IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 18, Issue 4 Ser. I (Jul. – Aug. 2022), PP 01-15
www.iosrjournals.org

Oscillation Criteria and Estimates of Solutions of Some Quasilinear Elliptic Equations Via Picone-Type Formulas

Tadie Tadie

I. Introduction

Date of Submission: 22-06-2022 Date of Acceptance: 04-07-2022

In the whole work , $E^n \equiv \mathbb{R}^n;\ n \geq 3$. We will be dealing with equations of the types

$$(H1)\begin{cases} (P): \quad P(u) := \nabla\cdot\Big(A(x)\Phi(\nabla u)\Big) + C(x)\phi(u) + F(x,u,\nabla u) = 0; \quad x \in E^n; \\ (Q): \quad p(y) := \Big(a(t)\phi(y')\Big)' + c(t)\phi(y) + f(t,y,y') = 0; adt > 0 \\ \text{where for some } \ \alpha > 0\ ,\ \phi(s) = \phi_\alpha(s) := |s|^{\alpha-1}s \ \text{ and } \ \Phi(S) = \Phi_\alpha(u) := |S|^{\alpha-1}S, \quad . \\ (1) \quad \text{The coefficients } \ a,\ A,\ c, \text{ and } \ C \ \text{ are contionuous and positive .} \\ (2) \quad \text{The perturbation functions } \ f \ \text{ and } \ F \ \text{ are continuous and satisfy :} \\ (i) \quad yf(t,y,y') \ \text{ and } \ uF(x,u,\nabla u) \ \text{ are strictly positive whenever } \ y|u| \neq 0 \ \text{ or} \\ (ii) \quad \forall(t,y,y'),\ (\forall(x,u,\nabla u)) \quad f(t,y,y')\ (\,resp.\ F(x,u,\nabla u)\,) \text{ keeps the same sign.} \end{cases}$$

When the operator contains perturbation terms $f(t,y,y')$ which is not a damping term (e.g. a multiple of $\phi(y')$ or that of $\Phi(\nabla u)$) , the following condition holds:

$$(H2) \qquad \lim_{|y|\to 0}\frac{f(t,y,y')}{\phi(y)} = 0 \quad (\text{ or } \lim_{|u|\to 0}\frac{F(x,u,\nabla u)}{\phi(u)} = 0\ ).$$

Here are some properties of the functions ϕ :

$$\begin{cases} s\phi_\alpha(s) = |s|^{\alpha+1}; \quad s\phi'_\alpha(s) = \alpha\phi_\alpha(s) \quad and \quad \phi_\alpha(st) = \phi_\alpha(s)\phi_\alpha(t); \\ S\Phi_\alpha(S) = |S|^{\alpha+1} \quad and \quad \Phi_\alpha(TS) = \Phi_\alpha(T)\Phi_\alpha(S). \end{cases}$$

The aims of this work is use some Picone-type formulas to establish oscillation criteria and some estimates of the solutions of the equations in (P) and *Q(above.

Definition 1.1. *We recall that a function $g \in C(\Omega);\ \Omega \subset E^n$ being an unbounded domaine*

a) is said to be (weakly) oscillatory in Ω if $\forall R > 0$, g has a zero inside $\Omega_R := \{x \in \Omega \mid |x| > R\}$.

b) g is said to be strongly oscillatory in Ω if it has a nodal set in any Ω_R; a nodal set of g , $D(g)$, being any

$D = D(g) := \{x \in \Omega \mid g|_{\partial D} = 0$ *and* $g(x) \neq 0$ *inside* $D\}$.

2000 *Mathematics Subject Classification.* 35B05, 35B50, 35J70.
Key words and phrases. p-Laplacian; Picone's identity; comparison methods.

c) An equation will be said to be oscillatory if any of its nontrivial solution is oscillatory.

Remark 1.2. .

1) The leading part of the equations here is their half linear part i.e.

$\nabla \cdot \Big(A(x)\Phi(\nabla u)\Big) + C(x)\phi(u)$ *or* $\Big(a(t)\phi(y')\Big)' + c(t)\phi(y)$

which is even in the sense that whenever u *(or* y *) is a solution, so is* $-u$ *(or* $-y$ *).*

2) It is to be noticed that the regularity hypotheses on the coefficients make any weakly oscillatory solution a strongly oscillationary one.

The condition (H2) implies that a regular solution cannot be compactly supported. (see [9] *)*

3) A solution of (Eq) will be said to be nonoscillatory if for some $R > 0$ *it keeps the same sign in* Ω_R*. The Picone-type formula we use in the proof requires the quantity* $\dfrac{f(t,y,y')}{\phi(y)}$ *(or* $\dfrac{F(x,u,\nabla u)}{\phi(u)}$ *) to remain positive. That explains (H1)2)(i) But (H1)2)(ii) could be used if the sign of the perturbation function does not depend on that of the unknown fuction.*

As when there are perturbation terms, the hypothesis of the perturbation term to keep the same sign could impose the use of $D(u^+)$ *or* $D(u^-)$ *for the need of having the sign of* $\dfrac{f(t,u.u')}{\phi(u)}$.

A Picone-type formula is a functional equation of two functions, $K(u,v)$ which holds in a domain only if one of the function here u, say, has no zero inside that domain. So given an equation $P(u) = 0,\ x \in \Omega$ we build an equation $Q(v) = 0$ with chosen coefficients related to those of $P(u)$ such that

1) v is oscillatory in Ω;

2) a 2-form $K(u,v) := \nabla[vK_1(u,v)] - K_2(u,v) = 0$ which makes sense only wherever $u \neq 0$ is built such that $K_2(S,w) \neq 0$ and keeps the same sign inside say, a $D(v) \subset \Omega_R$ for some $R > 0$. Because v has a nodal set in any such a Ω_R, $R > 0$, the integration of the 2-form over any such a $D(v)$ leads to an absurdity. This implying that u cannot remain non zero in any Ω_R.

2. Picone-type formulas

.

Given two positive functions $P,\ Q \in C(\mathbb{R})$, to extend the comparison theorem of C. Sturm for solutions $u,\ v$ of the Sturm-Liouville

$$-\Big(P(x)u'\Big)' + p_0(x)u = 0 \qquad \text{and} \qquad -\Big(Q(x)v'\Big)' + q_0(x)v = 0, \qquad x \in \mathbb{R},$$

M.Picone used the fact that if $u,\ v,\ Pu'$ and Qv' are differentiable then wherever $v \neq 0$

$$\begin{aligned}&\frac{d}{dx}\left[\frac{u}{v}\left\{vPu' - uQv'\right\}\right]\\ &= u\Big(Pu'\Big)' - \frac{u^2}{v}\Big(Qv'\Big)' = (P-Q)|u'|^2 + Q\Big(u' - \frac{u}{v}v'\Big)^2\\ &= (P-Q)|u'|^2 + Q\Big(u' - \frac{uv'}{v}\Big)^2\end{aligned}$$

(see [10]).

This implies that if $P \geq Q$ in I say, and there is a nodal set $D(u) \subset I$, v cannot be non zero all over I; otherwise the integration over $D(u)$ of the formula above would lead to an absurdity.

Since then such type of formula bears Picone's name.

Consider for $i = 1,\ 2$ the equations

$$\begin{cases} P_i(y) := \Big\{a_i(t)\phi(y_i')\Big\}' + c_i(t)\phi(y_i) + f_i(t, y_i, y_i') = 0, \quad t \in \mathbb{R}^+ \quad \text{and} \\ Q_i(u_i) := \quad \nabla\cdot\Big\{A_i(x)\Phi(\nabla u_i)\Big\} + C_i(x)\phi(u_i) + F_i(x, u_i, \nabla u_i) = 0, \quad x \in \mathbb{R}^n. \end{cases}$$

Let y_1 and y_2 be respectively solutions of $P_i(y_i),\ i = 1, 2$. Then wherever y_2 is non zero, a version of Picone's identity reads (through easy calculations)

$$\textbf{\textit{(P)}}\quad \begin{cases} \Big\{y_1 a_1(t)\phi(y_1') - y_1\phi(\dfrac{y_1}{y_2})a_2(t)\phi(y_2')\Big\}' = \\ = a_2(t)\zeta_\alpha(y_1, y_2) + \Big[a_1(t) - a_2(t)\Big]|y_1'|^{\alpha+1} + \Big[c_2(t) - c_1(t)\Big]|y_1|^{\alpha+1} \\ + |y_1|^{\alpha+1}\Big[\dfrac{f_2(t, y_2, y_2')}{\phi(y_2)} - \dfrac{f_1(t, y_1, y_1')}{\phi(y_1)}\Big] \end{cases}$$

where, $\forall\gamma > 0$, the two-form function ζ_γ is defined $\forall u, v \in C^1(\mathbb{R},\ \mathbb{R})$ by

$$(\mathbf{Z1}):\qquad \zeta_\gamma(u, v)\begin{cases} = |u'|^{\gamma+1} - (\gamma+1)u'\phi_\gamma(\dfrac{u}{v}v') + \gamma v'\dfrac{u}{v}\phi_\gamma(\dfrac{u}{v}v') \\ = |u'|^{\gamma+1} - (\gamma+1)u'\phi_\gamma(\dfrac{u}{v}v') + \gamma|\dfrac{u}{v}v'|^{\gamma+1} \end{cases}$$

is strictly positive for non null $u \neq v$ and null only if $u = \lambda v$ for some $\lambda \in \mathbb{R}$. Similarly, if u_1 and u_2 are respectively solutions of $Qu_i,\ i = 1, 2$, then wherever u_2

stays non zero, a version of Picone's identity reads

$$
(\boldsymbol{Q}) \quad \begin{cases} \nabla \cdot \left\{ u_1 A_1(x)\Phi(\nabla u_1) - u_1\phi(\frac{u_1}{u_2})A_2(x)\Phi(\nabla u_2) \right\} = A_2(x)Z_\alpha(u_1,u_2) \\ + \Big(A_1(x) - A_2(x)\Big)|\nabla u_1|^{\alpha+1} + \Big(C_2(x) - C_1(x)\Big)|u_1|^{\alpha+1} \\ + |u_1|^{\alpha+1}\left[\frac{F_2(x,u_2,\nabla u_2)}{\phi(u_2)} - \frac{F_1(x,u_1,\nabla u_1)}{\phi(u_1)}\right] \\ \text{where} \quad \forall \gamma > 0, \quad \forall u,v \in C^1(\mathbb{R}^n) \\ (\boldsymbol{Z2}): \quad Z_\gamma(u,v) := |\nabla u|^{\gamma+1} - (\gamma+1)\Phi_\gamma(\frac{u}{v}\nabla v)\cdot\nabla u + \gamma|\frac{u}{v}\nabla v|^{\gamma+1} \\ = |\nabla u|^{\gamma+1} - (\gamma+1)|\frac{u}{v}\nabla v|^{\gamma-1}\,\frac{u}{v}\nabla v\cdot\nabla u + \gamma|\frac{u}{v}\nabla v|^{\gamma+1}. \end{cases}
$$

We recall that $\forall \gamma > 0$ the two-form $Z_\gamma(u,v) \geq 0$ and is null only if $\exists k \in \mathbb{R};\ u = kv$. (see e.g. [1], [2] , [8]).

2.1. Direct application for one-dimensional equations. Consider in $\mathbb{R}^+$ the problems

$$
(E1) \begin{cases} (a) \quad \Big(a(t)\phi(y')\Big)' + c(t)\phi(y) + f(t,y,y') = 0; \quad t > 0 \\ (b) \quad \text{and its associate } \Big(a_m\phi(z')\Big)' + c_m\phi(z) = 0 \end{cases} \tag{2.1}
$$

where the hypotheses (H1) and (H2) hold. Assume that the problem (a) is not oscillatory (i.e. it has a regular solution having no zero inside $I_R := (R,\ \infty)$ for some $R > 0$. Define the positive constants $a_m,\ c_m$ as
$\forall m \in \mathbb{N}, \quad I_m := (R,\ mR).$, $a_m := \sup_{I_m}\{a(t)\}$ and $c_m := \inf_{I_m}\{c(t)\}$.
The equations with positive constant coefficients above are strongly oscillatory. In fact
$\Big(a_m\phi(z')\Big)' + c_m\phi(z) = a_m\Big[(\phi(z')' + q_m\phi(z)\Big] = 0; \quad q_m := \frac{c_m}{a_m}$ and
the solution of $(\phi(z')' + q_m\phi(z) = 0 \quad t > 0$ is "q_m−periodic "
We recall that the solution S_α of the problem

$$
\begin{cases} \Big[\phi_\alpha(u')\Big]' + \alpha\phi_\alpha(u) = 0; \qquad u(0) = 0;\ u'((0) = 1 \\ \text{has } D_\alpha; = [0,\ \pi_\alpha] \text{ as a nodal set and satisfies for } p = \alpha + 1 \\ \forall t > 0, \quad |S_\alpha(t)|^p + |S'_\alpha(t)|^p = 1; \quad S_\alpha(\pi_\alpha) = 0; \\ S_\alpha(t + \pi_\alpha) = -s_\alpha(t) \text{ where} \\ \pi_\alpha = 2\int_0^1 \frac{1}{(1-t^p)^1/p}dt = \frac{2\pi}{p\sin(\pi/p)}. \end{cases} \tag{2.2}
$$

(see e.g. [11])
If we set $\Sigma(t) := S_\alpha(\lambda t), \quad \lambda > 0$, then as from [5]

$$\begin{cases} \left(\phi(\Sigma'(t))\right)' + \lambda^{\alpha+1}\alpha\phi(\Sigma) = 0 \quad \text{and} \\ \Sigma(t + \frac{\pi_\alpha}{\lambda}) = -\Sigma(t). \end{cases} \tag{2.3}$$

Moreover for constant $a,\ c > 0,$ for the equation

$$\left(a\phi(u')\right)' + \alpha c\phi(u) = 0 \quad \text{or} \quad \left(\phi(u')\right)' + \alpha\frac{c}{a}\phi(u) = 0 \tag{2.4}$$

whence for this equation $\lambda := \left\{\frac{c}{a}\right\}^{1/(1+\alpha)}$ in the formula above. Thus if $D(u) := [T_1,\ T_2]$ denotes a nodal set of the equation,

$$diam[D(u)] = (T_2 - T_1) = \frac{\pi_\alpha}{\lambda} = \pi_\alpha\left\{\frac{a}{c}\right\}^{1/(1+\alpha)}. \tag{2.5}$$

Therefor if we assume that a regular solution of (E1)(a) is say , strictly positive in some I_R. We take m big enough for z to have several nodal sets in I_m . The Picone formula we choose in any $D(z) \subset I_m$ is

$$\begin{cases} \left\{za_m\phi(z') - z\phi(\frac{z}{y})a_m\phi(z')\right\}' = \\ = a_m\zeta_\alpha(z,y) + \left[a_m - a(t)\right]|z|^{\alpha+1} + \left[c(t) - c_m\right]|z|^{\alpha+1} \\ + |z|^{\alpha+1}\left[\frac{f(t,y,y')}{\phi(y)}\right] \end{cases} \tag{2.6}$$

The integration of the formula over any such $D(z)$

$$0 = \int_{D(z)} \left\{a_m\zeta_\alpha(z,y) + \left[a_m - a(t)\right]|z|^{\alpha+1} + \left[c(t) - c_m\right]|z|^{\alpha+1} + |z|^{\alpha+1}\left[\frac{f(t,y,y')}{\phi(y)}\right]\right\}dt$$

which is absurd as the integrand in the right side is strictly positive.

We then have the following

Theorem 2.1. *$\forall a, c \in C(\mathbb{R}^+)$ and $f \in C(\mathbb{R}^3)$ with $sf(t,s,s') \geq 0 \quad \forall s \in \mathbb{R}$, any nontrivial solution y of*

$$\left(a(t)\phi(y')\right)' + c(t)\phi(y) + f(t,y,y') = 0,\ t > 0$$

is strongly oscillatory.

.

Consider now in $\mathbb{R}^+$ the equations

$$\begin{aligned} &\left(a(t)\phi(y')\right)' + c(t)\phi(y) + f(t,y,y') = 0 \\ \text{and} \quad &\left(a(t)\phi(z')\right)' + C(t)\phi(z) = 0 \end{aligned} \tag{2.7}$$

where all the coefficients are strictly positive and continuous and f as in (H1)-(H2). We know that any non-trivial solutions y and z of those equations is strongly oscillatory .Let the set I be such that two nodal sets $D(z)$ and $D(y)$ overlap (providing that $c \geq C$) in I. From the following Picone formula

$$\left[\begin{array}{l} \Gamma(z,y) := \left\{ za\phi(z') - z\phi(\frac{z}{y})a\phi(z')\right\}' = \\ = \zeta_\alpha(z,y) + \left[c - C\right]|z|^{\alpha+1} + |z|^{\alpha+1}\left[\frac{f(t,y,y')}{\phi(y)}\right] \end{array}\right. \tag{2.8}$$

and the integration of the formula over $D(z)$ shows that y has to have a zero inside $D(z)$.

Theorem 2.2. *By a suitable translation $t \mapsto t + \xi, \quad \xi \in \mathbb{R}$ can be made such that the function $Y(t) := y(t+\xi)$ has the same singularity as z*
i.e. $Y'(t_0) = z'(t_0) = 0; \quad t_0 \in D(z)$ being the singulariry of z there.In this case,

$$D(Y) \subset D(z) \;\; whence \;\; diam(D(Y)) = dim(D(y)) \leq diam(D(z)). \tag{2.9}$$

Moreover if $\alpha \geq 1$ then

$$\max_{D(y)} \{y(t)\} = \max_{D(Y)} \{Y(t)\} \leq \max_{D(z)} \{z(t)\}. \tag{2.10}$$

Proof. Let $D(z) = [s_0, \; s_1]$. Without lost of generality assume that $s_0 < t_0 < t_1 < s_1$ where $Y'(t_0) = z'(t_0) = 0$ and $Y(t_1) = 0$. Because the integrand of the left hand side of the Picone formula above is zero at s_0 and at t_0, the integration over $(s_0, \; t_0)$, we get

$0 = \int_{s_0}^{t_0} \Gamma(z,y)dt = \int_{s_0}^{t_0} \{\zeta_\alpha(z,y) + \left[c - C\right]|z|^{\alpha+1} + |z|^{\alpha+1}\left[\frac{f(t,y,y')}{\phi(y)}\right]\}dt$

which is absurd, implying that Y has to have a zero inside $[s_0, \; t_0]$. as well i.e. $\exists t' \in (s_0, \; t_0)$ such that $D(Y) = [t', \; t_1]$.

The comparison of the maxima of the nodal sets follows from the fact that when $\alpha \geq 1$ then ϕ_α is monotonic increasing in the sense that

$\forall t, s \in \mathbb{R}, \qquad (t-s)(\phi(t) - \phi(s)) \geq 0.$

Assume that $J := \{t \in D(z) \mid Y(t) > z(t) \}$ has a non zero measure. Let $\tau > 0$ be chosen such that

for some $I := I_\tau \subset J, \quad w = z - Y + \tau > 0$ in I and $w|_{\partial I} = 0$. Then from the equations, because $w|_{\partial I} = 0$,

$\int_I w\left[a\phi(z') - a\phi(Y')\right]' dt = -\int_I w'(\,a\phi(z') - a\phi(Y')\,]dt$

$= \int_I (Y' - z')[a\phi(z') - a\phi(Y')]dt < 0.$

But $\quad w\left[a\phi(z') - a\phi(Y')\right]' = w\{-C\phi(z) + c\phi(Y) + f(t,Y,Y')\} > 0$ in I; thus

$\int_I w\left[a\phi(z') - a\phi(Y')\right]' dt > 0$, contradicting what we got above. Therefor J is empty and $Y \leq z$ in $D(z)$.

□

Consider for some constant $A, C > 0$ the problem

$$A\Big(\phi(z')\Big)' + C\alpha\phi(z) = 0, \quad t > 0; \quad z(0) = 0, \quad z'(0) = 1. \qquad (2.11)$$

With $\lambda := \alpha\dfrac{C}{A}$ simple calculations show that for a solution z

$$\forall t > 0, \qquad z'\Big(\phi(z')\Big)' + \lambda z'\phi(z) = \Big\{|z'|^{\alpha+1} + \lambda|z|^{\alpha+1}\Big\}' = 0 \qquad (2.12)$$

and from the initial conditions above, with $D(z) = D(z^+) := [0,\ \pi_\lambda]$ being the first nodal set of z,

$$\begin{cases} \max\limits_{D(z)} z(t) = \lambda^{-1/(1+\alpha)} = \left(\dfrac{A}{\alpha C}\right)^{1+\alpha} = z(\dfrac{\pi_\lambda}{2}); \\ \text{where } \pi_\lambda := \dfrac{\pi_\alpha}{\lambda}. \end{cases} \qquad (2.13)$$

This last paragraph is summarised in

Theorem 2.3. *Let* $a,\ c \in C(\ [0\ ,\ (0,\ +\infty)\ )$. *Then*

If $\exists q_0 > 0$ *such that* $\forall t > 0 \quad \dfrac{a(t)}{\alpha c(t)} < q_0$, *any nontrivial solution of*

$$\Big(a(t)\phi(u')\Big)' + \alpha c(t)\phi(u) = 0, \quad t > 0; \quad u(0) = 0;\ u'(0) = 1$$

is strongly oscillatory and bounded .

Moreover for any nodal set $D(u) \subset [T_0,\ T_1] := J$, *with* $A := \sup_J\{a(t)\}$ *and* $C := \inf_J\{c(t)\}$,

$$\begin{aligned} &diam\ \{D(u)\}\ \le \pi_\alpha\left[\frac{A}{\alpha C}\right]^{1/(\alpha+1)} \\ &and\ if\ \ \alpha \ge 1\ ,\quad \max_{D(u)}\{|u|\} \le \left[\frac{A}{\alpha C}\right]^{1/(\alpha+1)}. \end{aligned} \qquad (2.14)$$

2.2. **Multi dimensional cases.** .

In multidimensional case , an axis of the nodal sets is any straight line going through the successive zeros of the nodal sets $...x_k,\ x_{k+1},...$ and the diameter of a nodal set is the distance between its two zeros longside that line. For the equation

$$(Qm)\begin{cases} \nabla\cdot\Big\{A(x)\Phi(\nabla u)\Big\} + C(x)\phi(u) + F(x,u,\nabla u) = 0, \quad x \in E^3 \\ \text{we associate} \quad \nabla\cdot\Big\{A_1\Phi(\nabla v)\Big\} + C_1\phi(v) = 0 \\ \text{where } A,\ C,\ A_1,\ C_1 > 0, \quad \text{and } F \text{ fulfills the required} \\ \text{conditions in (H1)and (H2).} \end{cases} \qquad (2.15)$$

We take A_1 and C_1 constant. The corresponding Picone formula reads

$$\begin{cases} \nabla \cdot \left\{ vA(x)\Phi(\nabla v) - v\phi(\frac{v}{u})A_1\Phi(\nabla u) \right\} = AZ_\alpha(v,u) \\ + \left(A_1 - A\right)|\nabla v|^{\alpha+1} + \left(C - C_1\right)|v|^{\alpha+1} + |v|^{\alpha+1}\frac{F(x,u,\nabla u)}{\phi(u)} \end{cases} \tag{2.16}$$

Theorem 2.4. *If $A,\ C \in C(E^3)$ are strictly positive with bounded $\frac{A(x)}{C(x)}$, any nontrivial solution of*

$$\nabla \cdot \left\{ A(x)\Phi(\nabla u) \right\} + C(x)\phi(u) + F(x,u,\nabla u) = 0, \quad x \in E^3 \tag{2.17}$$

is strongly oscillatory and bounded. Moreover, as in one-dimensional case, if a $D(u)$ overlaps with a $D(v)$, then if $\alpha \geq 1$,

$$diam\{D(u)\} \leq diam\{D(v)\} \quad and \quad \max_{D(u)\bigcup D(v)} \left(|v| - |u|\right) \geq 0. \tag{2.18}$$

Proof. Assume that we have such a solution u such that say, $u > 0$ in some Ω_R. We set $\forall n \in \mathbb{N}$, $A_n := \sup_{I_n} \{A(x)\}$ and $C_n := \inf_{I_n} \{C(x)\}$ with $I_n := \{x \in \Omega_R$ such that $|x| \in [R,\ nR]$. Because $\frac{A(x)}{C(x)} < q$ for some $q > 0$, $\frac{A_n}{C_n}$ is also bounded. We take v_n an oscillatory solution of

$$\nabla \cdot \left\{ A_n\Phi(\nabla v) \right\} + C_n\phi(v) = 0; \quad v(0) = 0;\ |\nabla v|(0) = 1.$$

Replacing A_1 and C_1 respectively by A_n and C_n in the Picone formula above. From the Picone formula above, as before we get that u is oscillatory since u ; cannot remain strictly positive in any $D(v_n^+)$. The proof will be completed by the next Theorem.

□

The next Theorem concerns with **monotonic increasing operator** : the operator χ is said to be monotonic increasing in K, say if
$\forall S_1, \neq S_2 \in K, \qquad (S_1 - S_2)[\chi(S_1) - \chi(S_2)] > 0.$

Lemma Mc

If $h \in C^1(\mathbb{R}^1,\ \mathbb{R}^+)$ is increasing then the function $t \mapsto h(t)t$ is monotonic increasing in $\mathbb{R}$ and $\xi \mapsto h(|\xi|)\xi$ is monotonic increasing on $\mathbb{E}^n$.

This applies to the functions ϕ_α and Φ_α when $\alpha \geq 1$.

Proof. 1) Let $t, s \in \mathbb{R}$.

a) $(t-s)\left[h(|t|)t - h(|s|)s\right] = t^2h(|t|) + s^2h(|s|) - ts\left(h(|t|) + h(|s|)\right)$
which is positive if $ts \leq 0$.

b) $(t-s)\left[h(|t|)t - h(|s|)s\right] = (t-s)\left(t[h(|t|) - h(|s|)] + (t-s)h(|s|)\right)$
$= (t-s)^2h(|s|) + (t-s)t\left[h(|t|) - h(|s|)\right]$ which is positive if $t < s \leq 0$ or if $t > s \geq 0$.

2) For $K = E^n$, Let $\xi,\ \eta \in E^n$. Easy calculations show that

c) $H(\xi,\eta) := (\xi-\eta)\Big(h(|\xi|)\xi - h(|\eta|)\eta\Big)$

$= (\xi-\eta)\Big(\xi[h(|\xi|)-h(|\eta|)] + h(|\eta|)(\xi-\eta)\Big) = (\xi-\eta)^2 h(|\eta|)$

$+ \xi(\xi-\eta)\Big(h(|\xi|)-h(|\eta|)\Big)$ and this is positive if $|\xi| > |\eta|$.

d) $H(\xi,\eta) := (\xi-\eta)\Big(h(|\xi|)\xi - h(|\eta|)\eta\Big)$

$= (\xi-\eta)\Big[h(|\xi|)(\xi-\eta) + \eta\Big(h(|\xi|)-h(|\eta|)\Big)\Big] = (\xi-\eta)^2 h(|\xi|) +$

$(\xi-\eta)\eta\Big[h(|\xi|)-h(|\eta|)\Big]$ which is positive if $|\eta| > |\xi|$.

□

Theorem 2.5. *(A comparison result)*
Let u and v be oscillatory solution of (Qm) where $A_1 \equiv A(x)$ and $C(x) \geq C_1$ in some $\Omega \subset \Omega_R$. If there are two overlapping nodal sets $D(u)$ and $D(v)$ in Ω such that for some $x_0 \in D(u)\bigcap D(v)$ $\nabla u(x_0) = \nabla v(x_0) = 0$ then

$$D(u) \subset D(v) \quad \textit{and} \quad \max_{D(u)} |u(x)| \leq \max_{D(v)} |v(x)|.$$

Proof. Assume that $measure(U^+) := measure\{x \in D(u) \mid (u-v)(x) > 0\} > 0$.
Then we can find $\tau > 0$ such that $W(x) := (v-u)(x) + \tau > 0$ in some non negligible D_τ with $W|_{\partial D_\tau} = 0$. In that case, taking $A(x) = A_1 = A$,

$$\left| \begin{aligned} 0 &= \int_{D_\tau} W(x)\nabla\cdot\Big[A\Phi(\nabla v) - A\Phi(\nabla u)\Big]dx \\ &= -\int_{D_\tau} \nabla W\Big(A\Phi(\nabla v) - A\Phi(\nabla u)\Big)dx \qquad (i) \\ &= \int_{D_\tau} W(x)\Big[C(x)\phi(u) + F - C_1\phi(v)\Big]dx. \qquad (ii) \end{aligned} \right. \tag{2.19}$$

The (i) and (ii) above give

$-\int_{D_\tau} A\Big(\nabla v - \nabla u\Big)\Big[\Phi(\nabla v) - \Phi(\nabla u)\Big]dx$

$= \int_{D_\tau}(v-u+\tau)\Big(C\phi(u) - C_1\phi(v) + F\Big)dx$ which is absurd

because the first line is negative while the next line is positive; we cannot have such a $\tau > 0$. Therefore $D(u) \subset D(v)$.

□

In the one dimensional case, the investigation of the diameters and amplitudes had been facilitated because various elements like nodal sets were just closed intervals. Here for example the set $\{\, x \in \Omega \mid \nabla u(x) = 0 \,\}$ is not obviously visionalized. The way we chose to get around is to use as associate equation one whose solutions are radially symmetric. For some positive constant a and c wer look for such an equation from

$$\nabla\cdot\Big(a\Phi(\nabla u)\Big) + c\phi(u) = 0; \quad x \in E^n; \quad u(0) = 0;\ |\nabla u|(0) = 1. \tag{2.20}$$

We note here that when $\alpha \geq 1$, if a, c depend only on $r := |X|$, the solutions of this equation are radilly symmetric. (see e.g. [7]).
If $h \in C^1(E^n)$ is a radially symmetric function, i.e. $h(x) \equiv H(r)$, where
$r := |x| = \{\sum_1^n x_i^2\}^{1/2}$ and $X = [x_1, x_2, ...x_n]$ denoting the position vector ;
$\partial_{x_i} H(r) = \dfrac{x_i}{r} H'(r)$ and $\nabla H(r) = \dfrac{X}{r} H'(r)$.
Also if we have a vector function $\Psi \in C^1(E^n,\ E^n)$ of the form
$\psi(W) := Wg(|W|)$ with $g \in C^1(\mathbb{R}^+)$ then
if $u(x) := V(r)$, $\nabla u = \dfrac{X}{r} V'(r)$;

$$\begin{cases} \Psi(\nabla u) = \frac{X}{r} V' g(|V'|) \quad \text{and} \\ \nabla \cdot \Psi(\nabla u) = \nabla \cdot \left(\frac{X}{r} V'(|V'|) \right) = \frac{1}{r^{n-1}} \left\{ r^{n-1} \, V' g(|V'|) \right\}' \\ = \frac{n-1}{r} \, V' g(|V'|) + \frac{X}{r} \nabla \left[V' g(|V'|) \right]. \text{ So} \\ \nabla \cdot \left(a\Phi(\nabla u) \right) + c\phi(u) = 0; \quad x \in E^n; \quad u(0) = 0; \; |\nabla u|(0) = 1 \text{ is} \\ \left(a r^{n-1} \phi(V') \right)' + r^{n-1} c\phi(V) = 0, \quad t > 0; \quad V(0) = 0, \; V'(0) = 1. \end{cases} \tag{2.21}$$

We note that the equation

$\left(a r^{n-1} \phi(V') \right)' + r^{n-1} c\phi(V) = 0$ and

$$\left(a\phi(V') \right)' + c\phi(V) + \frac{n-1}{r} \phi(V') = 0 \tag{2.22}$$

are equivalent as the later is just a development of the former.

Moreover as we saw for the one dimension cases, in terms of investigating the diameter and the amplitudes of their solutions, the two above equations are equivalent. This is due to the fact that the damping term $\frac{n-1}{r}\phi(V')$ does not alter the results.

In this study, $\left(a\phi(V') \right)' + c\phi(V) = 0$ and $\left(a r^{n-1} \phi(V') \right)' + r^{n-1} c\phi(V) = 0$ are equivalent because the diameters and amplitudes depend on the same quantity $\frac{a}{c}$ or $\frac{ar^{n-1}}{cr^{n-1}}$. This is why we can use as associate equation the later equation without the r^{n-1} factor.

Let I_m be an intervall (possibly large) and define $A_m := \max_{I_m}[a(x)]$ and $C_m = \min_{I_m}[c(x)]$.

Let V_m be a nontrivial solution of

$$\left(A_m \phi(V') \right)' + C_m \phi(V) = 0 \quad t > o\,; \; V(0) = 0; \; V'(0) = 1 \tag{2.23}$$

such that $D(V_m) \subset I_m$.

We saw in the Theorem 2.3 that setting $A := A_m$ and $C := C_m$

$$diam \; \{D(v_m)\} \leq \pi_\alpha \left(\frac{A}{\alpha C} \right)^{1/(\alpha+1)} \text{ and if } \alpha \geq 1 \; \max_{D(V_m)} \{|V_m|\} \leq \left(\frac{A}{\alpha C} \right)^{1/(\alpha+1)}.$$

We then have the following

Theorem 2.6. *Let $A,\ C \in C(E^n)$ be positive functions and $\alpha \geq 1$. Then if $q(x) := \frac{a(x)}{c(x)}$ is bounded above. Then any nontrivial solution of*

$$\nabla \cdot \left\{ A(x)\Phi(\nabla u) \right\} + C(x)\phi(u) = 0; \quad x \in E^n \tag{2.24}$$

is strongly oscillatory and bounded. The same conclusions hold if a continuous perturbation term $F(x, u, \nabla u)$ is added to the equation provided that

$\forall S \in \mathbb{R} \quad \frac{F(x,s,\nabla u)}{\phi(s)} \geq 0$ or $F(x, s, \nabla u)$ does not change sign

and $|\frac{F(x,s,\nabla u)}{\phi(s)}| \searrow 0$ as $|s| \to 0$.

Moreover for such a solution U, say, define for a given nodal set $D := D(U)$, $A := \max_D \; [A(x)]$; and $\alpha C := \min_D \; [C(x)]$; then

$$diam\{D(V)\} \leq \pi_\alpha \left[\frac{A}{\alpha C} \right]^{1/(\alpha+1)} \quad \text{and} \quad \max_{D(U)} \{|U(x)|\} \leq \left[\frac{A}{\alpha C} \right]^{1/(\alpha+1)}. \tag{2.25}$$

3. Equations with damping terms

3.1. One-dimensional cases. .
We consider here equations of the type

$$\begin{cases} \Big(a(t)\phi(z')\Big)' + \alpha c(t)\phi(z) + B(t)\phi(z') + f(t,z,z') = 0; \quad t > 0; \\ \text{where} \quad \exists b \in C^1(\mathbb{R}); \quad b'(t) = B(t),\ \forall t > 0\ ; \\ \exists q > 0\ ; \quad \dfrac{a(t)}{c(t)} \leq q \text{ and } f \text{ is nonegative.} \end{cases} \tag{3.1}$$

We are going to build a Picone formula from this equation and the associate problem

$$\Big(a(t)\phi(y')\Big)' + \alpha c(t)\phi(y) = 0, \quad t > 0; \quad y(0) = 0,\ y'(0) = 1 \tag{3.2}$$

which is known to be oscillatory. Easy computations show that

1) $\Big(ay\phi(y')\Big)' = a|y'|^{\alpha+1} - c(t)\alpha|y|^{\alpha+1}$,

2) $\Big(ay\phi(\ \dfrac{y}{z}z'\ )\Big)' = a(1+\alpha)y'\phi(\ \dfrac{y}{z}z'\ ) - c\alpha|y|^{\alpha+1)}$
$-By\phi(\ \dfrac{y}{z}z'\ ) - y\phi(\dfrac{y}{z})f - a\alpha|\ \dfrac{y}{z}z'\ |^{1+\alpha}$ and

3) $\Big(by\phi(\ \dfrac{y}{z}z'\ )\Big)' = By\phi(\ \dfrac{y}{z}z'\ ) + b\Big(y\phi(\dfrac{y}{z}z')\Big)'$.

Then for the two equations above we set the Picone-type formula as follow:

$$\begin{cases} \Big[ay\phi(y') - (ay\phi(\ \dfrac{y}{z}z'\ ) - \Big(by\phi(\ \dfrac{y}{z}z'\ )\Big)\Big]' = \\ a\zeta_\alpha(y,z) + |y|^{\alpha+1}\dfrac{f}{\phi(z)} + b\Big\{\phi(\ \dfrac{y}{z}z'\ )\Big\}'. \end{cases} \tag{3.3}$$

We then have the

Theorem 3.1. *Given some positive* $a, c \in C(\mathbb{R}^+)$ *and* $b \in C^1(\mathbb{R})$, *any nontrivial regular solution of*

$$\Big(a(t)\phi(z')\Big)' + \alpha c(t)\phi(z) + B(t)\phi(z') + f(t,z,z') = 0; \quad t > 0 \tag{3.4}$$

where $B(t) := b'(t)$ *is strongly oscillatory. Moreover for such solutions,*

$$\begin{aligned} &maesure\ \{D(y)\bigcap D(z)\} > 0 \implies diam(D(z)) \leq diam(D(y)) \qquad and \\ &in\ addition\ if\ \ \alpha \geq 1 \qquad \max_{D(z)}|z(t)| \leq \max_{D(y)}\ |D(y)|. \end{aligned} \tag{3.5}$$

Proof. Assume that there is such a solution z which is strictly positive in $I_R := (R,\ \ +\infty)$. Assume that such a nontrivial solution z is strictly positive in say, $I_R := [R,\ +\infty)$. Obviously there are lots of nodal sets of y, an oscillatory solution of the associate problem. From the fact that we can substitute in the equation $b(t)$ by $b(t) + m$, for any arbitrary $m \in \mathbb{R}$ and the equation would be the same. In this case the Picone formula becomes

$$\begin{cases} \forall m \in \mathbb{R}, \qquad \left[ay\phi(y') - (ay\phi(\,\frac{y}{z}z'\,) - \Big((b+m)y\phi(\,\frac{y}{z}z'\,)\Big)\right]' \\ = a\zeta_\alpha(y,z) + |y|^{\alpha+1}\frac{f}{\phi(z)} + (b+m)\Big\{\phi(\,\frac{y}{z}z'\,)\Big\}' \\ = a\zeta_\alpha(y,z) + |y|^{\alpha+1}\frac{f}{\phi(z)} + b\Big\{\phi(\,\frac{y}{z}z'\,)\Big\}' + m\Big\{\phi(\,\frac{y}{z}z'\,)\Big\}'. \end{cases} \tag{3.6}$$

Therefor $D(y) \subset I_R$ being any nodal set of y, the integration over $D(y)$ of the formula gives

$$\forall m \in \mathbb{R}, \quad 0 = \int_{D(y)} \{a\zeta_\alpha(y,z) + |y|^{\alpha+1}\frac{f}{\phi(z)} + b\Big\{\phi(\,\frac{y}{z}z'\,)\Big\}'\}dt + \int_{D(y)} m\Big\{\phi(\,\frac{y}{z}z'\,)\Big\}' dt.$$

This can hold only if each integrand is zero. But if $\Big\{\phi(\,\frac{y}{z}z'\,)\Big\}' \equiv 0$ then the integrand at the left would not be zero. Thus z cannot be non zero in any $I_R, \quad R > 0$.

Estimates of the solutions.

For any $\mu > 0$, the function $u(t) := \mu z(t)$ solves

$$\begin{aligned} &\Big\{a\phi(u')\Big\}' + \alpha c\phi(u) + B(t)\phi(u') + F_\mu(t,u,u') = 0; \quad t > 0 \\ &\text{with } F_\mu(t,u,u') := \mu^\alpha f(t, \frac{u}{\mu}, \frac{u'}{\mu}). \end{aligned} \tag{3.7}$$

Because of the bounded $\dfrac{a(t)}{c(t)} < q \qquad \forall t > 0$, we choose μ such that the solution u above satisfies

$$\max_{t>0}\{|u(t)|,\ |u'(t)|\} \le 1 \text{ and let } H(t) := \sup_{(|s|+\xi|)\le 1} \{|B(t)\phi(S)|,\ |f(t,s,\xi)|\}. \tag{3.8}$$

It is clear that we can just use the Theorem 2.5 to get the estimates for the solution u above. We use for it the associate equation

$$\Big(a(t)\phi(z')\Big)' + \alpha c(t)\phi(z) + H(t) = 0,\ t > 0; \quad z(0) = 0,\ z'(0) = 1. \tag{3.9}$$

By using an usual Picone formula , taking as associate the equation without H we get easily that this problem above is strongly oscillatory with the same estimates as that with H.

Assume that two oscillatory solutions z and u respectively of

$\Big(a(t)\phi(z')\Big)' + \alpha c(t)\phi(z) + H(t) = 0,\ t > 0; \quad z(0) = 0,\ z'(0) = 1$ and

$\Big\{a\phi(u')\Big\}' + \alpha c\phi(u) + B(t)\phi(u') + F_\mu(t,u,u') = 0; \quad t > 0$ which satisfy

$$\begin{array}{ll}(s1) & \exists t_0 \in I(z,u) := D(z) \bigcap D(u) \quad | \quad z'(t_0) = u'(t_0) = 0; \\ (s2) & \exists \tau > 0 \text{ and } I_\tau \subset D(z) \bigcup D(u) \quad \text{such that } u(t) - z(t) < 0 \\ & \text{and } V(t) := u(t) - z(t) + \tau > 0 \quad \forall t \in I_\tau \text{ with } V|_{\partial I_\tau} = 0.\end{array} \tag{3.10}$$

Then $\int_{I_\tau} V(t)\Big(a\phi(u') - a\phi(z')\Big)' dt$

$= - \int_{I_\tau} V'(t)\Big[a\phi(u') - a\phi(z')\Big] dt \qquad$ (i)

$= \int_{I_\tau} V(t)\Big(\alpha c(t)\phi(z) + H(t) - \alpha c\phi(u) - B(t)\phi(u') - F_\mu(t,u,u')\Big) dt$

$= \int_{I_\tau} V(t)\Big(\alpha c(t)[\,\phi(z) - \phi(u)\,] + H(t) - [\,B(t)\phi(u') + F_\mu(t,u,u')\,]\Big) dt \qquad$ (ii)

From the definition of H and the fact that $V > 0$ in I_τ,

$- \int_{I_\tau} V'(t)\Big[a\phi(u') - a\phi(z')\Big] dt < 0$ while the quantity in (ii) is positive. Therefor

$$D(u) \subset D(z) \quad \text{and } u(t) \leq z(t) \quad \forall t \in D(z). \tag{3.11}$$

□

3.2. Damping terms equations in multi dimensional. .

We consider here in E^n the equations of the type

$$(E_d)\begin{cases} \nabla \cdot \Big\{A(x)\Phi(\nabla U)\Big\} + C(x)\phi(U) + B(x)\cdot\Phi(\nabla U) = 0. \\ \text{and} \quad \nabla \cdot \Big\{A(x)\Phi(\nabla V)\Big\} + C(x)\phi(V) = 0 \\ \text{where } \exists b \in C^1(E^n) \text{ such that } \nabla b(x) := B(x); \\ A \text{ and } C \text{ are as before.}\end{cases}$$

Easy calculations show the following:

1) $\nabla \cdot \Big(AV\Phi(\nabla V)\Big) = A|\nabla V|^{\alpha+1} - C|V|^{\alpha+1};$

2) $\nabla \cdot \Big\{AV\phi(\frac{V}{U})\Phi(\nabla U)\Big\} = A\nabla V\phi(\frac{V}{U})\Phi(\nabla U) +$

$\alpha A\Big\{\nabla V\phi(\frac{V}{U})\Phi(\nabla U) - \frac{V\phi(V)}{U\phi(U)}\nabla U\Phi(\nabla U)\Big\} - V\phi(\frac{V}{U})B(x)\cdot\Phi(\nabla U) - C(x)|V|^{\alpha+1}$

;

3) $\nabla\Big[b(x)V\phi(\frac{V}{U})\Phi(\nabla U)\Big] = V\phi(\frac{V}{U})B(x)\cdot\Phi(\nabla U) + b(x)\nabla\cdot\Big[V\phi(\frac{V}{U})\Phi(\nabla U)\Big].$

The corresponding Picone formula (wherever $U \neq 0$) is then

$$\begin{aligned}&\nabla \cdot \Big\{AV\Phi(\nabla V) - b(x)V\phi(\frac{V}{U})\Phi(\nabla U) - \\ &AV\phi(\frac{V}{U})\Phi(\nabla U)\Big\} \\ &\quad = A\ Z_\alpha(V,U) - b(x)\nabla\cdot\Big[V\phi(\frac{V}{U})\Phi(\nabla U)\Big].\end{aligned}$$

As we saw earlier, if we replace $b(x)$ by $b(x) + m$, $m \in \mathbb{R}$ in the equation (Ed), the equation is unchanged and the subsequent Picone formula reads

$$\begin{bmatrix} \nabla \cdot \Big\{AV\Phi(\nabla V) - (b(x)+m)V\phi(\frac{V}{U})\Phi(\nabla U) - AV\phi(\frac{V}{U})\Phi(\nabla U)\Big\} \\ = A\ Z_\alpha(V,U) - (b(x)+m)\nabla\cdot\Big[V\phi(\frac{V}{U})\Phi(\nabla U)\Big], \qquad \forall m \in \mathbb{R}.\end{bmatrix} \tag{3.12}$$

If we assume that $U > 0$ in Ω_R, for any $D(V) \subset \Omega_R$, the integration over $D(V)$ of the formule gives for any $m \in \mathbb{R}$

$$\begin{aligned}&\int_{D(V)} \nabla \cdot \Big\{AV\Phi(\nabla V) - b(x)V\phi(\frac{V}{U})\Phi(\nabla U) - \\ &AV\phi(\frac{V}{U})\Phi(\nabla U)\Big\} dx = 0 \\ &= \int_{D(V)} \Big(A\ Z_\alpha(V,U) - b(x)\nabla\cdot\Big[V\phi(\frac{V}{U})\Phi(\nabla U)\Big]\Big) dx \\ &- \int_{D(V)} m\,\nabla\cdot\Big[V\phi(\frac{V}{U})\Phi(\nabla U)\Big] dx.\end{aligned} \tag{3.13}$$

This can hold only if each integrand of the last lines is zero in $D(V)$. But if $\nabla \cdot \left[V\phi(\frac{V}{U})\Phi(\nabla U) \right] \equiv 0$ in $D(V)$, the first integrand will be strictly positive therefor U cannot be non zero throughout Ω_R.

We then have the following

Theorem 3.2. *Let* $A,\ C \in C(\mathbb{E}^n,\ (0,\ \infty)\ )$; *let* $b \in C^1(E^n)$ *and* $\nabla b(x) = B(x)$. *Then any nontrivial solution of*

$$Q(U) := \nabla \cdot \left\{ A(x)\Phi(\nabla U) \right\} + C(x)\phi(U) + B(x)\cdot \Phi(\nabla U) = 0, \quad x \in E^n$$

is oscillatory. Moreover, let $D(U)$ *be a nodal set of a nontrivial solution* U *. With* $A := \max_{D(U)} A(x)$ *and* $\alpha C := \inf_{D(U)}\ C(x)$,

$$diam[D(U)] \le \pi_\alpha \left[\frac{A}{\alpha C}\right]^{1/(\alpha+1)} \quad and \quad \max_{D(U)} |U(x)| \le \left[\frac{A}{\alpha C}\right]^{1/(\alpha+1)}. \tag{3.14}$$

if $\alpha \ge 1$.

Proof. Because the equation is homogeneous in the sense that $Q(U) = 0 \Longrightarrow Q(kU) = 0\ \forall k \in \mathbb{R}$, we take the solution U to swatisfy $\sup_{D(U)} \{|U(x)|,\ |\nabla U(x)|\ \} \le 1$.

We need to prove only the estimates. Having in hands the $D(U)$, let V be an oscillatory solution for

$$\nabla \cdot \left\{ A(x)\Phi(\nabla V) \right\} + C(x)\phi(V) - H(x) = 0 \quad x \in E^n; \quad V(0) = 0,\ |\nabla V(0)| = 1,$$

where H is a function satisfying $H(x) + B(x)\cdot \Phi(\nabla U) \ge 0$ in $D(U)$.

Without lost of generality, we assume that $\exists x_0 \in D(U) \bigcup D(V)$ such that $\nabla U(x_0) = \nabla V(x_0) = 0$. If $measure(\ \{x \in D(V) \mid V(x) - U(x) < 0\}) > 0$ then $\exists \tau > 0$ such that $D_\tau := \{x \in D(V) \mid w(x) = V(x) - U(x) + \tau > 0$ and $w|_{\partial D_\tau} = 0\}$ has also a positive measure.

$$\int_{D_\tau} \left\{ w(x)\nabla \cdot \Big(A(x)\Phi(\nabla U) - A(x)\Phi(\nabla V) \Big) \right\} dx$$

$$= -\int_{D_\tau} A(x) \left\{ \nabla(V-U)[\Phi(\nabla U) - \Phi(\nabla V)] \right\} dx \qquad (i)$$

$$= \int_{D_\tau} w(x) \Big(C[\phi(V) - \phi(U)] - H(x) - (\ B \cdot \Phi(\nabla U)\ ) \Big) dx.$$

The second line (i) is positive while the last is negative whence $measure(\ \{x \in D(V) \mid V(x) - U(x) < 0\}) = 0$ and $D(U) \subset D(V)$. The Theorem 2.6 completes the proof.

□

4. Equations with different parameters $\boldsymbol{\alpha}$

Consider for $\alpha,\ \beta > 0$ an $A,\ C \in C(E^n,\ (0, \quad \infty)\ )$ the equation

$$\nabla \cdot \left\{ A(x)\Phi_\alpha(\nabla U) \right\} + C(x)\phi_\beta(U) = 0; \quad x \in E^n. \tag{4.1}$$

For the Picone formula we chose as associate function

$$\nabla \cdot \Big\{ A(x)\Phi_\alpha(\nabla V) \Big\} + C_m \phi_\alpha(V) = 0; \quad x \in E^n.$$

where the positive constant C_m will be determined a posteriori. So

$$\left| \begin{aligned} &\nabla \cdot \Big\{ V A(x)\Phi_\alpha(\nabla V) - A(x) V \phi_\alpha\big(\frac{V}{U}\big)\Phi_\alpha(\nabla U) \Big\} \\ &= A(x) Z_\alpha(V, U) + |V|^{\alpha+1}\Big[C(x)|U|^{\beta-\alpha} - C_m \Big]. \end{aligned} \right. \tag{4.2}$$

Assume that there is a nontrivial solution U which remains strictly positive in some Ω_R. For any $m \in \mathbb{N}$, define $\theta(m) := \inf_{R<|x|<mR} \Big[C(x)|U(x)|^{\beta-\alpha} \Big]$ and take $C_m := \theta(m)$.

Based on the Picone formula above, we have the following result:

Theorem 4.1. *Given the positive functions* $A,\ C \in C(E^n)$, *if* $\exists q_0,\ q_1 > 0$ *such that* $0 < q_0 < \dfrac{A(x)}{C(x)} < q_1\ \forall x \in E^n$ *then* $\forall\ \alpha,\ \beta > 0$

$$\nabla \cdot \Big\{ A(x)\Phi_\alpha(\nabla U) \Big\} + C(x)\phi_\beta(U) = 0; \quad x \in E^n.$$

is strongly oscillatory. Moreover if $D(U)$ *is a nodal set of the solution* U,

$$\left. \begin{aligned} &\textit{for}\ \ q(U) := \max_{D(U)} \Big[\frac{A(x)}{C(x)} \Big], \quad \textit{diam}\ \{D(U)\} \le \pi_\alpha \Big[\frac{q(U)}{\alpha} \Big]^{1/(\alpha+1)} \\ &\textit{and} \quad \max_{D(U)} |U(x)| \le \Big[\frac{q(U)}{\alpha} \Big]^{1/(\alpha+1)}. \end{aligned} \right. \tag{4.3}$$

The setimates follow from Theorem 2.6.

Remark 4.2. .

This Picone-type formulae

1) is a powerful tool to establish the oscillation criteria for half linear quasilinear equations of the types (P) aand (Q) of the introduction;

2) provide estimates of the diameters and the amplitudes of their oscillatory solutions and these being independent of the admissible perturbations as displayed in (H1)-(H2).

Adknowlegments: To

Fotsi N. Tadie,Junior and Tecla Tadie, Karsten Shatue Tadie, Christoffer Mabou Kanstrup-Tadie and Steve L. Poka.

References

[1] Tadié , *Comparison results for Semilinear Elliptic Equations via Picone-type identities. Electron. J. Differential Equations* **2009**, No. 67: 1-7 (2009).

[2] Tadié, *Oscillation criteria for semilinear elliptic equations with a damping term in* $\mathbb{R}^n$. *Electron. J. Differential Equations* **2010**, No.51: 1-5 (2010).

[3] Tadié ; *Comparison Results and Estimates of Amplitude for Oscillatory Solutions of Some Quasilinear Elliptic Equations* , IOSR Journal of Mathematics (IOSR-JM) e-ISSN:2278-5728, p-ISSN;2319-765X. Vol 16, Issue 1 Ser.I(Jan-Febr 2020)pp48-60 www.isorjournals.org

[4] Tadié ; *ON STRONG OSCILLATION CRITERIA FOR BOUNDED SOLUTIONS FOR SOME QUASILINEAR SECOND ORDER ELLIPTIC EQUATIONS* , Communications in Mathematical Analysis, volume 13, Number 2, pp. 15-26 (2012) ISSN 1938-9787

[5] Tadié ; *Semilinear Second-Order Ordinary Differential Equations: Distances Between Consecutive Zeros of Oscillatory Solutions*, Constanda, Christian (ed.) et al., Integral methods in science and engineering. Theoretical and computational advances. Papers based on the presentations at the international conference, Zbl 06567057, Springer International Publishing Switzerland (2015).

[6] Tadié ; *Criteria and Estimates for decaying Oscillatory Solutions for Some Second Order Quasilinear O.D.Es ;* Elect. Jour. Differential Equations, Vol.**2017**, No 51: 1-5 (2017).

[7] B.Gidas, W.M. Ni and L. Nirenberg ; *Symmetry and related properties via the maximum principle* , Comm. Math.Phys. 68 (1979), 209-243.

[8] T. Kusano, J. Jaros, N. Yoshida; *A Picone-type identity and Sturmian comparison and oscillation theorems for a class of half-linear partial differential equations of second order*, Nonlinear Analysis, Vol. 40 (2000), 381-395.

[9] Pucci, P. , Serrin, J. , Zou,H., *A strong Maximum principle and a compact support principle for singular elliptic inequalities J. Math. Pures Appl.*, **78** (1999), 769-789.

[10] Mario Picone, *Sui valori eccezionali di un parametro do cui depende una equatione differenziale lineare ordinaria del secondo ordine*, Ann.Scuola Norm. Pisa 11 (1910) pp 1-141.

[11] Petr. Lukas Kotrla, *Differentiability properties of* $p-$*trigonometric functions Variational and Topological Methods: Theory, Applications...(2012)* ejde , Conference 21 (2014) pp101-127. ISSN:1072-6691.