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Application of Asymptotic Iteration Method in Solving Black-Scholes Equation for Different Forms of Interest Rates

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Abstract
This work deals with the application of asymptotic iteration method in solving Black-Scholes equation. The assumptions are first relaxed followed by the derivation of the closed-form solutions of the general Black-Scholes equation under option pricing for constant interest rate, the case where the interest rate is a linear function and where the interest rate is a reciprocal function.
Keywords*: Asymptotic iteration method, Black-Scholes equation, option pricing, interest rates.*

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I. Introduction

Asymptotic iteration method is atechnique used in solving analytically and approximately the linear second-order differential equation. This method is very useful in quantum mechanics especially in the area of eigenvalue problems.The Black-Scholes (B-S) model is a popular method for pricing options [1].Some of the assumptions of B-S equation do not hold in the real market, hence, Merton extended the model by removing some of the assumptions using stochastic calculus [2]. In general, closed form solution of Black-Scholes equation is very rare. Based on that, many numerical ways of pricing option based on the B-S model have been investigated. Han and Wu [3], Ehrhardt and Mickens [4], and Jeong et al. [5] used finite difference method on American option pricing governed by the B-S equation. The finite difference method for a generalized Black-Scholes equation was extended by Cen and Le [6]. Perelloa et al. [7] applied the Stratonovich calculus to derive the B-S equation. Wang [8] introduced fitted finite volume spatial discretization and an implicit time stepping method for B-S governing option pricing. Jodar et al. [9] applied Mellin transform to the solution of B-Sequation. Berkowitz [10] used ad hoc B-S approach to outperform the B-S formula out-of-sample. Li and Lee [11] developed a new successive over-relaxation method to calculate the B-S implied volatility. Yousuf et al. [12] used a new second-order exponential time differencing method for pricing American option with transaction cost. Lesmana and Wang [13] used an upwind finite difference method to the solution of nonlinear B-S equation under transaction costs. Tagliani and Milev [14] applied it in discrete monitored barrier options. Burkovska et al. [15] introduced a reduced basis method for pricing options based on B-S and Heston models. Chen et al. [16] presented a new operator splitting method for solving fractional B-S under American options. In the area of integrated methods, Allahviranloo and Behzadi [17] innovated the Adomian’s decomposition method, modified Adomian’s decomposition method, variational iteration method, modified variational iteration method, ho-motopy perturbation method, modified homotopy perturbation method and homotopy analysis method. Jena and Chakraverty [15] solved fractional B-S option pricing using an iterative method.

II. Asymptotic Iteration Method (AIM)

Consider the second order linear differential equation

$$y'' = P_0(x)y' + Q_0(x)y, \tag{1}$$

where $P_0(x)$ and $Q_0(x)$ are C^∞ functions. Under further differentiation, the invariant of $P_0(x)y' + Q_0(x)y$ in equation 91) is the main idea for AIM. Differentiating equation (1) gives

$$y''' = P_0'(x)y' + P_0(x)y'' + Q_0'(x)y + Q_0(x)y'. \tag{2}$$

Substituting equation (1) in equation (2) gives

$$\begin{aligned} y''' &= P_0'(x)y' + P_0(x)[P_0(x)y' + Q_0(x)y] + Q_0'(x)y + Q_0(x)y' \\ &= [P_0'(x) + Q_0(x) + P_0^2(x)]y' + [Q_0'(x) + Q_0(x)P_0(x)]y. \end{aligned} \tag{3}$$

With

$$P_1(x) = P_0'(x) + Q_0(x) + P_0^2(x) \tag{4}$$

and

$$Q_1(x) = Q_0'(x) + Q_0(x)P_0(x), \tag{5}$$

equation (3) becomes

$$y''' = P_1(x)y' + Q_1(x)y. \tag{6}$$

Comparing equations (1) and (6), the nth derivatives of equation (1) becomes

$$y^{(n+2)} = P_n(x)y' + Q_n(x)y, \quad n = 1,2,\cdots \tag{7}$$

where

$$P_n(x) = P_{n-1}'(x) + Q_{n-1}(x) + P_0(x)P_{n-1}(x) \; and \; Q_n(x) = Q_{n-1}'(x) + Q_0(x)P_{n-1}(x). \tag{8}$$

Let $P_0(x)$ and $Q_0(x)$ be $C^\infty(a,b)$ functions. From equation (7)

$$y^{(n+1)} = P_{n-1}(x)y' + Q_{n-1}(x)y, \tag{9}$$

$$y^{(n+2)} = P_n(x)y' + Q_n(x)y. \tag{10}$$

Divide equation (10) by equation (9) to have

$$\frac{y^{(n+2)}}{y^{(n+1)}} = \frac{P_n(x)y' + Q_n(x)y}{P_{n-1}(x)y' + Q_{n-1}(x)y}$$

$$= \frac{P_n(x)\left(y' + \frac{Q_n(x)}{P_n(x)}y\right)}{P_{n-1}(x)\left(y' + \frac{Q_{n-1}(x)}{P_{n-1}(x)}y\right)}. \tag{11}$$

We then introduce the asymptotic condition. If for some n

$$\frac{Q_n(x)}{P_n(x)} = \frac{Q_{n-1}(x)}{P_{n-1}(x)} \equiv \beta(x), \tag{12}$$

Then

$$\frac{y^{(n+2)}}{y^{(n+1)}} = \frac{P_n(x)}{P_{n-1}(x)}. \tag{13}$$

Since $\frac{y^{(n+2)}}{y^{(n+1)}} = \frac{d}{dx}\ln\left(y^{(n+1)}\right)$, equation (11) can be written thus

$$y^{(n+1)} = k_1 \exp\left(\int^x \frac{P_n(\tau)}{P_{n-1}(\tau)} d\tau\right)$$

$$= k_1 \exp\left(\int^x \frac{P_{n-1}'(\tau)}{P_{n-1}(\tau)} d\tau + \int^x \beta(\tau)\, d\tau + \int^x P_0(\tau) d\tau\right) \tag{14}$$

$$= k_1 P_{n-1} \exp\left(\int^x (\beta(t) + P_0(t))\, dt\right),$$

since $\int^x \frac{P_{n-1}'(\tau)}{P_{n-1}(\tau)} d\tau = \ln P_{n-1}(\tau)$ and $e^{\ln P_{n-1}(\tau)} = P_{n-1}(\tau)$. Substituting equation (14) in equation (9) gives

$$\frac{P_{n-1}}{P_{n-1}}y' + \frac{Q_{n-1}}{P_{n-1}}y = \frac{k_1 P_{n-1}}{P_{n-1}} \exp\left(\int^x (\beta + P_0) dt\right).$$

Hence,

$$y' + \beta y = k_1 \exp\left(\int^x (\beta + P_0)\, dt\right). \tag{15}$$

We then solve equation (15) which is a first order homogeneous differential equation. Considering the homogeneous part $y_c' + \beta(x)y_c = 0$, we have $\int^x \frac{dy_c}{y_c} = -\int \beta(x)\, dx$. Which gives

$$y_c = \exp\left(\int^x -\beta dt\right). \tag{16}$$

The general solution can be assumed to be

$$y = g(x) \exp\left(\int^x -\beta dt\right), \tag{17}$$

where $g(x)$ is an unknown function to be determined by direct substitution in equation (15). From equation (17) we have

$$y' = g'(x) \exp\left(\int^x -\beta dt\right) + g(x)(-\beta) \exp\left(\int^x -\beta dt\right). \tag{18}$$

Substituting equations (17) and (18) in equation (15) gives

$$g'(x) \exp\left(\int^x -\beta dt\right) - g(x)\beta \exp\left(\int^x -\beta dt\right) + \beta g(x) \exp\left(\int^x -\beta dt\right)$$

$= k_1 \exp\left(\int^x (\beta + P_0)\, dt\right)$. That is

$$g'(x) \exp\left(\int^x -\beta(t) dt\right) = k_1 \exp\left(\int^x (\beta(t) + P_0(t))\, dt\right),$$

or

$$g'(x) = k_1 \exp\left(\int^x (2\beta(t) + P_0(t))\, dt\right). \tag{19}$$

Direct integration of equation (19) gives

$$g(x) = k_2 + k_1 \int^x \exp\left(\int^t (2\beta(\tau) + P_0(\tau))\, d\tau\right) dt, \tag{20}$$

where k_1 and k_2 are constants of integration. Substituting equation (20) in equation (17) gives

$$y = k_2 \exp\left(-\int^x \beta(t)\, dt\right) + k_1 \exp\left(-\int^x \beta(t)\, dt\right) \int^x \exp\left(\int^t (2\beta(\tau) + P_0(\tau))\, d\tau\right) dt, \tag{21}$$

with $y_1(x) = \exp\left(-\int^x \beta(t)\, dt\right)$and

$$y_2(x) = \exp\left(-\int^x \beta(t)\, dt\right)\int^x \exp\left(\int^t (2\beta(\tau) + P_0(\tau))\, d\tau\right) dt$$
$$= y_1(x) \int^x \exp\left(\int^t (2\beta(\tau) + P_0(\tau))\, d\tau\right) dt.$$

III. Application

The partial differential equation of Black-Scholes is written as [19, 20]

$$\frac{\partial F(t,s)}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F(t,s)}{\partial s^2} + rs\frac{\partial F(t,s)}{\partial s} - rF(t,s) = 0, \quad (22)$$

where s is the stock price with random movement, σ is the positive constant volatility, r stands for short-term interest rate, F is the option price and t denotes time which is variable. In solving equation (22), we first transform it into an ordinary differential equation by proposing the following solution

$$F(t,s) = F(s)g(t), \quad (23)$$

where $F(s)$ depends only on s and $g(t)$ depends on$t \in [0,T]$, where 0 stands for present and T stands for expiry. Hence, differentiating equation (23) with respect to t gives

$$\frac{\partial F(t,s)}{\partial t} = F(s)\frac{dg(t)}{dt}. \quad (24)$$

Similarly, differentiating equation (23) with respect to s gives

$$\frac{\partial F(t,s)}{\partial s} = \frac{dF(s)}{ds} g(t) \text{ and } \frac{\partial^2 F(t,s)}{\partial s^2} = \frac{d^2 F(s)}{ds^2} g(t). \quad (25)$$

Substituting equations (23), (24) and (25) in equation (22) gives

$$F(s)\frac{dg(t)}{dt} + \frac{1}{2}\sigma^2 s^2 g(t)\frac{d^2F(s)}{ds^2} + rsg(t)\frac{dF(s)}{ds} - rg(t)F(s) = 0. \quad (26)$$

Divide equation (26) by $g(t)F(s)$ to have

$$\frac{1}{g(t)}\frac{dg(t)}{dt} = -\frac{1}{2}\sigma^2 s^2 \frac{1}{F(s)}\frac{d^2F(s)}{ds^2} - rs\frac{1}{F(s)}\frac{dF(s)}{ds} + r. \quad (27)$$

The left-hand side of equation (27) is a function of t while the right-hand side of the same equation is a function of s. Hence, by separation of variables, we have

$$\frac{1}{g(t)}\frac{dg(t)}{dt} = -\frac{1}{2}\sigma^2 s^2 \frac{1}{F(s)}\frac{d^2F(s)}{ds^2} - rs\frac{1}{F(s)}\frac{dF(s)}{ds} + r = \rho, \quad (28)$$

whereρ is the separation constant. From equation (28), we have two ordinary differential equations

$$\frac{dg(t)}{dt} = \rho g(t) \quad (29)$$

and

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2F(s)}{ds^2} + rs\frac{dF(s)}{ds} + (\rho - r)F(s) = 0. \quad (30)$$

Equation (29) is a first-order linear equation. Its solution is

$$g(t) = e^{\rho t}, \quad (31)$$

where $t \in [0,T]$ and $\rho > 0$.

3.1 Constant interest rate, r

Let r be a constant rate, from equation (30) we have

$$s^2\frac{d^2F(s)}{ds^2} + \frac{2r}{\sigma^2}s\frac{dF(s)}{ds} + \frac{2}{\sigma^2}(\rho - r)F(s) = 0. \quad (32)$$

Equation (32) can be called Euler-type ordinary differential equation. We then solve equation (32). Let the solution of equation (32) be $F(s) = s^\lambda$, then the first derivative becomes $F'(s) = \lambda s^{\lambda-1}$and the second derivative reads $F''(s) = \lambda(\lambda-1)s^{\lambda-2}$. Substituting in equation (32) gives

$$\left(\lambda(\lambda-1) + \frac{2r}{\sigma^2}\lambda + \frac{2}{\sigma^2}(\rho - r)\right)s^\lambda = 0. \quad (33)$$

But $s > 0$ for all λ, hence

$$\lambda(\lambda-1) + \frac{2r}{\sigma^2}\lambda + \frac{2}{\sigma^2}(\rho - r) = 0, \quad (34)$$

that is, $\lambda^2 - \lambda + \frac{2r}{\sigma^2}\lambda + \frac{2}{\sigma^2}(\rho - r) = 0$ or

$$\lambda^2 + \left(\frac{2r}{\sigma^2} - 1\right)\lambda + \frac{2}{\sigma^2}(\rho - r) = 0. \quad (35)$$

Equation (35) is quadratic in λ, solving it yields $\lambda = \frac{\sigma^2 - 2r \pm \sqrt{(2r+\sigma^2)^2 + 8\rho\sigma^2 - 8r\sigma^2}}{2\sigma^2}$.

Therefore, the solutions of equation (35) becomes

$$\lambda_1 = \frac{\sigma^2 - 2r - \sqrt{(2r+\sigma^2)^2 + 8\rho\sigma^2 - 8r\sigma^2}}{2\sigma^2} \text{ and } \lambda_2 = \frac{\sigma^2 - 2r + \sqrt{(2r+\sigma^2)^2 + 8\rho\sigma^2 - 8r\sigma^2}}{2\sigma^2}. \quad (36)$$

For two distinct real solutions, $(2r+\sigma^2)^2 + 8\rho\sigma^2 - 8r\sigma^2 > 0$, or $(2r+\sigma^2)^2 + 8\rho\sigma^2 > 8r\sigma^2$. The general solution is the linear combination $F(s) = k_1 s^{\lambda_1} + k_2 s^{\lambda_2}$. That is,

$F(s) = k_1 s^{\frac{\sigma^2 - 2r - \sqrt{(2r+\sigma^2)^2 + 8\rho\sigma^2 - 8r\sigma^2}}{2\sigma^2}} + k_2 s^{\frac{\sigma^2 - 2r + \sqrt{(2r+\sigma^2)^2 + 8\rho\sigma^2 - 8r\sigma^2}}{2\sigma^2}}$ where k_1 and k_2 are constants.

3.2 Interest rate, r is a linear function

Let the interest rate be a linear function, that is, $r = m - cs$, where $c > 0$ and $m > 0$. Equation (32) becomes

$$\sigma^2 s^2 F''(s) + (-2cs^2 + 2ms)F'(s) + \big(2cs + 2(\rho - m)\big)F(s) = 0. \qquad (37)$$

For $\rho \neq 0$, equation (37) has two singular points, $s = 0$ is a regular singular point with the indicial equation

$$\eta(\eta - 1) + \frac{2m}{\sigma^2}\eta + \frac{2(\rho - m)}{\sigma^2} = 0, \qquad (38)$$

and the point $s = \infty$ is an irregular singular point. Let $\eta_i, i = 1,2,\cdots$ be the exponents at the singularity for the regular singular point, $s = 0$ where $s_1 - s_2$ is not integer or zero. Hence, from the method of Frobenius, there exist two linearly independent solutions

$$F_1(s) = s^{\eta_1}\sum_{j=0}^{\infty}\alpha_j s^j, \quad F_2(s) = s^{\eta_2}\sum_{j=0}^{\infty}\beta_j s^j, \qquad (39)$$

where α_j and β_j are coefficients to be determined. To determine the coefficients, let the general solution be

$$F(s) = s^{\eta}\sum_{j=0}^{\infty} f_j s^j = \sum_{j=0}^{\infty} f_j s^{\eta+j}. \qquad (40)$$

Hence,

$$F'(s) = \sum_{j=0}^{\infty}(\eta + j)f_j s^{\eta+j-1} \qquad (41)$$

and

$$F''(s) = \sum_{j=0}^{\infty}(\eta + j)(\eta + j - 1)f_j s^{\eta+j-2}. \qquad (42)$$

Substituting equations (40), (41), and (42) in equation (37) gives

$$\sigma^2 s^2 \sum_{j=0}^{\infty}(\eta + j)(\eta + j - 1)f_j s^{\eta+j-2} + (-2cs^2 + 2ms)\sum_{j=0}^{\infty}(\eta + j)f_j s^{\eta+j-1} + \big(2cs + 2(\rho - m)\big)\sum_{j=0}^{\infty} f_j s^{\eta+j} = 0. \qquad (43)$$

Simplifying equation (43) gives

$$\sum_{j=0}^{\infty}\sigma^2(\eta + j)(\eta + j - 1)f_j s^{\eta+j} - \sum_{j=0}^{\infty}2c(\eta + j)f_j s^{\eta+j+1} + \sum_{j=0}^{\infty}2m(\eta + j)f_j s^{\eta+j} + \sum_{j=0}^{\infty}2cf_j s^{\eta+j+1} + \sum_{j=0}^{\infty}2(\rho - m)f_j s^{\eta+j} = 0. \qquad (44)$$

By shifting the indices for the power of s to be unified in equation (44), we have

$$\sum_{j=0}^{\infty}\sigma^2(\eta + j)(\eta + j - 1)f_j s^{\eta+j} - \sum_{j=1}^{\infty}2c(\eta + j - 1)f_{j-1}s^{\eta+j} + \sum_{j=0}^{\infty}2m(\eta + j)f_j s^{\eta+j} + \sum_{j=1}^{\infty}2cf_{j-1}s^{\eta+j} + \sum_{j=0}^{\infty}2(\rho - m)f_j s^{\eta+j} = 0.$$

Isolating the first terms starting from zero gives

$$\{\sigma^2\eta(\eta - 1) + 2m\eta + 2(\rho - m)\}f_0 s^{\eta} + \sum_{j=1}^{\infty}\{\sigma^2(\eta + j)(\eta + j - 1)f_j - 2c(\eta + j - 1)f_{j-1} + 2m(\eta + j)f_j + 2cf_{j-1} + 2(\rho - m)f_j\}s^{\eta+j} = 0. \qquad (45)$$

Collecting like terms in equation (45), we have

$$\{\sigma^2\eta(\eta - 1) + 2m\eta + 2(\rho - m)\}f_0 s^{\eta} + \sum_{j=1}^{\infty}[\{\sigma^2(\eta + j)(\eta + j - 1) + 2m(\eta + j) + 2(\rho - m)\}f_j + \big(2c - 2c(\eta + j - 1)\big)f_{j-1}]s^{\eta+j} = 0. \qquad (46)$$

The coefficient of the lowest power s^{η} of equation (46) is the indicial equation. That is $\{\sigma^2\eta(\eta - 1) + 2m\eta + 2(\rho - m)\}f_0 = 0$ or $\eta(\eta-1) + \frac{2m}{\sigma^2}\eta + \frac{2(\rho - m)}{\sigma^2} = 0$, which is the same as equation (38). The remaining terms produce the following two-term recurrence relation:

$$f_j = -\frac{-2c(j + \eta - 1) + 2c}{\sigma^2(j + \eta)(j + \eta - 1) + 2m(j + \eta) + 2(\rho - m)}f_{j-1}, f_0 \neq 0. \qquad (47)$$

The solution of equation (47) can be found recursively as:

$$f_j = f_0\prod_{i=0}^{j-1}\frac{2c(i + \eta) - 2c}{\sigma^2(i + 1 + \eta)(i + \eta) + 2m(i + 1 + \eta) + 2(\rho - m)}, \qquad (48)$$

where

$$\eta = \frac{\sigma^2 - 2m \pm \sqrt{(2m + \sigma^2)^2 + 8\rho\sigma^2 - 8m\sigma^2}}{2\sigma^2}. \qquad (49)$$

Equation (48) can be expressed in terms of the Pochhammer symbol defined interms of the gamma function $(\beta)_n = \beta(\beta + 1)\cdots(\beta + n - 1) = \frac{\Gamma(\beta + n)}{\Gamma(\beta)}$, with $f_0 = 1$ as

$$f_j = \frac{(2c)^j(\eta - 1)\sigma^{2-2j}(\eta)_{j-1}}{\left(\frac{3+2\eta}{2} + \frac{m}{\sigma^2} - \frac{\sqrt{(2m+\sigma^2)^2 + 8\rho\sigma^2 - 8m\sigma^2}}{2\sigma^2}\right)_{j-1}\left(\frac{3+2\eta}{2} + \frac{m}{\sigma^2} + \frac{\sqrt{(2m+\sigma^2)^2 + 8\rho\sigma^2 - 8m\sigma^2}}{2\sigma^2}\right)_{j-1}}, j = 1,2,\cdots. \qquad (50)$$

The general series solution $F(s) = s^{\eta}\big(1 + \sum_{j=1}^{\infty} f_j s^j\big)$ can be written in terms of the generalized hypergeometric function $2\mathbb{F}_2$ as

$$F(s)=s^{\eta}\{1+\frac{2c(\eta-1)s}{2\rho+2m\eta+\eta(\eta+1)\sigma^2}\ {}_2\mathbb{F}_2(1,\eta;\frac{m}{\sigma^2}+\eta+\frac{3}{2}-\frac{\sqrt{(2m+\sigma^2)^2+8\rho\sigma^2-8m\sigma^2}}{2\sigma^2},\frac{m}{\sigma^2}$$
$$+\eta+\frac{3}{2}+\frac{\sqrt{(2m+\sigma^2)^2+8\rho\sigma^2-8m\sigma^2}}{2\sigma^2};\ \frac{2c}{\sigma^2}s)\}. \quad (51)$$

Let $\eta_1=\eta=\eta_2$, then from equations (39), (49) and (51), the two linearly independent solutions of equation (51) are

$$F_1(s)=s^{\frac{\sigma^2-2m+\sqrt{(2m+\sigma^2)^2+8\rho\sigma^2-8m\sigma^2}}{2\sigma^2}}\ {}_1\mathbb{F}_1(\frac{\sqrt{(2m+\sigma^2)^2+8\rho\sigma^2-8m\sigma^2}}{2\sigma^2}-\frac{m}{\sigma^2}-\frac{1}{2};1$$
$$+\frac{\sqrt{(2m+\sigma^2)^2+8\rho\sigma^2-8m\sigma^2}}{\sigma^2};\frac{2c}{\sigma^2}s), \text{ and}$$
$$F_2(s)=s^{\frac{\sigma^2-2m-\sqrt{(2m+\sigma^2)^2+8\rho\sigma^2-8m\sigma^2}}{2\sigma^2}}\ {}_1\mathbb{F}_1(-\frac{\sqrt{(2m+\sigma^2)^2+8\rho\sigma^2-8m\sigma^2}}{2\sigma^2}-\frac{m}{\sigma^2}-\frac{1}{2};1$$
$$-\frac{\sqrt{(2m+\sigma^2)^2+8\rho\sigma^2-8m\sigma^2}}{\sigma^2};\frac{2c}{\sigma^2}s).$$

3.3 Interest rate, r is a reciprocal function

We consider the third case of the interest rate, r. Let $r=\beta s^{-\alpha}$ where β and α are real positive constants. From equation (30),

$$F''(s)=\frac{-2rs}{\sigma^2s^2}F'(s)-\frac{2(\rho-r)}{\sigma^2s^2}F(s)$$
$$=\frac{-2r}{\sigma^2s}F'(s)-\frac{2(\rho-r)}{\sigma^2s^2}F(s). \quad (52a)$$

Since $r=\beta s^{-\alpha}$, equation (52a) becomes

$$F''(s)=\frac{-2\beta s^{-\alpha}}{\sigma^2s}F'(s)-\frac{2(\rho-\beta s^{-\alpha})}{\sigma^2s^2}F(s)$$
$$=\frac{-2\beta}{\sigma^2s^{\alpha+1}}F'(s)-\frac{2(\rho s^{\alpha}-\beta)}{\sigma^2s^{\alpha+2}}F(s). \quad (52)$$

We then solve equation (52) using the asymptotic iteration method. Comparing equations (1) and (52) gives

$$\begin{cases} P_0(s)=\frac{-2\beta}{\sigma^2s^{\alpha+1}} \\ Q_0(s)=\frac{-2(\rho s^{\alpha}-\beta)}{\sigma^2s^{\alpha+2}} \end{cases} \quad (53)$$

From equation (8), with $P_{-1}=1$ and $Q_{-1}=0$, if $\alpha=1$, we have

$$P_1(s)=P_0'+P_0^2+Q_0$$
$$=\frac{4\beta}{\sigma^2s^3}+\frac{4\beta^2}{\sigma^4s^4}-\frac{2\rho}{\sigma^2s^2}+\frac{2\beta}{\sigma^2s^3} \quad (54a)$$
$$=\frac{-2\rho\sigma^2s^2+6\beta\sigma^2s+4\beta^2}{\sigma^4s^4}.$$

Similarly,

$$Q_1(s)=Q_0'+Q_0P_0$$
$$=\frac{4\rho}{\sigma^2s^3}-\frac{6\beta}{\sigma^2s^4}+\frac{4\beta(\rho s-\beta)}{\sigma^4s^5} \quad (54b)$$
$$=\frac{4\rho\sigma^2s^2+2(2\rho-3\sigma^2)\beta s-4\beta^2}{\sigma^4s^5}.$$

From equation (12), for $n=1$, the terminating condition implies that

$$\delta_1(s)=P_1(s)Q_0(s)-P_0(s)Q_1(s)$$
$$=\frac{-8\rho\sigma^2s^2\beta+4\rho^2\sigma^2s^3}{\sigma^6s^7} \quad (55)$$
$$=\frac{-4\rho(2\beta-\rho s)}{\sigma^4s^5}.$$

If $\rho=0$, $\delta_1(s)=0$ in equation (55). From equation (16), we have

$$y_1(s)=\exp\left(-\int^s\frac{Q_0(t)}{P_0(t)}dt\right)=\exp(\ln s)=s. \quad (56)$$

Further, $\alpha=1$ gives

$$P_2(s)=P_1'(s)+P_1(s)P_0(s)+Q_1(s)$$
$$=\frac{4\rho}{\sigma^2s^3}-\frac{18\beta}{\sigma^2s^4}-\frac{16\beta^2}{\sigma^4s^5}-\frac{2\beta(-2\rho\sigma^2s^2+6\beta\sigma^2s+4\beta^2)}{\sigma^6s^6}+\frac{4\rho\sigma^2s^2+2(2\rho-3\sigma^2)\beta s-4\beta^2}{\sigma^4s^5}$$
$$=\frac{-8\sigma^4s^3\rho+24\sigma^4s^2\beta+8\sigma^2s\beta^2-\rho\beta\sigma^2s^2+3\beta^2\sigma^2s+\beta^3}{\sigma^6s^6} \quad (57a)$$
$$=-\frac{8(\sigma^2s+\beta)(-\rho\sigma^2s^2+3\beta\sigma^2s+\beta^2)}{\sigma^6s^6}.$$
$$Q_2(s)=Q_1'(s)+Q_0(s)P_1(s)$$
$$=\left(\frac{4\rho\sigma^2s^2+2(2\rho-3\sigma^2)\beta s-4\beta^2}{\sigma^4s^5}\right)'+\left(\frac{-2(\rho s-\beta)}{\sigma^2s^3}\right)\left(\frac{-2\rho\sigma^2s^2+6\beta\sigma^2s+4\beta^2}{\sigma^4s^4}\right) \quad (57b)$$
$$=\frac{4\rho\sigma^2(\rho-3\sigma^2)s^3-8\sigma^2(4\rho-3\sigma^2)\beta s^2-8\beta^2(\rho-4\sigma^2)s+8\beta^3}{\sigma^6s^7}.$$

The terminating condition for $n = 2$ implies that

$$\delta_2(s) = P_2(s)Q_1(s) - P_1(s)Q_2(s)$$
$$= \frac{8\rho^2\sigma^8 s^8 - 24\rho^2\beta\sigma^6 s^7 - 24\beta\rho\sigma^8 s^7 + 8\rho^3\sigma^6 s^8}{\sigma^6 s^7}$$
$$= 8\rho(\rho\sigma^2 s - 3\rho\beta - 3\beta\sigma^2 + \rho^2 s) \qquad (58)$$
$$= -8\rho(\sigma^2 + \rho)(3\beta - \rho s).$$

Therefore, $\delta_2(s) = 0$ if $\rho = 0$ or $\rho = -\sigma^2$. For $\rho = -\sigma^2$,

$$y_2(s) = \exp\left(-\int^s \frac{Q_1(t)}{P_1(t)}dt\right) = \exp\left(-\int^s \frac{-2\sigma^2 t - \beta}{(\sigma^2 t + \beta)t}dt\right)$$
$$= e^{\ln(\sigma^2 t^2 + \beta t)}|^s \qquad (59)$$
$$= (\sigma^2 s + \beta)s.$$

Continuing the process for $\alpha = 1$, the terminating condition

$$\delta_{n+1}(s) = \prod_{c=0}^{n}\left(\rho + \frac{c(c+1)}{2}\sigma^2\right) \equiv 0\,,\ n = 0,1,2,\cdots \qquad (60)$$

leads to the necessary condition

$$\rho \equiv \rho_n = -\frac{n(n+1)}{2}\sigma^2,\ n = 0,1,2,\cdots \qquad (61)$$

for the existence of polynomial solutions of equation (52) for $\alpha = 1$. That is

$$\sigma^2 s^3 F''(s) + 2\beta s F'(s) + 2(\rho s - \beta)F(s) = 0. \qquad (62)$$

Therefore, we have the first polynomial solutions below

$$\begin{cases} n = 0,\ \rho = 0,\ \ y_1(s) = s, \\ n = 1,\quad \rho = -\sigma^2,\quad y_2(s) = s(\sigma^2 s + \beta), \\ n = 2,\quad \rho = -3\sigma^2,\quad y_3(s) = s(3\sigma^4 s^2 + 3\beta\sigma^2 s + \beta^2), \\ n = 3,\quad \rho = -6\sigma^2,\ \ y_4(s) = s(15\sigma^6 s^3 + 15\beta\sigma^4 s^2 + 6\beta^2\sigma^2 s + \beta^3), \\ n = 4,\quad \rho = -10\sigma^2,\ \ y_5(s) = s(105\sigma^8 s^4 + 105\beta\sigma^6 s^3 + 45\beta^2\sigma^4 s^2 + 10\beta^3\sigma^2 s + \beta^4), \\ n = 5, \rho = -15\sigma^2,\ \ y_6(s) = s(945\sigma^{10} s^5 + 945\beta\sigma^8 s^4 + 420\beta^2\sigma^6 s^3 + 105\beta^3\sigma^4 s^2 + 15\beta^4\sigma^2 s + \beta^5), \\ \vdots \qquad \vdots \qquad \vdots \end{cases} \qquad (63)$$

The above polynomial solution are generated using confluent hypergeometric function below up to a multiplication constant

$$y_n(s) = b^{-n-1}s^{n+1}\sigma^{2(n+1)}\ \ {}_1\mathbb{F}_1\left(-n; -2n; \frac{2b}{s\sigma^2}\right), n = 0,1,2,\cdots \qquad (64)$$

For $\alpha = 2$, we apply the asymptotic iteration process again. From equation (53), we have

$$\begin{cases} P_0(s) = \frac{-2\beta}{\sigma^2 s^3} \\ Q_0(s) = \frac{-2(\rho s^2 - \beta)}{\sigma^2 s^4} \end{cases} \qquad (65)$$

Therefore, from equation (8), we have

$$P_1(s) = P_0' + P_0 P_0 + Q_0$$
$$= \frac{8\beta}{\sigma^2 s^4} + \frac{4\beta^2}{\sigma^4 s^6} - \frac{2\rho s^2}{\sigma^2 s^4} \qquad (66a)$$
$$= \frac{-2\rho\sigma^2 s^4 + 8\beta\sigma^2 s^2 + 4\beta^2}{\sigma^4 s^6}.$$

Similarly,

$$Q_1(s) = Q_0' + Q_0 P_0$$
$$= \left(\frac{-2\rho s^2}{\sigma^2 s^4} + \frac{2\beta}{\sigma^2 s^4}\right)' + \left(\frac{-2\rho s^2}{\sigma^2 s^4} + \frac{2\beta}{\sigma^2 s^4}\right)\left(\frac{-2\beta}{\sigma^2 s^3}\right) \qquad (66b)$$
$$= \frac{4\rho\sigma^2 s^4 + 4\beta(\rho - 2\sigma^2)s^2 - 4\beta^2}{\sigma^4 s^7}.$$

The first iteration using equations (66a) and (66b) implies

$$\delta_1(s) = P_1(s)Q_0(s) - P_0(s)Q_1(s)$$
$$= \frac{4\rho^2\sigma^2 s^6 - 12\rho\beta\sigma^2 s^4}{\sigma^6 s^{10}}$$
$$= \frac{-4\rho(3\beta - \rho s^2)}{\sigma^4 s^6}.$$

Hence, $\delta_1(s) = 0$ if $\rho = 0$. We then have

$$y_1(s) = \exp\left(-\int^s \frac{Q_0(t)}{P_0(t)}dt\right) = \exp\left(-\int^s \frac{2\beta}{-2\beta t}dt\right)$$
$$= \exp\left(\int^s \frac{1}{t}dt\right) = s.$$

The second iteration gives

$$\delta_2(s) = P_2(s)Q_1(s) - P_1(s)Q_2(s)$$
$$= \frac{8\rho(\rho\sigma^2 s^4 + \rho^2 s^4 - 6\beta\sigma^2 s^2 - 6\beta\rho s^2 + 3\beta^2)}{\sigma^6 s^{10}},$$

which is not zero unless $\rho = 0$, and adds nothing new to the first iteration result. Therefore, the iteration process yields

$$\delta_3(s) \sim \rho(3\sigma^2 + \rho),$$
$$\delta_5(s) \sim \rho(3\sigma^2 + \rho)(10\sigma^2 + \rho),$$
$$\delta_7(s) \sim \rho(3\sigma^2 + \rho)(10\sigma^2 + \rho)(21\sigma^2 + \rho),$$
$$\delta_9(s) \sim \rho(3\sigma^2 + \rho)(10\sigma^2 + \rho)(21\sigma^2 + \rho)(36\sigma^2 + \rho),$$
$$\vdots \quad \sim \quad \vdots$$

In general, we have $\delta_{2n+1}(s) \sim \prod_{c=0}^{n}\left(\rho + \frac{2c(2c+1)}{2}\sigma^2\right)$, $n = 0,1,2,\cdots$. From equation (52), for $\alpha = 2$, we have

$$\sigma^2 s^4 F''(s) + 2\beta s F'(s) + 2(\rho s^2 - \beta)F(s) = 0. \qquad (67)$$

The necessary condition of the polynomial solutions of equation (67) is

$$\rho \equiv \rho_n = \frac{-2n(2n+1)\sigma^2}{2}, n = 0,1,2,\cdots.$$

The first polynomial solutions then become

$$\begin{cases} n = 0, \rho = 0, y_1(s) = s, \\ n = 1, \rho = -3\sigma^2, y_3(s) = s(3\sigma^2 s^2 + 2\beta), \\ n = 2, \rho = -10\sigma^2, y_5(s) = s(35\sigma^4 s^4 + 20\beta\sigma^2 s^2 + 4\beta^2), \\ n = 3, \rho = -21\sigma^2, y_7(s) = s(693\sigma^6 s^6 + 378\beta\sigma^4 s^4 + 84\beta^2\sigma^2 s^2 + 8\beta^3), \\ n = 4, \rho = -36\sigma^2, y_9(s) = s(19305\sigma^8 s^8 + 10296\beta\sigma^6 s^6 + 2376\beta^2\sigma^4 s^5 + 288\beta^3\sigma^2 s^2 + 16\beta^4), \\ \quad \vdots \qquad\qquad \vdots \end{cases} \qquad (68)$$

The polynomial solutions in equations (68) are again generated up to a multiplication constant by the confluent hypergeometric function given below:

$$y_n(s) = \beta^{-\frac{1}{2}(2n+1)} s^{2n+1} \sigma^{2n+1} \quad {}_1\mathbb{F}_1\left(-n; \frac{1}{2} - 2n; \frac{\beta}{s^2\sigma^2}\right); n = 0,1,2,\cdots. \qquad (69)$$

Similarly, for $\alpha = 3$, equation (52) becomes

$$\sigma^2 s^5 F''(s) + 2\beta s F'(s) + 2(\rho s^3 - \beta)F(s) = 0. \qquad (70)$$

The necessary condition for the polynomial solutions of equation (70) is

$\rho \equiv \rho_n = \frac{-3n(3n+1)\sigma^2}{2}, n = 0,1,2,\cdots$ with its polynomial solutions given by

$$y_n(s) = \beta^{-\frac{1}{3}(3n+1)} s^{3n+1} \sigma^{2(3n+1)/3} \quad {}_1\mathbb{F}_1\left(-n; \frac{2}{3} - 2n; \frac{2\beta}{3s^3\sigma^2}\right); n = 0,1,2,\cdots. \qquad (71)$$

Therefore, the general polynomial solutions of equation (52) for $\alpha = 1,2,3,\cdots$ are generated by the function

$$y_n(s) = \beta^{-\frac{1}{\alpha}(\alpha n+1)} s^{\alpha n+1} \sigma^{2(\alpha n+1)/\alpha} \quad {}_1\mathbb{F}_1\left(-n; \frac{\alpha-1}{\alpha} - 2n; \frac{2\beta}{\alpha s^\alpha \sigma^2}\right); n = 0,1,2,\cdots \qquad (72)$$

subject to the necessary condition

$$\rho \equiv \rho_n = \frac{-\alpha n(\alpha n+1)\sigma^2}{2}, n = 0,1,2,\cdots, \alpha = 1,2,3,\cdots. \qquad (73)$$

IV. Conclusion

The asymptotic iteration method was introduced to solve analytically and/or approximately the second-order linear homogenous differential equation. Considering the method to determine the complete solution, it is sufficient to calculate $y_1(x)$, since $y_2(x)$ can be obtained by knowing $y_1(x)$. We considered the three cases of the interest rate in the option pricing namely: constant rate, linear function and the reciprocal function. In the case of a constant rate, Black-Scholes equation resembles the Euler-type ordinary differential equation. When the interest rate is a reciprocal function, we rely on the asymptotic iteration method to investigate the analytic solution. Considering the case where the interest rate is a linear function, as the interest rate rises, the opportunity cost of holding money instead of investing in securities like stocks increases. This will in turn drop the stock prices. On the other hand, the opportunity cost of holding money rather than investing in securities like stocks decreases as the interest rate drops.

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