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Degree of Homogeneityof Finite Primitive Permutation Groups with Relatively Large Degrees VIA the Socle of the Groups

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Abstract
Finite transitive permutation groups of large degree possess socle section isomorphic to a particular given primitive groups. It was observed that groups of large degree had minimum base. Further indication showed that such groups are the symmetric and the alternating groups, except for the two Mathieu groups which are 5-transitive. The concept of socle form the basis in the determination of the degree of homogeneity of these groups.

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I. Introduction

Basically primitive group of large degree possess proper subgroups of relatively minimal bases. These groups with minimum base tend to show section isomorphism with other groups. The schreier conjecture was first investigated from the work of [15] which further paved way for some notable results as carried out in [1] and [9]. In order to determine the degree of homogeneity of transitive groups of relatively large degree, we require the theorem due to [14]. The emphasis on socle will give us avenue to avoid the geometrical approach of partitioning as was used in [12], and later extended by [10]. The t-orbit homogeneity from [5], also followed the same approach . The sets were partitioned in the form a tabloids where the orbit was used to determine the degree of homogeneity.

Preliminary results

In this aspect we give basic result which is useful in the attainment of the later results.

1.1 Definition

A block is a subset $\beta\ of\ \Omega$ such that for every $g \in G$, either

$g\beta = \beta$ or $g\beta \cap \beta = \emptyset$.

1.2 Definition

Let $G \leq sym(\Omega)$. A subset $\Delta \subseteq \Omega$ is a base for G if $G_{(\Delta)}= 1$

1.3 Definition let G be a primitive permutation group. We define the minima degree for G to be the $|supp(\alpha)|$for all $\alpha \in G$ and $\alpha \neq 1$

With these definitions given in [1], [2], and [9] we state a useful result which will proffer ways on how we can determine the degree and order of a permutation group G acting on the set Ω. We state the next theorem without proof that in a case G has a minimal degree, the base size is large

1.3Theorem

Let G be a proper primitive group of finite degree n. then G has a base size of at most $\frac{n}{2}$ and so $|G| \leq n(n-1) \dots \dots .. (n - \frac{n}{2} + 1)$.

It follows from the next result that primitive groups of relatively large degree has the alternating groups as its subgroups.

1.4 Theorem

Let G be a group acting primitively on a finite set Ω of size n and suppose that G has a Jordan complement of size m where $m \geq \frac{n}{2}$, then G is 3-transitive. Moreover if $m > \frac{n}{2}$ then$G \geq Alt(\Omega)$.

Proof

We proceed by induction on n toshow that G is 3-transitive. The result is easily verified. If $n \leq 4$.Suppose that $n \geq 4$, let $\emptyset = \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_k \ldots .(1)$ be a j- flag for G with $|\Delta_{k|}| = m$ then $k \geq 1$. Therefore with this we can say that $G_{\{\Delta_k\}} \cap G_{(\Delta_k)}$ acts transtively on $\Delta_k \backslash \Delta_i$ which force $|\Delta_k \backslash \Delta_i| > 1$ and so (1) above is also a j-fag for $G_{\{\Delta_k\}}$ acting on Δ_k , with $|\Delta_k| > t$ implying G is t- transitive. So conditions given shows that G is (m+1) – transitive. Thus we conclude that $G_{\{\Delta_k\}}$ is 3-transitive on Δ_k. Hence G is 3-transitive.

Now suppose that $m > {}^{n}/_{2}$ and assume that for $m \geq 5$. An induction similar to the one above shows that we may conclude that $H = G_\alpha$ restricted to Δ_k and it contains $Alt(\Delta_k)$. Thus the derived group $H^{'}$ restricted to Δ_k is equal to the simple group $Alt(\Delta_k)$ and $H^{'}$restricted to $\Omega \backslash \Delta_k$ has no homomorphic image isomorphic to $Alt(\Delta_k)$ because $m > n - m$ hence the kernel of the action of $H^{'}$ on $\Omega \backslash \Delta_k$ induces $Alt(\Delta_k)$ on Δ_k. Hence it shows that G contains a 3-cycle. Since G is primitive, then $G \geq Alt(\Omega)$ as asserted.

This shows that $sym(\Delta) \cong S_k$ with kernel $G_{(\Delta)}$ and the factor group $G_{\{\Delta\}}/G_{(\Delta)}$ is isomorphic to a subgroup of S_k . Next is a result due to [13]

1.5 Lemma

G is sharply n-homogeneous on Ω, for some $n \geq 1$.

Proof:

First observe that G cannot be k-homogeneous for all,$k \in N$, since otherwise there would be an infinite descending chain. $G, G_{x_1}, G_{x_1 x_2}, \ldots$ of subgroup (i.e. subgroup group with using the concept of socle. Thus to prove that H is homogeneous, it suffices to show that if for some k the group G is k-homogeneous but not sharply k-homogeneous on Ω, then G is (k+1)-homogeneous. We prove this by induction on k.

We now suppose that G is k-homogeneous but not sharply k-homogeneous. We claim that (G_x, Γ) is not sharply (k-1)-homogeneous. Suppose otherwise and pick $x_1 <, \ldots < x_{k-1}$in Γ. Then $G_{xx,\ldots x-1}$ fixes $\{2: y < z\}$, so $G_{yx_1}, \ldots\ x_{k-1} < G_{xx_1} \ldots x_{k-1}$ since these groups are conjugate, this contradicts the descending chain condition on the subgroup. It follows by induction that (G_x, Γ) is k-homogeneous and hence (G, Ω) is (k+1)-homogeneous as required.

II. Main Results

2.1 **Theorem 1**

Let G be a k-transitive permutation group. For k≥ 5 and any subset $\Omega^{'}$ of Ω with $|\Omega^{'}|$ =k, and let H=$G_{\Omega'}$ be the setwise stabilizer of G such that H=soc(G) and invariant in G .Then G acts homogeneously on Ω.

Proof

Since G is k-transitive for k≥ 5 , then the only 5-transitive groups are the symmetric, alternating groups,M_{12} and M_{24} groups, and so G is one of them. Therefore by Theorem 3, it implies$G_{\Omega'}$ is k-1 –transitive on $\Omega^{'} \backslash \alpha$ for Ω . Hence G is k-transitive or has a chain of Jordan complements of size k.

Also, if soc(G)=$G_{\Omega'}$ it follows H is k-homogeneous which also k-1-homogeneity.Finally it shows that $\Omega^{'}$ acts k-transitively on Ω.

2.2 Theorem

Let H be a permutation group and G a transitive extension of H, then G is k-homogeneous.

Proof

Let $H \leq G$ be a subgroup of Ω that is k-transitive on $\Omega = \Omega^{'} \cup \{\Psi\}$ such that$\Omega' \neq \Omega$, suppose that H = soc(G) and H = G_α. We see that the stabilizer of H is equal to G_α. Thus $H \leq sym(\Omega)$ and H is k-transitive by Lemma 1.5. If G is homogeneous, then for any $y \in G$the double coset of $G = G_\alpha y, G_\alpha \cup \ldots\ G_\alpha y_{r-1} G_\alpha$ has rank r with r orbits. Hence for any $x \in sym(\Omega^{'})$ which does not fix a point in Ψ, we have that$G =< H, x >$ implying that it is k-homogeneous.

III. Conclusion

The idea of socle helped a lot in determining the degree a level of homogeneity of groups which degree of primitivity is large. These groups showed section isomorphism with other groups.

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