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On Bi-Interior Γ-Hyperideals Of Γ-Semihypergroups

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Abstract
In this paper, the concept of bi-interior Γ-hyperideals of a Γ-semihypergroup is introduced and made the characterization of bi-interior Γ-hyperideals with the help of Γ-hyperideals, bi- Γ-hyperideals, quasi- Γ-hyperideals of a Γ-semihypergroup. Also characterization of minimal bi-interior Γ-hyperideals is done with respect to minimal left Γ-hyperideals and minimal right Γ-hyperideals of Γ-semihypergroup.
Keywords: *Bi-Γ-hyperideal, Quasi Γ-hyperideal, Minimal Γ-hyperideal.*

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I. Introduction

In mathematics lot of study has been made on classical algebraic structure in which the composition of two elements is an element. In 1934, the French Mathematician Marty has introduced the theory of hyperstructure in which composition of two elements is a set. Although the mathematicians across the word has another approach to study mathematics by studying the concepts of classical algebraic structure to hyperstructure theory still study in hyperstructure was not famous initially. But when it is found that the theory of hyperstructure has vast applications in various streams of science, hyperstructure theory has been widely studied. In 2003, Corsini and Leoreanu [1] have given application of theory of hyperstructures in various subjects like: geometry, cryptography, artificial intelligence, relation algebras, automata, median algebras, relation algebras, fuzzy sets and codes.

Davvaz et. Al. [2-3] introduced and studied the notion of a Γ-semihypergroup as a generalization of semigroups, semihypergroups and Γ- semigroups. Also Pawar et al. [14], introduced the notion of a regular Γ-semihyperring and got main findings in this perspective. In [11,12], Patil and Pawar studied strongly prime Γ-semihyperrings and Prime, semiprime ideals in a Γ-semihyperring. In [5] Ansari and Pawar, introduced the notion studied bi-ideals in Γ-semihypergroup and duo Γ-semihypergroup.

The main objective of this paper is to introduced and to study the concepts of classical algebraic theory to hyperstructure theory. We made analogous study of bi-interior ideal from Γ-semiring on the line of [15]. In section-2, some preliminaries are given which are useful for us to take intensive idea about paper while reading. In section-3, we introduced the concept of bi-interior Γ-hyperideal of Γ-semihypergroups and proved some properties regarding the bi-interior Γ-hyperideal.

II. Preliminaries

Here we present some useful definitions further reader is requested to refer [1-3].

Definition 2.1. [1] Let H be a non-empty set and $o : H \times H \to \wp^*(H)$ be a hyperopertion, where $\wp^*(H)$ is the family of all non-empty subsets of H. The couple (H, o) is called a hypergroupoid.

For any two non-empty subsets A and B of H and $x \in H$,

$AoB = \cup_{a\in A, b\in B}\, aob$, $Ao\{x\} = Aox$ and $\{x\}oA = xoA$.

Definition 2.2. [1] A hypergroupoid (H, o) is called a semihypergroup if for all $a, b. c$ in H we have $(aob)oc = ao(boc)$.

In addition, if for every $a \in H$, $aoH = H = Hoa$, then (H, o) is called a hypergroup **Definition 2.3.** [1] **Let** S be a non-empty set and Γ be a non-empty set of binary operation on S. Then S is called a Γ-semigroup if:

(1) $s_1\alpha s_2 \in S$.

(2) $(s_1\alpha s_2)\beta s_3 = s_1\alpha(s_2\beta s_3)$

For all $s_1, s_2, s_3 \in S$ and all $\alpha, \beta \in \Gamma$.

In [6], Kehayopulu added the following property in the definition

(3) If $s_1, s_2, s_3, s_4 \in S, \gamma_1, \gamma_2 \in \Gamma$, are such that $s_1 = s_3$, $\gamma_1 = \gamma_2$ and $s_2 = s_4$ then $s_1\alpha s_2 = s_3\alpha s_4$.

Definition 2.4. [3] Let S and Γ be two non-empty sets. Then S is called a Γ-semihypergroup if every $\gamma \in \Gamma$ is a hyperoperation on S that is $x\gamma y \subseteq S$ for every $x, y \in S$ such that, for $a, b, c, d \in S$, $\gamma_1, \gamma_2 \in \Gamma$, $a = c, b = d, \gamma_1 = \gamma_2$ imply $a\gamma_1 b = c\gamma_2 d$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ we have the associative property
$x\alpha(y\beta z) = (x\alpha y)\beta z$,

Which means that, for all $\alpha,\beta \in \Gamma$ and $x,y,z \in S$ we have
$\bigcup_{u\in(x\alpha y)} u\beta z = \bigcup_{v\in(y\beta z)} x\alpha v$.
It is clear from above definition, if every $\gamma \in \Gamma$ is an ordinary operation, then S is a Γ-semigroup
Let A and B be two non-empty subsets of a Γ-semihypergroup S and $\gamma \in \Gamma$, we denote the following:
$A\gamma B = \bigcup_{a\in A, b\in B} a\gamma b$,
Also,
$A\Gamma B = \{x \mid x \in a\alpha b, a \in A, b \in B, \alpha \in \Gamma\}$.

Definition 2.5. [3] A non-empty subset A of Γ-semihypergroup S is said to be a Γ-sub semihypergroup if $A\Gamma A \subseteq A$.
Definition 2.6. [3] A non-empty subset A of Γ-semihypergroup S is said to be a right (left) Γ-hyperideal if $A\Gamma S \subseteq A$ ($S\Gamma A \subseteq A$).
If A is both right and left Γ-hyperideal of S, then we say that A is a two sided Γ-hyperideal or simply a Γ-hyperideal of S.
We can say Γ-semihypergroup S is said to be a simple Γ-semihypergroup if there is no Γ-hyperideal in other than S itself.
Definition 2.7. [5] A non-empty set B of a Γ-semihyperring S is called bi-Γ-hyperideal of S if B is a Γ-subsemihypergroup of S and $B\Gamma S\Gamma B \subseteq B$.
Definition 2.8. [6] Let H be a Γ-semihypergroup S and Q a non-empty subset of H. Then Q is called a quasi Γ-hyperideal of S if $(H\Gamma Q)\bigcap(Q\Gamma H) \subseteq Q$.
Definition 2.9. [4] An element $x \in S$ is said be a regual element Γ-semihypergroup S if there exists $y \in S$ and $\alpha,\beta \in \Gamma$ such that $x \in x\alpha y\beta x$ or equivalently if $x \in x\Gamma S\Gamma x$.
A Γ-semihypergroup S is said to be a regular if every element of S is a regular.
Theorem 2.10. [4] Let S be a Γ-semihypergroup. Then S is regular if and only if for any left Γ-hyperideal I and for any right Γ-hyperideal J of S. Then $\cap J = J\Gamma I$.
For Γ-semihypergroup S one can define the identity element as $a \in a\alpha s, a \in s\alpha a$, for any $s \in S, \alpha \in \Gamma$. In this paper we consider the Γ-semihypergroup is with identity element.

III. Bi-Interior Γ-Hyperideals of Γ-semihypergroups

In this section introducing bi-interior Γ-hyperideals in a Γ-semihypergroup, we studied basic properties in this respect and made characterization of bi-interior Γ-hyperideals of a Γ-semihypergroup with respect to various hyperideals of a Γ-semihypergroup on the line of [15].
Definition 3.1. A non-empty subset B of a Γ-semihypergroup S is said to be bi-interior Γ-hyperideal of S if B is a Γ-subsemihypergroup of S and $S\Gamma B\Gamma S \bigcap B\Gamma S\Gamma B \subseteq B$.
Definition 3.2. A Γ-semihypergroup S is said to be simple bi-interior Γ-semihypergroup if S has no bi-interior Γ-hyperideal other than S itself.
Theorem 3.3. Arbitrary intersection of bi-interior Γ-hyperideals of a Γ-semihypergroup S is a bi-interior Γ-hyperideal.
Proof: - Let $\{B_\alpha\}_{\alpha\in\Delta}$ be indexed family of bi-interior Γ-hyperideals of a Γ-semihypergroup S and $B = \bigcap_{\alpha\in\Delta} B_\alpha$. Then clearly B is Γ-subsemihypergroup of S and $S\Gamma B_\alpha \Gamma S \bigcap B_\alpha \Gamma S\Gamma B_\alpha \subseteq B_\alpha$, for all $\alpha \in \Delta$. Thus we get,

$$\{S\Gamma(\bigcap_{\alpha\in\Delta} B_\alpha)\, \Gamma S\} \bigcap \{\left(\bigcap_{\alpha\in\Delta} B_\alpha\right) \Gamma S\Gamma \left(\bigcap_{\alpha\in\Delta} B_\alpha\right)\} \subseteq (\bigcap_{\alpha\in\Delta} B_\alpha)$$

Therefore we get, arbitrary intersection of bi-interior Γ-hyperideals of a Γ-semihypergroup S is a bi-interior Γ-hyperideal.

Theorem 3.4. Let S be a Γ-semihypergroup. Then the following are hold.
1. Every left Γ-hyperideal is a bi-interior Γ-hyperideal of S.
2. Every right Γ-hyperideal is a bi-interior Γ-hyperideal of S.
3. Every quasi Γ-hyperideal is a bi-interior Γ-hyperideal of S.
4. Intersection of a right Γ-hyperideal and a left Γ-hyperideal of S is a bi-interior Γ-hyperideal of S.
5. If B is a bi-interior Γ-hyperideal S, then $B\Gamma S$ and $S\Gamma B$ are bi-interior Γ-hyperideals of S.
6. If B is a bi-interior Γ-hyperideal and T is a Γ-subsemihypergroup of S, then $B \cap T$ is a bi-interior Γ-hyperideal of Γ-semihypergroup of S.

Proof: Proof are straightforward so omitted.

Theorem 3.5. Let S be a simple Γ-semihypergroup. Then every bi-interior Γ-hyperideal is a bi- Γ-hyperideal of S.

Proof: Let S be a simple Γ-semihypergroup and B be a bi-interior Γ-hyperideal of S. Then $S\Gamma B\Gamma S \cap B\Gamma S\Gamma B \subseteq B$ and $S\Gamma B\Gamma S$ is an ideal of S. Since S is a simple Γ-semihypergroup, so we get $S\Gamma B\Gamma S = S$. Hence we get, $S\Gamma B\Gamma S \cap B\Gamma S\Gamma B = S \cap B\Gamma S\Gamma B = B\Gamma S\Gamma B \subseteq B$. Hence the theorem.

Theorem 3.6. Let S be a Γ-semihypergroup. Then $a\Gamma S\Gamma a$ is a bi-interior Γ-hyperideal of S, where a is any element of S.

Proof: Let S be a Γ-semihypergroup and $a \in S$. Then
$S\Gamma(a\Gamma S\Gamma a)\Gamma S \cap (a\Gamma S\Gamma a)\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq S \cap a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma a$.
Thus we get, $a\Gamma S\Gamma a$ is a bi-interior Γ-hyperideal of S.

Theorem 3.7. Let S be a Γ-semihypergroup. Then $S\Gamma a\Gamma S \cap a\Gamma S\Gamma a$ is bi-interior Γ-hyperideal of S , where a is any element of S.

Proof: Proof is straightforward By the Theorems 3.3, 3.4 and 3.6.

Theorem 3.8. Let S be a simple Γ-semihypergroup. Then S is a simple bi-interior Γ-semihypergroup.if and only if $S\Gamma a\Gamma S \cap a\Gamma S\Gamma a = S$, for all $a \in S$.

Proof: Suppose that S is a simple bi-interior Γ-semihypergroup and $a \in S$. We know that by the Theorem 3.7, $S\Gamma a\Gamma S \cap a\Gamma S\Gamma a$ is a bi-interior Γ-hyperideal of S. Hence $S\Gamma a\Gamma S \cap a\Gamma S\Gamma a = S$, for all $a \in S$.
Conversely, suppose that $S\Gamma a\Gamma S \cap a\Gamma S\Gamma a = S$, for all $a \in S$. Let B be a bi-interior Γ-hyperideal of Γ-semihypergroup S and a is any element of B. Then $S = S\Gamma a\Gamma S \cap a\Gamma S\Gamma a \subseteq S\Gamma B\Gamma S \cap B\Gamma S\Gamma B \subseteq B$. Therefore we get, $S = B$. Hence the theorem.

Theorem 3.9. Let S be a regular Γ-semihypergroup. Then every bi-interior Γ-hyperideal of S is an Γ-hyperideal of S.

Proof: Let S be a regular Γ-semihypergroup and B be a bi-interior Γ-hyperideal of S. Then $S\Gamma B\Gamma S \cap B\Gamma S\Gamma B \subseteq B$ and $B\Gamma S \subseteq B\Gamma S\Gamma B$, $B\Gamma S \subseteq S\Gamma B\Gamma S$. Therefore we get, $B\Gamma S \subseteq S\Gamma B\Gamma S \cap B\Gamma S\Gamma B \subseteq B$. Similarly we can show, $S\Gamma B \subseteq S\Gamma B\Gamma S \cap B\Gamma S\Gamma B \subseteq B$. Hence Theorem.

Theorem 3.10. Every bi-Γ-hyperideal of a Γ-semihypergroup S is a bi-interior Γ-hyperideal of S.

Proof: - Let B be a bi-Γ-hyperideal of a Γ-semihypergroup S. Then $B\Gamma S\Gamma B \subseteq B$. Therefore $B\Gamma S\Gamma B \cap S\Gamma B\Gamma S \subseteq B\Gamma S\Gamma B \subseteq B$. Hence every bi-Γ-hyperideal of a Γ-semihypergroup S is a bi-interior Γ-hyperideal of S.
The concepts of interior ideals are studied in various algebraic structures. In Γ-semihypergroup S one can define interior ideal I which is a Γ-subsemihypergroup of S and $S\Gamma I\Gamma S \subseteq I$.

Theorem 3.11. Every Interior ideal of a Γ-semihypergroup S is a bi-Γ-hyperideal of S.

Proof: Let I be a interior ideal of a Γ-semihypergroup S.
Then $S\Gamma I\Gamma S \subseteq B$. Therefore $I\Gamma S\Gamma I \cap S\Gamma I\Gamma S \subseteq S\Gamma I\Gamma S \subseteq I$. Hence the theorem.

The minimal left (right) Γ-hyperideal L of a Γ-semihypergroup S means there is no left (right) Γ-hyperideal of S which is properly contained in L. Also, the minimal bi-interior Γ-hyperideal B of S means there is no bi-interior Γ-hyperideal of S which is properly contained in B.

Theorem 3.12. If I is a minimal left Γ-hyperideal and J is a minimal right Γ-hyperideal of a Γ-semihypergroup S, then $B = J\Gamma I$ is a minimal bi-interior Γ-hyperideal of S.

Proof: Let I be a minimal left Γ-hyperideal and J be a minimal right Γ-hyperideal of Γ-semihypergroup S. Then $S\Gamma(J\Gamma I)\Gamma S \cap (J\Gamma I)\Gamma S\Gamma(J\Gamma I) \subseteq J\Gamma I = B$ .Hence $B = J\Gamma I$ is a bi-interior Γ-hyperideal of S.
Suppose that A is a bi-interior Γ-hyperideal of S such that $A \subseteq B$. $S\Gamma A \subseteq S\Gamma B = S\Gamma J\Gamma I \subseteq I$, since I is a left Γ-hyperideal of S. Thus we get, $S\Gamma A = I$ since I is minimal left Γ-hyperideal of S. Similarly we get, $A\Gamma S = J$. Now, $B = A\Gamma S\Gamma S\Gamma A \subseteq A\Gamma S\Gamma A$ and $B = J\Gamma I \subseteq J\Gamma S\Gamma A \subseteq S\Gamma A \subseteq S\Gamma S$. Therefore $B \subseteq A\Gamma S\Gamma A \cap S\Gamma S \subseteq A$. Therefore we get, $A = B$. Hence the theorem.

Theorem 3.13. The intersection of a bi-interior Γ-hyperideal B of Γ-semihypergroup S and a Γ-subsemihypergroup A of S is a bi-interior Γ-hyperideal of S.

Proof: Let B be a bi-interior Γ-hyperideal of Γ-semihypergroup S and A be a Γ-subsemihypergroup of S. Suppose $C = B \cap A$. Then C is a Γ-subsemihypergroup of S, $C \subseteq A$ and $C\Gamma A\Gamma C \subseteq A\Gamma A\Gamma A \subseteq A$....(1). $C\Gamma A\Gamma C \subseteq B\Gamma A\Gamma B \subseteq B\Gamma S\Gamma B$, $S\Gamma C\Gamma S \subseteq S\Gamma B\Gamma S$. It gives that $C\Gamma A\Gamma C \cap S\Gamma C\Gamma S \subseteq B\Gamma S\Gamma B \cap S\Gamma B\Gamma S \subseteq B$, $C\Gamma A\Gamma C \cap S\Gamma C\Gamma S \subseteq A$ from (1). Therefore $C\Gamma A\Gamma C \cap S\Gamma C\Gamma S \subseteq B \cap A = C$. Hence the theorem.

Theorem 3.14. Let A and C be a Γ-subsemihypergroup of Γ-semihypergroup S and $B = A\Gamma C$. If A is the left Γ-hyperideal then B is a bi-interior Γ-hyperideal of S.

Proof: Let A and C be a Γ-subsemihypergroup of Γ-semihypergroup S and $B = A\Gamma C$. Suppose A is the left Γ-hyperideal of S. Then $B\Gamma S\Gamma B = A\Gamma C\Gamma S\Gamma A\Gamma C \subseteq A\Gamma C = B$. Therefore we get, $B\Gamma S\Gamma B \cap S\Gamma B\Gamma S \subseteq B\Gamma S\Gamma B \subseteq B$. Hence the theorem.

Corollary 3.15. Let A and C be a Γ-subsemihypergroup of Γ-semihypergroup S and $B = C\Gamma A$. If C is the right Γ-hyperideal then B is a bi-interior Γ-hyperideal of S.

Theorem 3.16. Let S is a regular Γ-semihypergroup if and only if $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any bi-interior Γ-hyperideal B, Γ-hyperideal I and left Γ-hyperideal L of S.
Proof: Suppose S is a regular Γ-semihypergroup, B is bi-interior Γ-hyperideal, I is a Γ-hyperideal and L is a left Γ-hyperideal of S respectively. Let $a \in B \cap I \cap L$. Then $a \in a\Gamma S\Gamma a$ since S is a regular. $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq B\Gamma S\Gamma I\Gamma S\Gamma L \subseteq B\Gamma I\Gamma L$. Hence we get, $B \cap I \cap L \subseteq B\Gamma I\Gamma L$.
Conversely suppose that $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any bi-interior Γ-hyperideal B, Γ-hyperideal I and left Γ-hyperideal L of S. If R is any right Γ-hyperideal and L is any left Γ-hyperideal of S, we get $R \cap L \subseteq R \cap S \cap L \subseteq R\Gamma S\Gamma L \subseteq R\Gamma L$, considering, S is an Γ-hyperideal and R is an bi-interior Γ-hyperideal. Also we have, $R\Gamma L \subseteq R, R\Gamma L \subseteq L$. Thus we get, $R\Gamma L \subseteq R \cap L$. Therefore from both inclusions we get, $R \cap L = R\Gamma L$.

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