LHS = $\sum_{i=0}^{3}(-1)^i J_{4+2i}^2 = J_4^2 - J_6^2 + J_8^2 - J_{10}^2 = 5^2 - 21^2 + 85^2 - 341^2$

$= 25 - 441 + 7225 - 116281 = -109472$

RHS = $\sum_{i=0}^{3}(-1)^i J_{2i}^2 - J_4[4\eta_1^* J_2 + \beta_1^*]$

$= J_0^2 - J_2^2 + J_4^2 - J_6^2 - J_4[4\eta_1^* J_2 + \beta_1^*]$ (100)

Substituting $n = 0$ in (82), we have

$\beta_1^* - \beta_0^* = 2^5(200-1) = 6368$

$\Rightarrow \beta_1^* = 6368 + \beta_0^* = 6368 + 23 = 6391$ (101)

Using (92),(101) and the values of Jacobsthal numbers in (100), we get

RHS $= 0^2 - 1^2 + 5^2 - 21^2 - 5[4 \times 3855 \times 1 + 6391] = -109472$

This identity is verified since the value of either LHS or RHS is equal to (-109472).

IX. Third set of Identities satisfied by Jacobsthal numbers

The following set of identities satisfied by Jacobsthal numbers are mentioned in [3]. These identities are verified by taking examples and showing the LHS of identity is equal to its RHS. The Table no.1 is referred for the values of Jacobsthal numbers.

Define

$\tau_0^{**} = 1$ and $(\tau_{n+1}^{**} - \tau_n^{**}) = 101 \times 2^{4n+1} + 25 \times 2^{4n+3}(2^{4n} - 1)$ (102)

for $n = 0,1,2, \dots\dots etc.$

$\beta_0^{**} = 13$ and $(\beta_{n+1}^{**} - \beta_n^{**}) = 401 \times 2^{4n+3} + 100 \times 2^{4n+5}(2^{4n} - 1)$ (103)

for $n = 0,1,2, \dots\dots etc.$

$\beta_0^{***} = 49$ and $(\beta_{n+1}^{***} - \beta_n^{***}) = 799 \times 2^{4n+4} + 25 \times 2^{4n+9}(2^{4n} - 1)$ (104)

for $n = 0,1,2, \dots\dots etc.$

Jacobsthal numbers satisfy the following identities.

1. For every $n \geq 0$ & $k \geq 0$,

$$8J_{2k}\{2(\gamma_{n+1}^* - \gamma_n^*)J_{2k-2} + \tau_{n+1}^{**} - \tau_n^{**}\} = J_{2k+4n+5}^2 - J_{2k+4n+3}^2 - J_{4n+5}^2 + J_{4n+3}^2$$

(105)

Verification: For $n = 1$ & $k = 2$,

LHS $= 8J_4\{2(\gamma_2^* - \gamma_1^*)J_2 + \tau_2^{**} - \tau_1^{**}\}$ (106)

Substituting $n = 1$ in (102), we get

$\tau_2^{**} - \tau_1^{**} = 101 \times 2^5 + 25 \times 2^7(2^4 - 1)$

$= 101 \times 32 + 25 \times 128 \times 15 = 51232$ (107)

Using (87) and (107) in (106), we get

LHS $= 8 \times 5\{2(61681 - 241) \times 1 + 51232\} = 6964480$

Now for $n = 1$ & $k = 2$,

RHS $= J_{13}^2 - J_{11}^2 - J_9^2 + J_7^2$

$= 2371^2 - 683^2 - 171^2 + 43^2$

$= 7458361 - 466489 - 29241 + 1849 = 6964480$

This identity is verified since the value of either LHS or RHS is equal to 6964480.

2. For every $n \geq 0$ & $k \geq 0$,
$$8J_{2k}\{2(\eta^*_{n+1} - \eta^*_n)J_{2k-2} + \beta^{**}_{n+1} - \beta^{**}_n\} = J^2_{2k+4n+7} - J^2_{2k+4n+5} + J^2_{4n+5} - J^2_{4n+7} \quad (108)$$

Verification: For $n = 1$ & $k = 2$,
$$\text{LHS} = 8J_4\{2(\eta^*_2 - \eta^*_1)J_2 + \beta^{**}_2 - \beta^{**}_1\} \quad (109)$$
From (83), we have
$$\eta^*_2 = \frac{2^{24}-1}{17} = \frac{16777216-1}{17} = 986895 \quad (110)$$
Substituting $n = 1$ in (103) we get
$$\beta^{**}_2 - \beta^{**}_1 = 401 \times 2^7 + 100 \times 2^9(2^4 - 1) = 819328 \quad (111)$$
Using (92), (110) and (111) in (109) we obtain
$\text{LHS} = 8 \times 5\{2(986895 - 3855) \times 1 + 819328\} = 111416320$
Now for $n = 1$ & $k = 2$,
$\text{RHS} = J^2_{15} - J^2_{13} + J^2_9 - J^2_{11}$
$= 10923^2 - 2731^2 + 171^2 - 683^2 = 111416320$
This identity is verified since the value of either LHS or RHS is equal to 111416320.

3. For every $m \geq 0$ & $k \geq 0$,
$$\sum_{i=0}^{m}(-1)^i J^2_{2k+2i+1} = \sum_{i=0}^{m}(-1)^i J^2_{2i+1} + 8J_{2k}\ [2\gamma^*_n J_{2k-2} + \tau^{**}_n]$$
if $m = 2n$ and $n = 0,1,2, \ldots.. etc.$ (112)

Verification: when $n = 1$, we have $m = 2n = 2$, then
LHS = $\sum_{i=0}^{2}(-1)^i J^2_{2k+2i+1}$ and RHS = $\sum_{i=0}^{2}(-1)^i J^2_{2i+1} + 8J_{2k}[2\gamma^*_1 J_{2k-2} + \tau^{**}_1]$
Now taking $k = 2$, we get
LHS = $\sum_{i=0}^{2}(-1)^i J^2_{4+2i+1} = J^2_5 - J^2_7 + J^2_9 = 11^2 - 43^2 + 171^2 = 27513$
RHS = $\sum_{i=0}^{2}(-1)^i J^2_{2i+1} + 8J_4[2\gamma^*_1 J_2 + \tau^{**}_1]$
$= J^2_1 - J^2_3 + J^2_5 + 8J_4[2\gamma^*_1 J_2 + \tau^{**}_1]$ (113)
Substituting $n = 0$ in (102), we get
$\tau^{**}_1 - \tau^{**}_0 = 101 \times 2 + 25 \times 2^3(2^0 - 1) = 202$
$\Rightarrow \tau^{**}_1 = 202 + \tau^{**}_0 = 202 + 1 = 203$ [Refer (102)]
Using the value of γ^*_1 from (87) and the above value of τ^{**}_1 and the values of Jacobsthal numbers in (113), we have
RHS $= 1^2 - 3^2 + 11^2 + 8 \times 5[2 \times 241 \times 1 + 203] = 27513$
This identity is verified since the value of either LHS or RHS is equal to 27513.

4. For every $m \geq 0$ & $k \geq 0$,
$$\sum_{i=0}^{m}(-1)^i J^2_{2k+2i+1} = \sum_{i=0}^{m}(-1)^i J^2_{2i+1} - 8J_{2k}\ [2\eta^*_n J_{2k-2} + \beta^{**}_n]$$
if $m = 2n + 1$ and $n = 0,1,2, \ldots.. etc.$ (114)

Verification: when $n = 1$, we have $m = 2n + 1 = 3$, then
LHS = $\sum_{i=0}^{3}(-1)^i J^2_{2k+2i+1}$ and RHS = $\sum_{i=0}^{3}(-1)^i J^2_{2i+1} - 8J_{2k}[2\eta^*_1 J_{2k-2} + \beta^{**}_1]$
Now taking $k = 2$, we get
LHS = $\sum_{i=0}^{3}(-1)^i J^2_{4+2i+1} = J^2_5 - J^2_7 + J^2_9 - J^2_{11}$
$= 11^2 - 43^2 + 171^2 - 683^2 = -438976$
RHS = $\sum_{i=0}^{3}(-1)^i J^2_{2i+1} - 8J_4[2\eta^*_1 J_2 + \beta^{**}_1]$
$= J^2_1 - J^2_3 + J^2_5 - J^2_7 - 8J_4[2\eta^*_1 J_2 + \beta^{**}_1]$ (115)
Substituting $n = 0$ in (103), we have
$\beta^{**}_1 - \beta^{**}_0 = 401 \times 2^3 + 100 \times 2^5(2^0 - 1) = 3208$
$\Rightarrow \beta^{**}_1 = 3208 + \beta^{**}_0 = 3208 + 13 = 3221$ (116)
Using (92),(116) and the values of Jacobsthal numbers in (115), we get
RHS $= 1^2 - 3^2 + 11^2 - 43^2 - 8 \times 5[2 \times 3855 \times 1 + 3221] = -438976$
This identity is verified since the value of either LHS or RHS is equal to (- 438976).

5. For every $n \geq 0$ & $k \geq 0$,
$J_{2k}\{8(\eta^*_{n+1} - \eta^*_n)J_{2k-2} + \beta^{***}_{n+1} - \beta^{***}_n\}$

$$= J_{2k+4n+6}J_{2k+4n+7} - J_{2k+4n+4}J_{2k+4n+5} + J_{4n+5}J_{4n+4} - J_{4n+6}J_{4n+7} \quad (117)$$

Verification: For $n = 0$ & $k = 2$,

$$\text{LHS} = J_4\{8(\eta_1^* - \eta_0^*)J_2 + \beta_1^{***} - \beta_0^{***}\} \quad (118)$$

Substituting $n = 0$ in (104), we get

$$\beta_1^{***} - \beta_0^{***} = 799 \times 2^4 + 25 \times 2^9(2^0 - 1) = 12784 \quad (119)$$

Using (92) and (119) and the values of Jacobsthal numbers in (118) we obtain

LHS = $5\{8(3855 - 15) \times 1 + 12784\} = 217520$

For $n = 0$ & $k = 2$,

RHS = $J_{10}J_{11} - J_8J_9 + J_5J_4 - J_6J_7$

$= 341 \times 683 - 85 \times 171 + 11 \times 5 - 21 \times 43 = 217520$

This identity is verified since the value of either LHS or RHS is equal to 217520.

6. For every $m \geq 0$ & $k \geq 0$,

$$\sum_{i=0}^{m}(-1)^i J_{2k+2i}J_{2k+2i+1} = \sum_{i=0}^{m}(-1)^i J_{2i}J_{2i+1} - J_{2k}\ [8\eta_n^* J_{2k-2} + \beta_n^{***}]$$

if $m = 2n + 1$ and $n = 0,1,2, \ldots\ etc.$ (120)

Verification: For $n = 1$ & $k = 2$,

LHS = $\sum_{i=0}^{3}(-1)^i J_{4+2i}J_{5+2i} = J_4J_5 - J_6J_7 + J_8J_9 - J_{10}J_{11}$

$= 5 \times 11 - 21 \times 43 + 85 \times 171 - 341 \times 683 = -219216$

RHS = $\sum_{i=0}^{3}(-1)^i J_{2i}J_{2i+1} - J_4[8\eta_1^* J_2 + \beta_1^{***}]$

$$= J_0J_1 - J_2J_3 + J_4J_5 - J_6J_7 - J_4[8\eta_1^* J_2 + \beta_1^{***}] \quad (121)$$

Using (104) in (119) we have

$$\beta_1^{***} = 12784 + \beta_0^{***} = 12784 + 49 = 12833 \quad (122)$$

Using (92), (122) and the values of Jacobsthal numbers in (121) we obtain

RHS = $0 \times 1 - 1 \times 3 + 5 \times 11 - 21 \times 43 - 5[8 \times 3855 \times 1 + 12833]$

$= -219216$

This identity is verified since the value of either LHS or RHS is equal to (- 219216).

X. Conclusion

Jacobsthal and Jacobsthal-Lucas numbers have interesting properties. Jacobsthal and Jacobsthal-Lucas numbers are related to each other. These numbers can be represented by matrices. Jacobsthal numbers can be expressed in terms of Fibonacci numbers and vice versa. The identities satisfied by Jacobsthal and Jacobsthal-Lucas numbers can be derived from their Binet formulas. This article may inspire curious mathematicians to explore it further.

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