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The Form of The Friendly Number of 10

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Abstract
Any positive integer n other than 10 with abundancy index $^9/_5$ must be a square with at least 6 distinct prime factors, the smallest being 5, and my new argument about the form of the friendly number of 10 is $\frac{25(5^{2c+1}-1)(8n+1-2c)}{4}$, if exist. Further at least one of the prime factors must be congruent to 1 modulo 3 and appear with an exponent congruent to 2 modulo 6 in the prime factorization of n.

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I. Introduction

For a positive integer n, the sum of the positive divisors of n is denoted by $\sigma(n)$;the ratio $\frac{\sigma(n)}{n}$ is known as the abundancy index of n or sometimes the ratio denoted as $\hat{\sigma}(n)$ or $I(n)$.A pair (a,b) is called a friendly pair if $\hat{\sigma}(a) = \hat{\sigma}(b)$ in this case, it is also common to say that b is a friend of a or simply that b and a are friend. Perfect numbers have abundancy index 2, and thus all friendly numbers with abundancy index less than 2 are often called deficient, while numbers whose abundancy index are greater than 2 are called abundant. The original problem was to show that the density of friendly integers ℕ,is unity and the density of solitary numbers (numbers with no friends) is zero.

We used two main approaches: one was an analysis of the $\frac{\sigma(n)}{n}$ function. While the other used number theoretic arguments to find a representation for a friend of 10.In[1], it was shown that 10 is the smallest number where it is unknown whether there are any friends of it. We will assume a basic understanding of the function $\sigma(n)$. This can be found in [2]. For more on number theoretic techniques, see [3].And the question "Is 10 a solitary number" is still unanswered. If 10 does have a friend, the following may be of use in finding it.

1. Elementary Properties of Abundancy Index:

Let a and b be positive integers. In what follows, all primes are positive.

1. $I(a) \geq 1$with equality only if a=1

2. If a divides b then $I(a) \leq I(b)$ with equality only if a=b

3. If $p_1, p_2, \ldots\ldots, p_k$ are distinct primes and $e_1, e_2, \ldots\ldots, e_k$ are positive integers then

$$I\left(\prod_{j=1}^{k} p_j^{e_j}\right) = \prod_{j=1}^{k}\left(\sum_{i=0}^{e_j} p_j^{-i}\right)$$

$$= \prod_{j=1}^{k} \frac{p_j^{e_j+1} - 1}{p_j^{e_j}(p_j - 1)}$$

These formulae follow from well-known analogues for σ:

$$\sigma\left(\prod_{j=1}^{k} p_j^{e_j}\right) = \prod_{j=1}^{k}\left(\sum_{i=0}^{e_j} p_j^{i}\right)$$

$$= \prod_{j=1}^{k} \frac{p_j^{e_j+1} - 1}{(p_j - 1)}$$

Property 3 directly implies a property of I shared by σ.

4. I is weakly multiplicative (meaning, if $\gcd(a,b) = 1$, then $I(ab) = I(a)I(b)$.

5. Suppose that $p_1, p_2, \dots \dots, p_k$ are distinct primes, $q_1, q_2, \dots \dots, q_k$ are distinct primes $e, e_2, \dots \dots, e_k$ are positive integers and $p_j \le q_j, j = 1,2,3 \dots \dots, k$ then

$$I\left(\prod_{j=1}^{k} p_j^{e_j}\right) \ge I\left(\prod_{j=1}^{k} q_j^{e_j}\right)$$

With equality only if $p_j = q_j, j = 1,2, \dots \dots, k$. This follows form 3 and the observation that if $e \ge 1$,then $\frac{x^{e+1}-1}{x^e(x-1)}$ is a decreasing function of x on $(1, \infty)$.

6. If the distinct prime factors of a are $p_1, p_2, \dots \dots, p_k$, then $I(a) < \prod_{j=1}^{k} \frac{p_j}{p_j - 1}$. Although related to 5,7 is most easily seen by applying3 and the observation that for $p > 1$,

$$\frac{p^{e+1} - 1}{p^e(p-1)} = \frac{p - \frac{1}{p^e}}{p - 1}$$

Increases to $\frac{p}{p-1}$ as $e \to \infty$.

II. Theorem 1:

If n is a friend of 10 then n is an odd square number with at least 6 distinct prime factors, the smallest being 5. Further, at least one of $n's$ prime factors must be congruent to 1 modulo 3, and appear in the prime power factorization of n to a power congruent to 2 modulo 6. If there is only one such prime dividing n, then it appears to a power congruent to 8 modulo 18 in the factorization of n.

Proposition 1.$I(an) > I(n)$ for$a > 1$

Proof. In general a can share prime factors with n. Let $a = uv$ where $\gcd(a, n) = u, \gcd(v, n) = 1$.We thus have $I(an) = I(un)I(v)$ by Elementary properties of Abundancy Index.

$$I(an) = I(un)I(v) > I(un)$$

$$I(un) > I(n)$$

Thus,

$$I(an) > I(n) \text{ for } a > 1$$

Lemma 1. A friend of n cannot be a multiple of n. That is $\frac{\sigma(n)}{n} \neq \frac{\sigma(an)}{an}$ for $a > 1$

Proof. This follows directly from proposition1. Since an is a multiple of n. $\frac{\sigma(an)}{an} > \frac{\sigma(n)}{n}$,

So $\frac{\sigma(n)}{n} \neq \frac{\sigma(an)}{an}$ for $a > 1$

Corollary 1. A friend of 10 cannot be the form $n = 2^a 5^b m$.Thus a friend of 10 cannot be an even integer.

Proof. For $a, b > 1. 2^a 5^b$ is a multiple of 10, and $\frac{\sigma(n)}{n} > \frac{9}{5}$

Therefore n is not a friend of 10.A friend of 10 must be of the form $5^b m$, So a friend of 10 can't be an even integer.

Corollary 2. A friend of 10 must be square of some numbers: $n = 5^{2b} \prod_{j=1}^{k} p_j^{2e_j}$

Proof. Suppose $\frac{\sigma(n)}{n} = \frac{9}{5}$ and $n = 5^b d$, Where $d = \prod_{j=1}^{k} p_j^{e_j}$ then

$$\frac{\sigma(n)}{n} = \frac{9}{5}$$

$$5.\sigma(5^b)\sigma\left(\prod_{j=1}^{k} p_j^{e_j}\right) = (9).5^b \prod_{j=1}^{k} p_j^{e_j}$$

$$5.\sigma(5^b).\sigma(p_1^{e_1}).\sigma(p_2^{e_2}).\sigma(p_3^{e_3}) \dots \sigma(p_k^{e_k}) = (9).5^b.p_1^{e_1}.p_2^{e_2}.p_3^{e_3} \dots p_k^{e_k}$$

$$(1 + 5 + \cdots 5^b)(1 + p_1 + \cdots + p_1^{e_1}) \dots (1 + p_k + \cdots + p_k^{e_k}) = (9).5^{b-1}.p_1^{e_1}.p_2^{e_2}.p_3^{e_3}.p_k^{e_k}$$

Form the corollary 1, we have $p_i > 2$ for any $i \leq k$.So the right side must be odd.To obtain this, we must have that every sum on the left side is also odd. Since each p_i^x is odd, We must have an odd number of terms in the sum for the whole sum to be odd. To obtain this, each e_i and b must be even. Thus n must be the square of some numbers.

Proposition 2. If $I(N^2) = \frac{9}{5}, then\ 3 \nmid N^2$

Proof. From the corollary 2, we have that if $I(N^2) = \frac{9}{5}\ and\ 3 \mid N^2$. Then

$$N^2 = 3^{2a} 5^{2b} \prod_{j=1}^{k} p_j^{2e_j}$$

Where 2,3,5$\nmid \prod_{j=1}^{k} p_j^{2e_j}$. It is easy to verify that $I(3^2 5^4)$ and $I(3^4 5^2) > \frac{9}{5}$. Combining this with Lemma 1, we have that $I\left(3^2 5^4 \prod_{j=1}^{k} p_j^{2e_j}\right) andI\left(3^4 5^2 \prod_{j=1}^{k} p_j^{2e_j}\right) > \frac{9}{5}$

Thus, the problem reduces to a single case: $I\left(3^2 5^2 \prod_{j=1}^{k} p_j^{2e_j}\right) = \frac{9}{5}$

$$5.\sigma(3^2).\sigma(5^2)\sigma\left(\prod_{j=1}^{k} p_j^{2e_j}\right) = 9.3^2.5^2.\prod_{j=1}^{k} p_j^{2e_j}$$

$$(13).(31).\sigma\left(\prod_{j=1}^{k} p_j^{2e_j}\right) = (3^4).(5).\prod_{j=1}^{k} p_j^{2e_j}$$

We can see that $13,31 \mid \prod_{j=1}^{k} p_j^{2e_j}$,Thus $\prod_{j=1}^{k} p_j^{2e_j} = 13^{2c}31^{2d}\prod_{i=1}^{h} p_i^{2e_i}$

Where $2,3,5,13,31 \nmid \prod_{i=1}^{h} p_i^{2e_i}$. We will divide by 13,31 immediately.

$$\sigma(13^{2c})\sigma(31^{2d})\sigma\left(\prod_{i=1}^{h} p_i^{2e_i}\right) = 3^4(5)(13^{2c-1})(31^{2d-1})\prod_{i=1}^{h} p_i^{2e_i}$$

$$I\left(\prod_{i=1}^{h} p_i^{2e_i}\right) = \frac{405}{\underbrace{\frac{\sigma(13^{2c})}{13^{2c-1}}}_{\geq 14}.\underbrace{\frac{\sigma(31^{2d})}{31^{2d-1}}}_{\geq 32}} < \frac{405}{448} < 1$$

This is a contradiction with Elementary properties of Abundancy Index (property no.1).

Hence, $3 \nmid N^2$ whenever N^2 is a friend of 10.

Proposition 3. If $I(n) = \frac{9}{5}$, then $n = 5^{2a}\prod_{j=1}^{k} p_j^{e_j}$ where $k \geq 4$

Proof.

Using the previous proposition and elementary properties of Abundancy Index (property no.6), We will construct the largest value of $I(n^e)$ with 4 distinct primes, Let $n = (5)(7)(11)(13)$

Here is a largest value with 4 distinct primes because from proposition 2. We have that $p_i \geq 7$.

To maximize the value of $I(n^e)$ we let $e \to \infty$.Hence,

$$\lim_{e\to\infty} I(n^e) = \frac{5.7.11.13}{4.6.10.12} = \frac{1001}{576} < \frac{9}{5}$$

So, there must be at least 5 distinct primes in the factorization of$n$.

Proposition 4. If $I(n) = \frac{9}{5}$, then $n = 5^{2a}\prod_{j=1}^{k} p_j^{e_j}$ where $k \geq 5$

Proof.

We can use the same technique as in the last proposition to show that only 3 cases could work if n were represented as 5 distinct primes. These are $n = (5^a7^b11^c13^d17^f)^2$ but this does not work because

The smallest it could be is: $I(5^2 7^2 11^2 13^2 17^2) > \frac{9}{5}$. Case2 gives $n = (5^a 7^b 11^c 13^d 19^f)^2$, but this does not work because the smallest it could be: $I(5^2 7^2 11^2 13^2 19^2) > \frac{9}{5}$. The final case takes a little more work: $n = (5^a 7^b 11^c 13^d 23^f)^2$ We can see that if $a > 1$, then $I(n) > \frac{9}{5}$, so $a = 1$. Let us examine when $I(n) = \frac{9}{5}$, then

$$5\sigma(5^2)\sigma(7^{2b})\sigma(11^{2c})\sigma(13^{2d})\sigma(23^{2f}) = 9(5^2)(7^b 11^c 13^e 23^f)^2$$

$$31\sigma(7^{2b})\sigma(11^{2c})\sigma(13^{2d})\sigma(23^{2f}) = 9(5)(7^b 11^c 13^e 23^f)^2 = l$$

Clearly, $31 \nmid l$. Since the left hand side is some integer, this results is a contradiction.
Hence a friend of 10 must be composed of at least 6 distinct primes.

Finally, since

If $I(N^2) = 5^{2a} \prod_{j=1}^{k} p_j^{2e_j}$, then

$$5\sigma(N^2) = 5(1 + 5 + \cdots + 5^{2a}) \prod_{j=1}^{k} \left(\sum_{i=0}^{2e_j} p_j^i \right) = 9N^2,$$

We have that $9 | \sigma(N^2)$. If $p \equiv 2 (mod\ 3)$, then $\sigma(p^{2e}) \equiv 1\ (mod\ 3)$ for any positive integer e.

Consequently, some $p_j \equiv 1\ (mod\ 3)$, and $\sigma\left(p_j^{2e_j}\right) \equiv 0\ (mod\ 3)$ implies $2e_j + 1 \equiv 0\ (mod\ 3)$.

Thus $e_j = 3t + 1$ for some integer t, so $2e_j = 6t + 2$

If p_j is only such prime dividing N^2, then $\sigma\left(p_j^{2e_j}\right) \equiv 0\ (mod\ 9)$. Checking the possibilities $p_j \equiv 1,4\ or\ 7 (mod\ 9)$, one finds that $2e_j \equiv 8 (mod\ 18)$

III. Theorem 2:

If N^2 is a friend of 10 then N^2 must be in the form of $\frac{25(5^{2c+1}-1)(8n+1-2c)}{4}$.

Proof.

Let, the friendly number of 10 is N^2

$$N^2 = 5^{2c} \prod_{j=1}^{k} p_j^{2e_j}$$

$$\frac{\sigma(N^2)}{N^2} = \frac{9}{5}$$

As N^2 is a square number from corollary 2., So it is in the form of $(4a+1)$

$$\because 5|(4a+1)$$

$$\therefore N^2 \text{is in the form of } (20a-15)$$

$$\frac{\sigma(20a-15)}{20a-15} = \frac{9}{5}$$

$$\Rightarrow \sigma(20a-15) = -27 + 36a$$

$$\sigma(20a-15) \equiv -27 \ (mod\ 36)$$

&

$$\sigma(20a-15) \equiv -27 \ (moda)$$

Again $(20a-15)$ is a square number so$a = 5a_0(a_0-1)+2$

After putting $a = 5a_0(a_0-1)+2$, we get $(20a-15) = (10a_0-5)^2$

Last digit of $5a_0(a_0-1)+2$ is always 2 so

$\frac{\sigma(20a-15)+27}{36} = \frac{\sigma[(10a_0-5)^2]+27}{36} = a_1$ here the last digit of $a_1$is must be 2

$$\sigma[(10a_0-5)^2] \equiv -27 \ (mod\ 36)$$

$$\sigma\left(5^{2c}\prod_{j=1}^{k} p_j^{2e_j}\right) \equiv -27(\ mod\ 36)$$

Here the last digit of the quotient is 2

Hence,

$$\sigma\left(\prod_{j=1}^{k} p_j^{2e_j}\right) \equiv 45 - 90c\ (\ mod\ 360)$$

$$\Rightarrow \sigma\left(\prod_{j=1}^{k} p_j^{2e_j}\right) = 360n + 45 - 90c$$

$$\Rightarrow \sigma(5^{2c})\sigma\left(\prod_{j=1}^{k} p_j^{2e_j}\right) = \frac{(5^{2c+1}-1)(360n+45-90c)}{4}$$

$$\Rightarrow \sigma\left(5^{2c}\prod_{j=1}^{k} p_j^{2e_j}\right) = \frac{(5^{2c+1}-1)(360n+45-90c)}{4}$$

$$\because \frac{\sigma(N^2)}{N^2} = \frac{9}{5}$$

$$\Rightarrow \frac{\sigma(5^{2c}\prod_{j=1}^{k} p_j^{2e_j})}{5^{2c}\prod_{j=1}^{k} p_j^{2e_j}} = \frac{9}{5}$$

$$\Rightarrow \frac{(5^{2c+1}-1)(360n+45-90c)}{4(5^{2c}\prod_{j=1}^{k} p_j^{2e_j})} = \frac{9}{5}$$

$$\Rightarrow 5^{2c}\prod_{j=1}^{k} p_j^{2e_j} = \frac{25(5^{2c+1}-1)(8n+1-2c)}{4}$$

$$\Rightarrow N^2 = \frac{25(5^{2c+1}-1)(8n+1-2c)}{4}$$

So, we have come to the conclusion that the friendly number of 10 is must be in the form of

$\frac{25(5^{2c+1}-1)(8n+1-2c)}{4}$ if exists.

$$N^2 = \frac{25(5^{2c+1}-1)(8n+1-2c)}{4}$$

From proposition 2.3 $\nmid N^2$

$$3 \nmid (8n+1-2c)$$

$$\therefore n \neq 3n_0 + (c-2)$$

Just for fun, we introduce a new definition.

Definition 1:(Theoretical Friend). A sequence n_e is a theoretical Friend of m if:

$$\lim_{e\to\infty} I(n_e) = I(m)$$

Proposition 5.10 has at least one theoretical Friend, namely $n_e = 3^e.5$

Proof.

$$\lim_{e\to\infty} I(n_e) = \lim_{e\to\infty} \frac{\sigma(3^e).\sigma(5)}{3^e.5}$$

$$= \left(\frac{3}{2}\right).\left(\frac{6}{5}\right) = I(10)$$

For further reading on the topic of $I(n)$ and $\sigma(n)$, see [5] and [4]. See [5]for information concerning when $\sigma(n) = k$ has exactly m solutions (Sierpi'nski conjecture). See [4]for a more in-depth study of $\sigma(n)$ and on the distribution and density of numbers of this form.

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