Stochastic Modelling Of Influenza Epidemic

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Abstract
In many tropical countries, influenza epidemic has remained a health concern. Allocating resources for its treatment and control has always targeted certain period of the year when its incidence is believed to be highest. In this study, we have used the stochastic rates of changes insix classes of the population N(t) to find suitable Markov jump process from which a stochastic differential equation was obtained for the symptomatic infectious class and solved by Ito's formula. Analysis of the sample path of the infectives shows that influenza is endemic in Nigeria throughout the year with occasional change in epidemic size.
Keywords: *Infectives, Poisson process, Brownian motion, Stochastic process.*

I.   Introduction

Influenza is a highly infectious viral disease of the lung,nose and throat which"causes between 250,000 – 500,000 deaths annually world-wide out of 3 to 5 million cases of severe illness annually"[6]. Nigeria is one of the worst hit by influenza epidemic.In 2018, the World Health Organisation(WHO) reported that influenza and pneumonia deaths reached 288,619 or 14.89% of total deaths in Nigeria[8].Its

symptoms include: fever, muscle pain, chills, and cough (dry or with phloem), fatigue, loss of appetite, runny nose or sneezing. It can also include, headache, vomiting, sore throat, shortness of breath or swollen lymph nodes. The disease has incubation period of 2 days on the average but can range from one to four days.

The disease is spread by inhalation of respiratory droplets of infected persons through cough or sneezing. It can also be contracted by a healthy person in skin-to-skin contact, saliva exchange or by touching contaminated surfaces.One in three influenza infected individuals is asymptomatic" [2] while"50% of seasonal influenza infections may be asymptomatic" [7]. Access, adherence to preventive information and timely treatment are very crucial to the control of influenza epidemic. Contrary to the usual practice of concentrating control measures on the cold seasons, this research focusses on using stochastic differential equations (SDE's) to findthe best time of the year when health authorities could deplore the most resources in the treatment and control of influenza.

II.   Methodology

We use a multitype form of Bartlett's compartmental epidemic model [1]. Therefore, we have a stochastic epidemic model with 6 compartments, each of which can be described by a linear stochastic differential equation with additive noise and multidimensional Brownian motion. The population N(t), $t \geq 0$ is divided into 6 non-overlapping categories such that

$N(t) = S_1(t) + S_2(t) + I_1(t) + I_1(t) + T(t) + D(t)$

At the initial time, $S_1(0) = s_1$, $S_2(0) = s_2$, $I_1(0) = i_1$, $I_2(0) = i_2$, $D(0) = T(0) = 0$

The classes and their sizes at time t include two susceptible classes namely: Informed Susceptible ($S_1(t)$) and Uniformed Susceptible ($S_2(t)$); two infectious classes namely: Symptomatic Infectives($I_1(t)$) and Asymptomatic Infectives ($I_2(t)$), and two removed classes namely: The Treated T(t) and the Dead (D(t)).

Members of the informed class have adequate knowledge ofinfluenza preventive measures such as regular washing of hands, maintaining social distancing, personal hygiene and they are assumed to abide by them. The class of the uninformed, $S_2(t)$ are individuals who are either unaware of influenza preventive measures or are aware but fail to follow the rules. An uninformed person can become informed and move to the informed category at the rate γ. The two susceptible classes can gain member through birth at the rate $\lambda_1$or lose members through non-influenza-related death at the rate λ_2.

The Symptomatic Infectious Class $I_1(t)$ consists of members of the population who are infected by influenza and clearly manifest its notable symptoms such as cough chills sneezing, muscle aches, runny nose, headache and fatigue. A person becomes a member of this class when a contact is made between an infected person and a member of either of the two susceptible classes, therefore resulting in infection at the Poisson

rateβ_1. The compartment$I_2(t)$ of the population consists of individuals who have contracted influenza but either show mild symptoms or do not manifest the influenza symptoms at all. They are assumed to also take part in the spread of the infection unnoticed. A member of the informed susceptible class or uninformed susceptible class can move to the $I_2(t)$ compartment when infectious contact is made with an infectious person at the Poisson rate β_2.

T(t) is the compartment containing people who have been treated either by medication or quarantine and are assumed to no longer participate in the transmission of the disease. A member of $I_1(t)$ or $I_2(t)$ will transit to this class at the rate $\mu_1$when hegets treated.The section of the population is for those who died as a result of the infection are represented, in name and size, by D(t). Obviously, a dead person can no longer participate in the spread of the disease. However, a member of the population can be in this compartment at the rate μ_2 if he died of influenza or rate λ_2 if he died of other causes.

The increments in each of the classes in disjoint time interval [t, t + Δt] are independent. Thus, for $\beta > 0$ and any compartment Y(t) of the model, the increment $\Delta Y(t)$ has the probabilities

P[Y(t + Δt) − Y(t) = 1] = $\beta\Delta t$ + o(Δt) P[Y(t + Δt) − Y(t) > 1] = o(Δt) P[Y(t + Δt) − Y(t) = 0] = 1 - $\beta\Delta t$ + o(Δt)

Whereβ, the Poisson rate, is an important factor of the stochastic rate of transition of an individual from Susceptible class to Infective class given by $\beta\frac{SI}{N}$, known as the stochastic infection rate where β defines the mean number of infectious contacts made by mean infectious individuals per unit time, N is the total population .Hence, using the stochastic rates as in [6]the probabilitiesof changes the compartmentsin a short time$\Delta t$are:

$P\{\Delta S_1(t)\} = [\lambda_1 N(t) \,_+ \gamma S_2 - \lambda_2 S1(t) - \left(\frac{\beta_1}{N}I_1(t)S_1(t) + \frac{\beta_2}{N}I_2(t)S_1(t)\right)]\Delta t \,_+ o(\Delta t)$

$P\{\Delta S_2(t)\} = [\lambda_1 N(t) - \lambda_2 S_2(t) - \gamma S_2 \,_- \left(\frac{\beta_1}{N}I_1(t)S_2(t) + \frac{\beta_2}{N}I_2(t)S_2(t)\right)]\Delta t \,_+ o(\Delta t)$

$P\{\Delta I_1(t)\} = [\theta_1 N(t) + \lambda_1 I_1(t) + \frac{\beta_1}{N}I_1(t)S_1(t) + \frac{\beta_1}{N}I_1(t)S_2(t) - \left(\mu_1 I_1(t) + \mu_2 I_1(t) + \lambda_2 I_2(t)\right)]\Delta t + o(\Delta t)$

$P\{\Delta I_2(t)\} = [\theta_2 N(t) + \lambda_1 I_2(t) + (\frac{\beta_2}{N}I_2(t)S_1(t) + \frac{\beta_2}{N}I_2(t)S_2(t)$
$\qquad -(\mu_1 I_2(t) + \mu_2 I_2(t) + \lambda_2 I_2(t))]\Delta t + o(\Delta t)$ $\qquad\qquad\qquad$ (1)

$P\{\Delta T(t)\} = [\mu_1 I_1(t) + \mu_1 I_2(t) - \lambda_2 T(t)]\Delta t + o(\Delta t)$

$P\{\Delta D(t)\} = [\mu_2 I_1(t) + \mu_2 I_2(t)]\Delta t + o(\Delta t)$

The changes in the various compartments will yield deterministic equations and later the corresponding stochastic equations, firstby adding to the increments in S(t), $I_1(t)$, $I_2(t)$, T(t) and D(t), the conditional expectation of each disease state given the value of the process at the beginning of [t, t + Δt]. Thus,

$\Delta S_1(t) = [\lambda_1 N(t) \,_+ \gamma S_2 - \lambda_2 S_1(t) - \left(\frac{\beta_1}{N}I_1(t)S_1(t) + \frac{\beta_2}{N}I_2(t)S_1(t)\right)]\Delta t \,_+ \Delta Z_1 - \Delta Z_3 - \Delta Z_4$

$\Delta S_2(t) = [\lambda_1 N(t) - \lambda_2 S_2(t) - \gamma S_2 \,_- \left(\frac{\beta_1}{N}I_1(t)S_2(t) + \frac{\beta_2}{N}I_2(t)S_2(t)\right)]\Delta t \,_+ \Delta Z_2 - \Delta Z_3 - \Delta Z_4$

$\Delta I_1(t) = [\theta_1 N(t) + \lambda_1 I_1(t) + \frac{\beta_1}{N}I_1(t)S_1(t) + \frac{\beta_1}{N}I_1(t)S_2(t) - \left(\mu_1 I_1(t) + \mu_2 I_1(t) + \lambda_2 I_2(t)\right)]\Delta t$
$\qquad + \Delta Z_1 + \Delta Z_2 - \Delta Z_3$ $\qquad\qquad\qquad$ (2)

$\Delta I_2(t) = [\theta_2 N(t)\lambda_1 I_2(t) + (\frac{\beta_2}{N}I_2(t)S_1(t) + \frac{\beta_2}{N}I_2(t)S_2(t) - (\mu_1 I_2(t) + \mu_2 I_2(t) + \lambda_2 I_2(t))]\Delta t$
$\qquad + \Delta Z_1 - \Delta Z_2 - \Delta Z_4$

$\Delta T(t) = [\mu_1 I_1(t) + \mu_1 I_2(t) - \lambda_2 T(t)]\Delta t + \Delta Z_5$

$\Delta D(t) = [\mu_2 I_1(t) + \mu_2 I_2(t)]\Delta t + \Delta Z_6$

Where ΔZ_i, i = 1, 2, . . . ,6 are the differences of the Poisson increment resulting frombirth and death in the Informed Susceptible, Uninformed Susceptible, Symptomatic Infective, Asymptomatic Carrier, The Treated and The Dead Classes respectively

All the $\Delta Z_i, i = 1, 2, \ldots, 6$are normally distributed with mean zero and variances ($\lambda_1 N(t) \,_+ \gamma S_2 - \lambda_2 S1(t))\Delta t$,

$(\lambda_1 N(t) - \lambda_2 S_2(t) -. \gamma S_2)\Delta t$, $[\theta_1 N(t) + \lambda_1 I_1(t) + \frac{\beta_1}{N}I_1(t)S_1(t) + \frac{\beta_2}{N}I_1(t)S_2(t) - \lambda_2 I_1(t)]\Delta t$,$[\theta_2 N(t) + \lambda_1 I_2(t) + ($

$\frac{\beta_1}{N}I_2(t)S_1(t) + \frac{\beta_2}{N}I_2(t)S_2(t) - \lambda_2 I_2(t))]\Delta t$, $[\mu_1 I_1(t) + \mu_1 I_2(t) - \lambda_2 T(t)]\Delta t$ and $[\mu_2 I_1(t) + \mu_2 I_2(t)]\Delta t$

respectively.

Dividing through (2) by Δt, letting ΔZ_i, i = 1,2,. . ., 6 go to zero and taking limit as $\Delta t \to 0$

yields the deterministic framework for the desired stochastic differential equations for the symptomatic infectiveswhich is our focus.

$$\frac{dI_1(t)}{dt} = [\theta_1 N(t) + \lambda_1 I_1(t) + \frac{\beta_1}{N} I_1(t)S_1(t) + \frac{\beta_1}{N} I_1(t)S_2(t) - (\mu_1 I_1(t) + \mu_2 I_1(t) + \lambda_2 I_2(t))] \quad (3)$$

To derive the stochastic model corresponding to the deterministic form, we rely on the strong law of large numbers to estimate an expected population $E(N(t)) = N$ from the stochasticprocess $N(t)$.However, our main concern now is on the pattern of changes in the infectious classes. So, an SDE corresponding to $I_1(t)$ will suffice for this study.Next, we find the diffusion approximation to the Markov jump process (3) by normalizing the processes [3]. That is,expressing each of the state variables as a proportion of the expected total population and replacing each $\frac{\Delta Z_i}{N}$ by a multiple of the Brownian motion increment ΔW_i having standard deviation equal to that of the corresponding Poisson increment. In particular,

$$\frac{d}{dt}\frac{I_1(t)}{N} = [\theta_1 + \lambda_1 \frac{I_1(t)}{N} + \frac{\beta_1}{N}\frac{I_1(t)}{N}\frac{S_1(t)}{N} + \frac{\beta_1}{N}\frac{I_1(t)}{N}\frac{S_2(t)}{N} - (\mu_1 \frac{I_1(t)}{N} + \mu_2 \frac{I_1(t)}{N} + \lambda_2 I_1(t))]$$
$$+ \sigma_1 \frac{dW_1}{dt} + \sigma_2 \frac{dW_2}{dx} - \sigma_3 \frac{dW_3}{dt} \quad (4)$$

Since the Brownian motion is nowhere differentiable, we clear the equation (4)of dtto have the real-valued stochastic differential equation corresponding to the symptomatic infectious class.

$$d(\frac{I_1(t)}{N}) = [\theta_1 + \lambda_1 \frac{I_1(t)}{N} + \frac{\beta_1}{N}\frac{I_1(t)}{N}\frac{S_1(t)}{N} + \frac{\beta_1}{N}\frac{I_1(t)}{N}\frac{S_2(t)}{N} - (\mu_1 \frac{I_1(t)}{N} + \mu_2 \frac{I_1(t)}{N} + \lambda_2 I_1(t))]dt + \sigma_1 dW_1$$
$$+ \sigma_2 dW_2 - \sigma_3 dW_3 \quad (5)$$

Representing the processes$\frac{S_1(t)}{N}, \frac{S_2(t)}{N}, \frac{I_2(t)}{N}, \frac{T(t)}{N}, \frac{D(t)}{N}$ by the X_1, X_2, X_4, X_5, $X_6$and $\frac{I_1(t)}{N}$ by X we have the stochastic differential equations associated with eachcompartment. In particular,

$$dX = [\theta_1 - \lambda_2 X + \beta_1 XX_1 + \beta_1 XX_2 - (\mu_1 + \mu_2 X)]dt + G_1 dW_1 + G_2 dW_2 + G_3 dW_3, X(0) = x$$

III. Results And Analysis

The initial size of each disease state is important to epidemic size at time t. So expressingdX in terms of the initial sizes of the compartment associated with it by evaluating theX$_i$'s at t = 0 so that the SDE becomes a linear SDE whose explicit solutions can be found by Ito formular. Thus,

$$dX = [\theta_1 - \lambda_2 X + \beta_1 Xx_1 + \beta_1 Xx_2 - (\mu_1 X + \mu_2 X)]dt + G_1 dW_1$$
$$+ G_2 dW_2 - G_3 dW_3, \quad X(0) = x \text{ where the diffusion coefficients are}$$

$$G_1 = \sqrt{\lambda_1 - \lambda_2 x_1 + \gamma x_2}, \quad G_2 = \sqrt{\lambda_1 - (\lambda_2 + \gamma)x_2}, \quad G_3 = \sqrt{\beta_1 x_3 x_1 + \beta_1 x_3 x_2}.$$

Thus,

$$dX = [(\theta_1 + (-\lambda_2 + \beta_1 x_1 + \beta_1 x_2 - (\mu_1 + \mu_2))X]dt + G_1 dW_1$$
$$+ G_2 dW_2 + G_3 dW_3, \quad X(0) = x$$
$$= [b_1 + b_2 X]dt + \boldsymbol{H}.\boldsymbol{dW(t)}, X(0) = x$$

Where b$_1$ =θ_1, b$_2$ = $(-\lambda_2 + \beta_1 x_1 + \beta_1 x_2) - (\mu_1 + \mu_2)$, **H = (**G$_1$, G$_2$, G$_3$**)**
and$\boldsymbol{dW(t)}$ **=** (dW$_1$,dW$_2$, dW$_3$)with the fundamental solution F$_{0,t}$ =exp($\int_0^t b_2 ds$ = exp(b$_2$t).

Using Ito's formularwith the transformation Y = U(t, X(t))= exp(-b$_2$t)dY = $\frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dX + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}|\mathbf{H}|^2 dt$ where the vector = (G$_1$, G$_2$, G$_3$) =**H** $= \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x}((b_1 + b_2 X)dt + G_2 dW_2(t) - G_3 dW_3(t) - G_4 dW_4(t)) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}|\mathbf{H}|^2 dt$ $= [\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}(b_1 + b_2 X) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}|\mathbf{H}|^2]dt + \frac{\partial u}{\partial x}(G_1 dW_1(t) + G_2 dW_2(t) + G_3 dW_3(t))$

Since U(t, X(t)) = $e^{-b_2 t}X$, we have $\frac{\partial u}{\partial x} = e^{-b_2 t}$ and $\frac{\partial^2 u}{\partial x^2} = 0$

$$d(e^{-b_2 t}X) = [\frac{\partial}{\partial t}e^{-b_2 t}X + e^{-b_2 t}b_1 + e^{-b_2 t}b_2 X]dt$$
$$+ e^{-b_2 t}(G_1 dW_1(t) + G_2 dW_2(t) + G_3 dW_3(t))$$

But $\frac{\partial}{\partial t}e^{-b_2 t} = -b_2 e^{-b_2 t} := -b_2 F_{t_0,t}^{-1}$, so

$$d(e^{-b_2 t}X) = e^{-b_2 t}b_1 dt + e^{-b_2 t}(G_2 dW_2(t) - G_3 dW_3(t) - G_4 dW_4(t))$$

$$e^{-b_2 t}X = F_{t_0,t_0}^{-1}X(0) + \int_{t_0}^t e^{-b_2 s}b_1 ds + \int_{t_0}^t e^{-b_2 s}G_2 dW_2(s)ds - \int_{t_0}^t e^{-b_2 s}G_3 dW_3(s) - \int_{t_0}^t e^{-b_2 s}G_4 dW_4$$

For $F_{t_0,t_0}^{-1} = e^{-\int_{t_0}^{t_0} b_2 ds} = e^0 = 1$

$$e^{-b_2 t}X = X(0) + \int_{t_0}^t e^{-b_2 s}b_1 ds + \int_{t_0}^t e^{-b_2 s}G_1 dW_1(s) + \int_{t_0}^t e^{-b_2 s}G_2 dW_2(s) - \int_{t_0}^t e^{-b_2 t}G_3 dW_3(s)$$

$X = e^{b_2 t}\{x - \frac{b_1}{b_2}(exp(-b_2 t) - 1) + \int_{t_0}^t e^{-b_2 s}G_1 dW_1(s) + \int_{t_0}^t e^{-b_2 s}G_2 dW_2(s) - \int_{t_0}^t e^{-b_2 s}G_3 dW_3(s)\}$ Therefore,

$I_1(t) = Ne^{b_2 t} \{x - \frac{b_1}{b_2}(exp(-b_2 t) - 1) + \int_{t_0}^{t} e^{-b_2 s} G_1 dW_1(s) + \int_{t_0}^{t} e^{-b_2 s} G_2 dW_2(s)$ -

$\int_{t_0}^{t} e^{-b_2 s} G_3 dW_3(s)\}$ since $X = \frac{I_1(t)}{N}$ and $x = X(0) = x = \frac{I_1(0)}{N}$ Using Gaussian random number generator for the increments $\Delta W_i = W_{(i+1)h} - W_{ih}, i = 0, 1, 2 \dots, n - 1$ of a standard Wiener process we simulate the sample path of $I_1(t)$ against time (in weeks) for n = 104, corresponding to 52 weeksas follows.
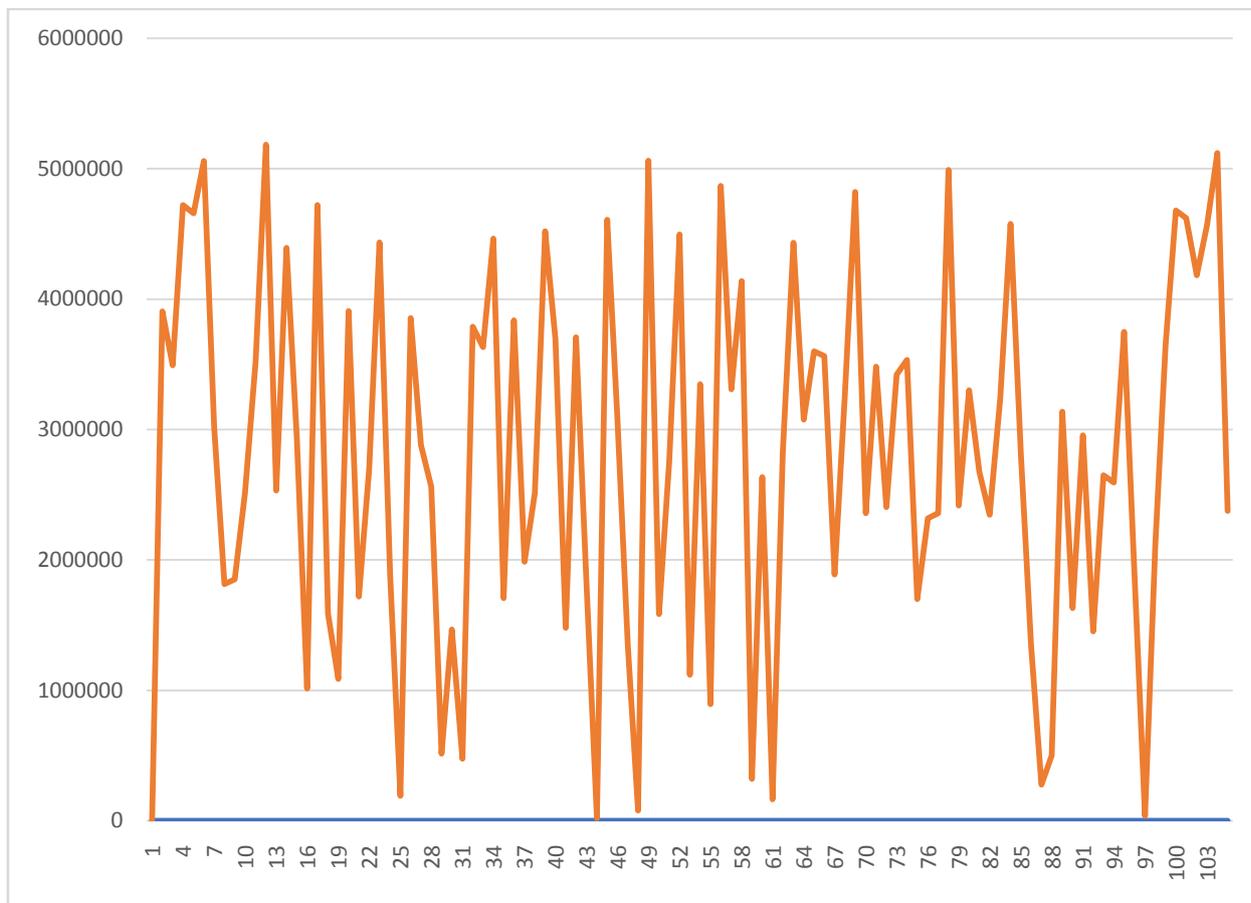

Fig. 1: Sample path of $I_1(t)\{b_1 = \theta_1 = 0$, $b_2 = -0.42768$, $x=x_3 = 1.7371 \times 10^{-5}$, $\lambda_1 = 7.0875 \times 10^{-4}$, $\lambda_2 = 2.1423 \times 10^{-4}$, $\gamma = 0.0075$, $\beta_1 = 1.7372 \times 10^{-5}$, $\beta_2 = 8.68602 \times 10^{-6}$, $x_1 = 0.075$, $x_2 = 0.0011731$ [4]. We assume the mother-to-child transmission rate $\theta_1 = 0$.

Fig. 1 shows that $I_1(t)$ increases with time as susceptible individuals (informed or uninformed) get infected at rate β_1 and β_2 respectively It also increases as new symptomatic infected people migrate into the population at rate λ_1. Its sources of decrease are: treatment at the rate μ_1, influenza related death at the rate μ_2,and death from other causes at the rate λ_2. Due to the competing effects ofits interaction with the other four compartments, the graph in Fig. 1 shows an oscillation around an approximately horizontal line from the beginning of the year to the end.

IV. Conclusion

This trend implies that influenza is endemic in Nigeria throughout the year with periodic rise and fall resulting from both environmental stochasticity and human controllable factors such as prevention through adequate information and effective treatment. Therefore, adequate control measures should be in place all year round.

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