IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 18, Issue 5 Ser. I (Sep. – Oct. 2022), PP 29-33
www.iosrjournals.org

General Form of Integral Solutions to the Ternary Non-Homogeneous Cubic Equation

$$x^2+y^2+xy+x-y+1=(m^2+3n^2)z^3$$

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Abstract:
The purpose of this paper is to obtain a general form of non-zero distinct integral solutions of ternary non-homogeneous cubic diophantine equation $x^2+y^2+xy+x-y+1=(m^2+3n^2)z^3$.

Keywords: *Ternary cubic equation, Non-homogeneous cubic, Integer solutions*

Date of Submission: 23-09-2022 Date of Acceptance: 08-10-2022

Notations:

T_n -Triangular number of rank n , Ob_n -Oblong number of rank n

Th_n -Tetrahedral number of rank n , PP_n -Pentagonal Pyramidal number of rank n

J_n -Jacobsthal number of rank n , j_n -Jacobsthal-Lucas number of rank n

I. Introduction:

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, cubic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-4]. In this context, one may refer [5-24] for various problems on the cubic diophantine equations with three variables, where, in each of the problems, different sets of non-zero integer solutions are obtained. However, often we come across homogeneous and non-homogeneous cubic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining a general form of non-trivial integral solutions of the non-homogeneous cubic equation with three unknowns given by $x^2+y^2+xy+x-y+1=(m^2+3n^2)z^3$.

Method of Analysis:

The ternary non-homogeneous cubic diophantine equation to be solved for its distinct non-zero integral solution is

$$x^2+y^2+xy+x-y+1=(m^2+3n^2)z^3 \quad (1)$$

where m, n are not simultaneously zero.

Introduction of the linear transformations

$$x=u+v-1, y=u-v+1 \quad (2)$$

in (1) leads to

$$v^2+3u^2=(m^2+3n^2)z^3 \quad (3)$$

Let v_0, u_0, z_0 be any given non-zero integer solution to (3) so that

$$v_0^{\,2} + 3u_0^{\,2} = (m^2 + 3n^2) z_0^{\,3} \qquad (4)$$

Now, consider

$$v = A v_0 \pm 3B u_0, u = B v_0 \mp A u_0 \qquad (5)$$

where A, B are non-zero integers to be determined such that (5) satisfies (3).
Substituting (5) in (3),we have

$$\text{L.H.S. of } (3) = (v_0^{\,2} + 3u_0^{\,2})(A^2 + 3B^2) \qquad (6)$$

In view of (4), it is seen that

$$\text{L.H.S. of } (1) = (m^2 + 3n^2) z_0^{\,3} (A^2 + 3B^2) \qquad (7)$$

On comparing the R.H.S. of (3) and (7), note that we have to choose A and B so that $(A^2 + 3B^2)$ is a perfect cubical integer. Choosing

$$A = a(a^2 + 3b^2), B = b(a^2 + 3b^2) \qquad (8)$$

it is seen that

$$A^2 + 3B^2 = (a^2 + 3b^2)^3$$

and thus, one obtains

$$z = (a^2 + 3b^2) z_0 \qquad (9)$$

Substituting (8) in (5) , we have

$$\left.\begin{aligned} u &= b(a^2 + 3b^2) v_0 \mp a(a^2 + 3b^2) u_0, \\ v &= a(a^2 + 3b^2) v_0 \pm 3b(a^2 + 3b^2) u_0 \end{aligned}\right\} \qquad (10)$$

In view of (2) , we get

$$\left.\begin{aligned} x &= (b + a)(a^2 + 3b^2) v_0 + (a^2 + 3b^2)(\mp a \pm 3b) u_0 - 1, \\ y &= (b - a)(a^2 + 3b^2) v_0 \mp (a^2 + 3b^2)(a + 3b) u_0 + 1 \end{aligned}\right\} \qquad (11)$$

Thus, (9) and (11) represent the general form of integral solutions to (1).

Note :

It is worth to mention that

$A^2 + DB^2$ is a perfect cubical integer when

$$A = a(a^2 - 9b^2), B = b(3a^2 - 3b^2)$$

In this case, the general form of integer solution to (1) is given by

$$\left.\begin{aligned} x &= [a(a^2 - 9b^2) + b(3a^2 - 3b^2)] v_0 \pm [-a(a^2 - 9b^2) + 3b(3a^2 - 3b^2)] u_0 - 1 \\ y &= [-a(a^2 - 9b^2) + b(3a^2 - 3b^2)] v_0 \mp [a(a^2 - 9b^2) + 3b(3a^2 - 3b^2)] u_0 + 1 \\ z &= (a^2 + 3b^2) z_0 \end{aligned}\right\} \qquad (12)$$

A few examples are presented in the following Table:

Table-Examples

m	n	u_0	v_0	z_0	a	b	x	y	z
2	1	1	2	1	1	1	23	-15	4
							7	**17**	4
							-9	25	4
							-25	**9**	4
1	**0**	**0**	1	1	1	1	**-9**	**9**	4
							15	-15	4

To analyze the nature of solutions, one has to go in for particular values of m, n, u_0, v_0, z_0 .Here we present the solutions to (1) and a few properties for $m = 1, n = 0, v_0 = 1, u_0 = 0, z_0 = 1$ from (12). The solutions to (1) under consideration is

$$\left.\begin{aligned} x &= a^3 - 9ab^2 + 3a^2b - 3b^3 - 1 \\ y &= 3a^2b - 3b^3 - a^3 + 9ab^2 + 1 \\ z &= a^2 + 3b^2 \end{aligned}\right) \quad (13)$$

We observe the following relations among the solutions.

1) $x + y$ is a Nasty number when $a = \left(2^{4\alpha-2} + 1\right)q^2, b = 2^{2\alpha} q^2$

2) $\frac{3b(x - y + 2)}{2}$ is a Nasty number.

3) When $a = 3b$, $x - y + 2 = 0$ and therefore $x^3 - y^3 + 8 = -6xy$

4) When $a = 3b$, $2z$ is a Nasty number.

5) When $a = 2b$, $y + bz - x - 2$ and $\frac{3(x + y)}{2}$ are cubical integers.

6) When $a = 3b$, $36(x + y)$ is a cubical integer and $z - 6b^2(y - x) = 0$

7) $\frac{a(x + y)}{6}$ is a Nasty number.

8) $a(x + y)$ is 6 times the area of the Pythagorean Triangle $\left(2ab, a^2 - b^2, a^2 + b^2\right)$

9) $x - y + 6az + 2$ is a cubical integer.

10) $(x - y + 2)^2 = 4(z - 3b^2)(z - 12b^2)^2$

11) Representing the solutions x, y, z in (13) by the notations $x(a,b), y(a,b), z(a,b)$ respectively, the following relations are observed.

a) $x(a,b) + x(-a,-b) = -2$

b) If $a > 3b, \frac{3b}{2}[x(a,b) - x(-a,b)]$ is a Nasty number.

c) $x(-a,b) + y(a,-b) = 0$

d) $\frac{a}{6}[y(a,b) - y(a,-b)]$ is a Nasty number.

e) $x(a,-b) + y(a,b) = 0$

f) $x(-a,b) + x(a,-b) = -2$

g) $y(-a,b) - x(a,b) = 2$

h) $y(a,-b) + y(9 - a,b) = 2$

i) If $a > b, \frac{a}{6}[y(a,b) + x(a,b)]$ is a Nasty number.

j) $x(-a,b) - y(a,b) = -2$

k) $x(a,b) + y(a,-b) = 0$

l) $a[y(a,1) - y(a,-1) = 12\ (PP_a - T_a)$

m) $a[y(a,1) - y(a,-1)] = 12(PP_a - 6Ob_a)$

n) $a[y(a,1) + x(a,1)] = 36Th_{(a-1)} = a[y(a,1) - y(a,-1)]$

o) $y(2^n,1) + x(2^n,1) = 6(3J_{2n}) = 6(j_{2n} - 2)$

p) $b[y(1+2b,b) + x(1+2b,b) + y(1,b) + x(1,b) = 48(T_b)^2$

q) $z(a + 3b, a - b) = 4a^2 + 12b^2 = 4z$

r) $z(a + 6b, a - 2b) - 4z(a,b) = (6b)^2$, a perfect square.

s) $z(a + 6b, a - 2b) - 16z \equiv 0 \pmod{12}$

t) $z(a+9b,a-3b)-4z$ is a Nasty number.

II. Conclusion:

In this paper, we have made an attempt to find a general form of non-zero distinct integer solutions to the non-homogeneous cubic equation with three unknowns given by $x^2+y^2+xy+x-y+1=(m^2+3n^2)z^3$. To conclude, one may search for other choices of general form of integer solutions to the cubic equation with three unknowns in title.

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