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Numerical Solution Of A Fuzzy Eco-Epidemiological Model By The Fourth-Order Runge-Kutta Method

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Abstract :
Fuzzy set A is a subset of set S with membership function $A(x)$ that has membership degree $\alpha \in [0,1]$. Fuzzy numbers are needed to help overcome the uncertainty that occurs, so the idea of transforming ordinary differential equations into differential equations with fuzzy initial value problems to solve the problem. The transformation of ordinary differential equations into fuzzy differential equations is based on the definition of Hukuhara differential equations. The numerical solution analysis is compared with the fourth-order Runge-Kutta method. Based on the analysis, for $\alpha < 1$, the (2,2,2)-differentiable form is obtained which provides a solution that is in accordance with biological conditions, while the results in other forms do not show results that are close to the solution of the ordinary fourth-order Runge-Kutta method. For $\alpha = 1$, all forms of the Hukuhara differential derivative have the same value as the ordinary fourth-order Runge-Kutta method.
Keywords *: differential equations, eco-epidemiologi, fuzzy, fourth order Runge-Kutta, Hukuhara generalized derivative*

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I. Introduction

Recently, Greenhalgh [1] proposed an eco-epidemic model which is basically a predator-prey model with disease amongst the prey and ratio-dependent functional response for both infected and susceptible prey. The predator and prey in this model are pelicans and tilapia. It is assumed that the disease can be spread through direct contact on the prey. The predator population is denoted by $Y(t)$. In the presence of vibrio infection, the total prey population at time t , i.e. $X(t)$, is divided into two classes, namely susceptible tilapia ($S(t)$) and infected tilapia ($I(t)$), that is

$$X(t) = S(t) + I(t).$$

Assuming that only susceptible tilapia breed, in the absence of infected tilapia from breeding, the growth rate is given as a logistic function with carrier capacity k. Both susceptible and infected tilapia are subject to predation by pelicans, with pelicans preferring to feed on infected tilapia due to tilapia becoming weakened by infection and rising to the sea surface in search of oxygen. This makes infected tilapia easier to catch and thus more attractive to pelicans.
Based on the assumptions, the basic equation of the model is

$$\begin{aligned} \frac{dS}{dt} &= rS\left(1 - \frac{S+I}{k}\right) - \lambda SI - \frac{pYS}{mY+S}, \\ \frac{dI}{dt} &= \lambda SI - \frac{cYI}{mY+I} - \gamma I, \\ \frac{dY}{dt} &= \delta Y\left(1 - \frac{hY}{I+S}\right), \end{aligned} \tag{1}$$

where
r : growth rate of tilapia species in the subpopulation,
λ : disease transmission coefficient,
γ : per capita mortality rate of infected prey,
p : pelican search rate for susceptible tilapia
δ : per capita growth rate of pelicans,
h : a constant relating the density dependent mortality of the predator population,
m : a positive constant.
c : pelican search rate for infected tilapia.

The model parameters used to express the rate of population change and the initial values in model (1) are assumed to have certain values. In many cases, the initial values or parameters in the model are uncertain and heterogeneity in the population is very likely to occur. Uncertainty that occurs in a real-world phenomenon can be caused by data deficiencies, measurement errors or when determining the initial value of differential equations. Therefore, fuzzy differential equations are introduced that can overcome this uncertainty [2]. Fuzzy set theory, proposed by Zadeh [3], can be used to account for uncertainty in biological data. The application of fuzzy logic and fuzzy sets in biological systems offers a lot of potential, but to date there is not much research in this field. Several studies that discuss epidemic models that take into account parameter uncertainty and population heterogeneity, including Mondal et al. in 2015 which discussed the dynamic behavior of epidemic models with fuzzy transmission [4]. In 2017, Verma et al. proposed a fuzzy model to study the spread of influenza [5]. Motivated by the above discussion, in this paper we consider model (1) with fuzzy initial condition.

Eco-epidemiological model (1) with fuzzy initial initial values is a system of nonlinear fuzzy differential equations, where its general exact solution is not known. Hence, we usually solve the problem numerically. Ma et al. in [6] introduced an Euler method to solve fuzzy differential equation. Sekar and Prabhavathi in [7] proposed a leapfrog method when solving the fuzzy differential equation. The Runge-Kutta method is recently introduced by Abbasbandy et al. [8].

In this paper, we apply the Runge-Kutta proposed in [8] to solve model (1) with fuzzy initial value. For this aim, we review fuzzy differential equation in Section II. Then the review of Runge-Kutta method for fuzzy differential equation is presented in Section III. We implement the Runge-Kutta method to solve model (1) with fuzzy initial value and discuss the numerical results in Section IV. Finally, Section V provide some conclusions.

II. Fuzzy Differential Equation

Definition 1 [2] :A fuzzy subset A of a set S is defined by a membership function written as $A(x)$ with values [0,1] for each x in S, so $A(x)$ is a function that maps S to [0,1]. If $A(x)$ is always equal to one or zero, then the subset A is called a crisp set. In the crisp case, $A(x)$ is called the characteristic function (or indicator function) and is denoted by χ_A. If $\chi_A(x) = 0$, then x is not a member of A, otherwise if $\chi_A(x) = 1$, then x is a member of A. A fuzzy subset is a generalization where an element in S has partial membership in A characterized by a degree in the interval [0,1], for example when $A(x) = 0.6$, then the membership value of x in A is 0.6.

A fuzzy number N is a fuzzy subset of real numbers if it satisfies:
1. $\exists x: N(x) = 1$,
2. $[N]_\alpha$ is closed and finite interval for $0 < \alpha \leq 1$.

The set of all fuzzy numbers will be denoted by R_F
A special type of fuzzy number M is called a triangular fuzzy number, defined by $a_1 < a_2 < a_3$ such that:
1. $M(x) = 1 \ at \ x = a_2$.
2. The graph of $M(x)$ on $[a_1, a_2]$ is a straight line from $(a_1, 0)$ to $(a_2, 1)$ and also on $[a_2, a_3]$ the graph is a straight line from $(a_2, 1)$ to $(a_3, 0)$.
3. $M(x) = 0$ for $x \leq a_1$ or $x \geq a_3$.

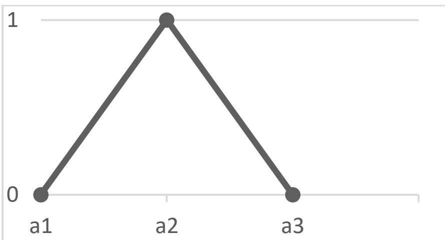


Figure 2.1. Triangular fuzzy numbers written as $M = (a_1, a_2, a_3)$ for triangular fuzzy numbers.

If in Figure (1) there is at least one graph that is not a straight line (curve), then it is said to be a triangular fuzzy number and is written $M \approx (a1, a2, a3)$.

Definition 2 [9]: Let be $u, v \in R_F$. If there exist $w \in R_F$, such that $u = v + w$, then w is called Hukuhara difference (H-*difference*) from u and v, denoted by $u \ominus v$. Some properties of H-*difference* include:
1. Hukuhara difference is not defined for pairs of fuzzy numbers such that the support of a fuzzy number has a diameter larger than the reduced one.
2. If $u \ominus v = \{0\}$, then $u \ominus v = \{0\}$.
3. $(u + v) \ominus v = u$.

4. H-*difference* unique and its $\alpha - level$ is $[u \ominus v] = [u_{1\alpha} - v_{1\alpha}, u_{2\alpha} - v_{2\alpha}]$.

Definition 3 [2]: Let be $f: [a,b] \to R_F$, with f is Hukuhara Differentiable (*H-Differentiable*) at x_0 if

$\lim_{h \to 0^+} \frac{f(x_0+h) \ominus f(x_0)}{h}$ and $\lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0-h)}{h}$ exist and equal to $f'(x_0)$.

The definition of the Hukuhara derivative is very limited, if a fuzzy differential equation $(x) = c \cdot g(x)$, where c is a fuzzy number and $g: [a,b] \to R^+$is a function with $g'(x) < 0$, then F is not differentiable. To avoid this, a more general definition of derivative for fuzzy mapping is introduced.

Definition 4 [9]: Consider a mapping $f: (a,b) \to R_F$ and $x_0 \in (a,b)$ where f is a strongly differential generalization at x_0 if there exist $f'(x_0) \in R_F$ such that:

1. for $h > 0$ sufficiently small, $\exists f(x_0+h) \ominus f(x_0), f(x_0) \ominus f(x_0-h)$ and satisfies the limit $\lim_{h \to 0^+} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0)$
2. for $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0+h), f(x_0-h) \ominus f(x_0)$ and satisfies the limit $\lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0+h)}{(-h)} = \lim_{h \to 0^+} \frac{f(x_0-h) \ominus f(x_0)}{(-h)} = f'(x_0)$
3. Untuk $h > 0$ sufficiently small, $\exists f(x_0+h) \ominus f(x_0), f(x_0-h) \ominus f(x_0)$ and satisfies the limit $\lim_{h \to 0^+} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0-h) \ominus f(x_0)}{(-h)} = f'(x_0)$
4. for $h > 0$ sufficiently small, $\exists f(x_0+h) \ominus f(x_0), f(x_0-h) \ominus f(x_0)$ and satisfies the limit
$$\lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0+h)}{(-h)} = \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0).$$

A function which is strongly differentiable according to cases (1) and (2) in Definition 4, will respectively denoted by (1)-differentiable and (2)-differentiable.

Lemma 1 [9]: If $u(t) = (x(t), y(t), z(t), w(t))$ is a trapezoidal fuzzy number-valued function, then:
If u is (1)-differentiable (Hukuhara differentiable) then $u' = (x', y', z', w')$.
If u is (2)-differentiable, then $u' = (w', z', y', x')$.

Theorem 2 [9]: Given $f[a,b] \to R_F$, with $[f(x)]_\alpha = [f_{1\alpha}(x), f_{2\alpha}(x)]$ for each $\alpha \in [0,1]$

1. If f is (1)-differentiable, then $f_{1\alpha}$ and $f_{2\alpha}$ are differentiable functions and $[f'(x)]_\alpha = [f'_{1\alpha}(x), f'_{2\alpha}(x)]$.
2. If f is (2)-differentiable, then$f_{1\alpha}$ and $f_{2\alpha}$ are differentiable functions and dan $[f'(x)]_\alpha = [f'_{2\alpha}(x), f'_{1\alpha}(x)]$

III. The Fourth-Order Runge-Kutta Method

In this Section, we review the Runge-Kutta method proposed by by Abbasbandy et al. [8]. Here we consider an initial value problem

$$\begin{cases} y_1' = f_1(t, y_1, \dots, y_n) \\ \quad\vdots \\ y_n' = f_n(t, y_1, \dots, y_n) \\ y_1(0) = y_1^{[0]} = a_1, \dots, y_n(0) = y_n^{[0]} = a_n, \end{cases} \tag{2}$$

where f_i $(1 \le i \le n)$ is a continuous mapping from $R_+ \times R^n$ to R and $y_i^{[0]}$ is a fuzzy number in E with α-level interval

$\left[y_1^{[0]}\right]_\alpha = [\underline{y_i}^{[0]}(\alpha), \overline{y_i}^{[0]}(\alpha)]$ for $i = 1, \dots, n$ and $0 < \alpha \le 1$. If

$$\underline{y_i}(t,\alpha) = min\{f_i(t, u_1, \dots, u_n); u_j \epsilon [\underline{y_i}^{[0]}(\alpha), \overline{y_i}^{[0]}(\alpha)]\} = \underline{f_i}(t, y(t,\alpha)),$$
$$\overline{y_i}\,(t,\alpha) = max\{f_i(t, u_1, \dots, u_n); u_j \epsilon [\underline{y_i}^{[0]}(\alpha), \overline{y_i}^{[0]}(\alpha)]\} = \overline{f_i}(t, y(t,\alpha)),$$

and

$$\underline{y_i}\,(0,\alpha) = \underline{y_i}^{[0]}(\alpha), \overline{y_i}\,(0,\alpha) = \overline{y_i}^{[0]}(\alpha),$$

then $\boldsymbol{y} = (y_1, \dots, y_n)^t$ on interval I is a fuzzy solution of equation (2).

For a given α, a system of differential equations with fuzzy initial value on R^{2n} can be obtained. To get the solution, it is necessary to verify that a fuzzy number $y_i(t) \in R_f$ is defined on the interval $[\underline{y_j}(t,\alpha), \overline{y_j}(t,\alpha)]$. Given $\underline{y}^{[0]}(\alpha) = \left(\underline{y_1}^{[0]}(\alpha), \dots, \underline{y_n}^{[0]}(\alpha)\right)^t$ and $\overline{y}^{[0]}(\alpha) = \left(\overline{y_1}^{[0]}(\alpha), \dots, \overline{y_n}^{[0]}(\alpha)\right)^t$, based on the existing indicators, the system of equations (3) can be written by assuming

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{F}\big(t, \boldsymbol{y}(t)\big), \\ \boldsymbol{y}(0) = \boldsymbol{y}^{[0]} \in R_f^n. \end{cases} \tag{3}$$

By taking $\boldsymbol{y}(t,\alpha) = [\underline{\boldsymbol{y}}(t,\alpha), \overline{\boldsymbol{y}}(t,\alpha)]$ dan $\boldsymbol{y}'(t,\alpha) = [\underline{\boldsymbol{y}'}(t,\alpha), \overline{\boldsymbol{y}'}(t,\alpha)]$ where

$$\underline{\boldsymbol{y}}(t,\alpha) = \left(\underline{y_1}(t,\alpha), \dots, \underline{y_n}(t,\alpha)\right)^t,$$
$$\overline{\boldsymbol{y}}(t,\alpha) = \left(\overline{y_1}(t,\alpha), \dots, \overline{y_n}(t,\alpha)\right)^t,$$
$$\underline{\boldsymbol{y}'}(t,\alpha) = \left(\underline{y_1}'(t,\alpha), \dots, \underline{y_n}'(t,\alpha)\right)^t,$$
$$\overline{\boldsymbol{y}'}(t,\alpha) = \left(\overline{y_1'}(t,\alpha), \dots, \overline{y_n'}(t,\alpha)\right)^t,$$

and assuming $\boldsymbol{F}\big(t, \boldsymbol{y}(t,\alpha)\big) = [\,\underline{\boldsymbol{F}}\big(t, \boldsymbol{y}(t,\alpha)\big), \overline{\boldsymbol{F}}\big(t, \boldsymbol{y}(t,\alpha)\big)]$ with

$$\underline{\boldsymbol{F}}(t, \boldsymbol{y}(t,\alpha)) = \left(\underline{f_1}(t, \boldsymbol{y}(t,\alpha)), \dots, \underline{f_n}(t, \boldsymbol{y}(t,\alpha))\right)^t,$$
$$\overline{\boldsymbol{F}}(t, \boldsymbol{y}(t,\alpha)) = \left(\overline{f_1}(t, \boldsymbol{y}(t,\alpha)), \dots, \overline{f_n}(t, \boldsymbol{y}(t,\alpha))\right)^t,$$

the fuzzy solution of equation (2) on interval I for all $\alpha \in (0,1]$ can be written as

$$\begin{cases} \underline{\boldsymbol{y}'}(t,\alpha) = \underline{\boldsymbol{F}}(t, \boldsymbol{y}(t,\alpha)), \\ \overline{\boldsymbol{y}'}(t,\alpha) = \overline{\boldsymbol{F}}(t, \boldsymbol{y}(t,\alpha)), \\ \underline{\boldsymbol{y}}(0,\alpha) = \underline{\boldsymbol{y}}^{[0]}(\alpha), \overline{\boldsymbol{y}}(0,\alpha) = \overline{\boldsymbol{y}}^{[0]}(\alpha), \end{cases} \tag{4}$$

or

$$\begin{cases} \boldsymbol{y}'(t,\alpha) = \boldsymbol{F}\big(t, \boldsymbol{y}(t,\alpha)\big), \\ \boldsymbol{y}(0,\alpha) = \boldsymbol{y}^{[0]}(\alpha). \end{cases} \tag{5}$$

To find the numerical solution of equation (2) by the fourth-order Runge-Kutta method [2], we first define

$$\underline{k_{i1}}(t, \boldsymbol{y}(t,\alpha)) = \min\left\{f_i(t, s_1, \dots, s_n); s_j \epsilon \left[\underline{y_i}(\alpha), \overline{y_i}(\alpha)\right]\right\},$$
$$\overline{k_{i1}}(t, \boldsymbol{y}(t,\alpha)) = \max\left\{f_i(t, s_1, \dots, s_n); s_j \epsilon \left[\underline{y_i}(\alpha), \overline{y_i}(\alpha)\right]\right\},$$
$$\underline{k_{i2}}(t, \boldsymbol{y}(t,\alpha)) = \min\left\{f_i\left(t + \tfrac{h}{2}, s_1, \dots, s_n\right); s_j \epsilon = \left[\underline{z_{j1}}(t, \boldsymbol{y}(t,\alpha), h), \overline{z_{j1}}(t, \boldsymbol{y}(t,\alpha), h)\right]\right\},$$
$$\overline{k_{i2}}(t, \boldsymbol{y}(t,\alpha)) = \max\left\{f_i\left(t + \tfrac{h}{2}, s_1, \dots, s_n\right); s_j \epsilon = \left[\underline{z_{j1}}(t, \boldsymbol{y}(t,\alpha), h), \overline{z_{j1}}(t, \boldsymbol{y}(t,\alpha), h)\right]\right\},$$
$$\underline{k_{i3}}(t, \boldsymbol{y}(t,\alpha)) = \min\left\{f_i\left(t + \tfrac{h}{2}, s_1, \dots, s_n\right); s_j \epsilon = \left[\underline{z_{j2}}(t, \boldsymbol{y}(t,\alpha), h), \overline{z_{j2}}(t, \boldsymbol{y}(t,\alpha), h)\right]\right\},$$
$$\overline{k_{i3}}(t, \boldsymbol{y}(t,\alpha)) = \max\left\{f_i\left(t + \tfrac{h}{2}, s_1, \dots, s_n\right); s_j \epsilon = \left[\underline{z_{j2}}(t, \boldsymbol{y}(t,\alpha), h), \overline{z_{j2}}(t, \boldsymbol{y}(t,\alpha), h)\right]\right\},$$
$$\underline{k_{i4}}(t, \boldsymbol{y}(t,\alpha)) = \min\left\{f_i(t, s_1, \dots, s_n); s_j \epsilon = \left[\underline{z_{j3}}(t, \boldsymbol{y}(t,\alpha), h), \overline{z_{j3}}(t, \boldsymbol{y}(t,\alpha), h)\right]\right\},$$
$$\overline{k_{i4}}(t, \boldsymbol{y}(t,\alpha)) = \max\left\{f_i(t, s_1, \dots, s_n); s_j \epsilon = \left[\underline{z_{j3}}(t, \boldsymbol{y}(t,\alpha), h), \overline{z_{j3}}(t, \boldsymbol{y}(t,\alpha), h)\right]\right\},$$

$1 \le i, j \le n$, such that

$$\underline{z_{j1}}(t, \boldsymbol{y}(t,\alpha), h) = \underline{y_j}(t,\alpha) + \frac{h}{2}\underline{k_{j1}}\big(t, \boldsymbol{y}(t,\alpha)\big),$$
$$\overline{z_{j1}}(t, \boldsymbol{y}(t,\alpha), h) = \overline{y_j}(t,\alpha) + \frac{h}{2}\overline{k_{j1}}\big(t, \boldsymbol{y}(t,\alpha)\big),$$
$$\underline{z_{j2}}(t, \boldsymbol{y}(t,\alpha), h) = \underline{y_j}(t,\alpha) + \frac{h}{2}\underline{k_{j2}}(t, \boldsymbol{y}(t,\alpha), h),$$
$$\overline{z_{j2}}(t, \boldsymbol{y}(t,\alpha), h) = \overline{y_j}(t,\alpha) + \frac{h}{2}\overline{k_{j2}}(t, \boldsymbol{y}(t,\alpha), h),$$

$$\underline{z_{j3}}(t,\boldsymbol{y}(t,\alpha),h) = \underline{y_j}(t,\alpha) + \frac{h}{2}\underline{k_{j3}}(t,\boldsymbol{y}(t,\alpha),h),$$
$$\overline{z_{j3}}(t,\boldsymbol{y}(t,\alpha),h) = \overline{y_j}(t,\alpha) + \frac{h}{2}\overline{k_{j3}}(t,\boldsymbol{y}(t,\alpha),h).$$

Taking into account the following relation

$$F_i(t,\boldsymbol{y}(t,\alpha),\mathrm{h}) = \underline{k_{i1}}(t,\boldsymbol{y}(t,\alpha)) + 2\underline{k_{i2}}(t,\boldsymbol{y}(t,\alpha),h) + 2\underline{k_{i3}}(t,\boldsymbol{y}(t,\alpha),h) + \underline{k_{i4}}(t,\boldsymbol{y}(t,\alpha),h),$$
$$G_i(t,\boldsymbol{y}(t,\alpha),\mathrm{h}) = \overline{k_{i1}}(t,\boldsymbol{y}(t,\alpha)) + 2\overline{k_{i2}}(t,\boldsymbol{y}(t,\alpha),h) + 2\overline{k_{i3}}(t,\boldsymbol{y}(t,\alpha),h) + \overline{k_{i4}}(t,\boldsymbol{y}(t,\alpha),h),$$

and partitioning the interval $[0,T]$ into N sub intervals with equally spaced discrete points $\{t_0 = 0, t_1, \dots, t_N = T\}$. If the exact and approximation solutions of the i-th piece α at $t_m, 0 \le m \le N$ are denoted by $[\underline{y_i}^{[m]}(\alpha), \overline{y_i}^{[m]}(\alpha)]$ and $[\underline{w_i}^{[m]}(\alpha), \overline{w_i}^{[m]}(\alpha)]$, then the numerical solution at the i-th coordinate of the piece α, by the Runge-Kutta method is

$$\underline{w_i}^{[m+1]}(\alpha) = \underline{w_i}^{[m]}(\alpha) + \frac{h}{6}F_i(t_m,\boldsymbol{w}^m(t,\alpha),\mathrm{h}), \qquad \underline{w_i}^{[0]}(\alpha) = \underline{y_i}^{[0]}(\alpha),$$
$$\overline{w_i}^{[m+1]}(\alpha) = \overline{w_i}^{[m]}(\alpha) + \frac{h}{6}G_i(t_m,\boldsymbol{w}^m(t,\alpha),\mathrm{h}), \qquad \overline{w_i}^{[0]}(\alpha) = \overline{y_i}^{[0]}(\alpha),$$

with

$$[w_i(t)]_\alpha = \left[\underline{w_i}(t,\alpha), \overline{w_i}(t,\alpha)\right],$$
$$\boldsymbol{w}^{[m]}(\alpha) = \left[\underline{\boldsymbol{w}}^{[m]}(\alpha), \overline{\boldsymbol{w}}^{[m]}(\alpha)\right],$$
$$\underline{\boldsymbol{w}}^{[m]}(\alpha) = (\underline{w_i}(t,\alpha), \dots, \underline{w_n}(t,\alpha))^t,$$
$$\overline{\boldsymbol{w}}^{[m]}(\alpha) = (\overline{w_i}(t,\alpha), \dots, \overline{w_n}(t,\alpha))^t.$$

Given

$$\boldsymbol{F}^*(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) = \frac{1}{6}\left(F_1(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}), \dots, F_n(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h},)\right)^t$$
$$\boldsymbol{G}^*(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) = \frac{1}{6}\left(G_1(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}), \dots, G_n(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h},)\right)^t.$$

The approximation of α-cuts solution of equation (5) by the Runge-Kutta method is

$$w^{[m+1]}(\alpha) = w^{[m]}(\alpha) + hH(t_m, w^m(t,\alpha),\mathrm{h}), \qquad w^{[0]}(\alpha) = y^{[0]}(\alpha)$$

where

$$H(t_m, w^m(t,\alpha),\mathrm{h}) = [\boldsymbol{F}^*(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}), \boldsymbol{G}^*(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h})],$$

and

$$\boldsymbol{F}^*(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) = \frac{1}{6}\left[\underline{\boldsymbol{k_1}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) + 2\underline{\boldsymbol{k_2}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) + 2\underline{\boldsymbol{k_3}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) + \underline{\boldsymbol{k_4}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h})\right],$$
$$\boldsymbol{G}^*(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) = \frac{1}{6}\left[\overline{\boldsymbol{k_1}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) + 2\overline{\boldsymbol{k_2}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) + 2\overline{\boldsymbol{k_3}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) + \overline{\boldsymbol{k_4}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h})\right]$$

and also

$$\underline{\boldsymbol{k_j}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) = \left(\underline{k_{1j}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}), \dots, \underline{k_{nj}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h})\right)^t$$
$$\overline{\boldsymbol{k_j}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}) = \left(\overline{k_{1j}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h}), \dots, \overline{k_{nj}}(t,\boldsymbol{w}^{[m]}(\alpha),\mathrm{h})\right)^t.$$

IV. Numerical Results and Analysis

We now implement the Runge-Kutta method to solve model (1) with fuzzy initial value. In the case of crisp initial value problem, we consider initial value and values of parameter as in Table 1.

Table 1 Parameters and their value [1].

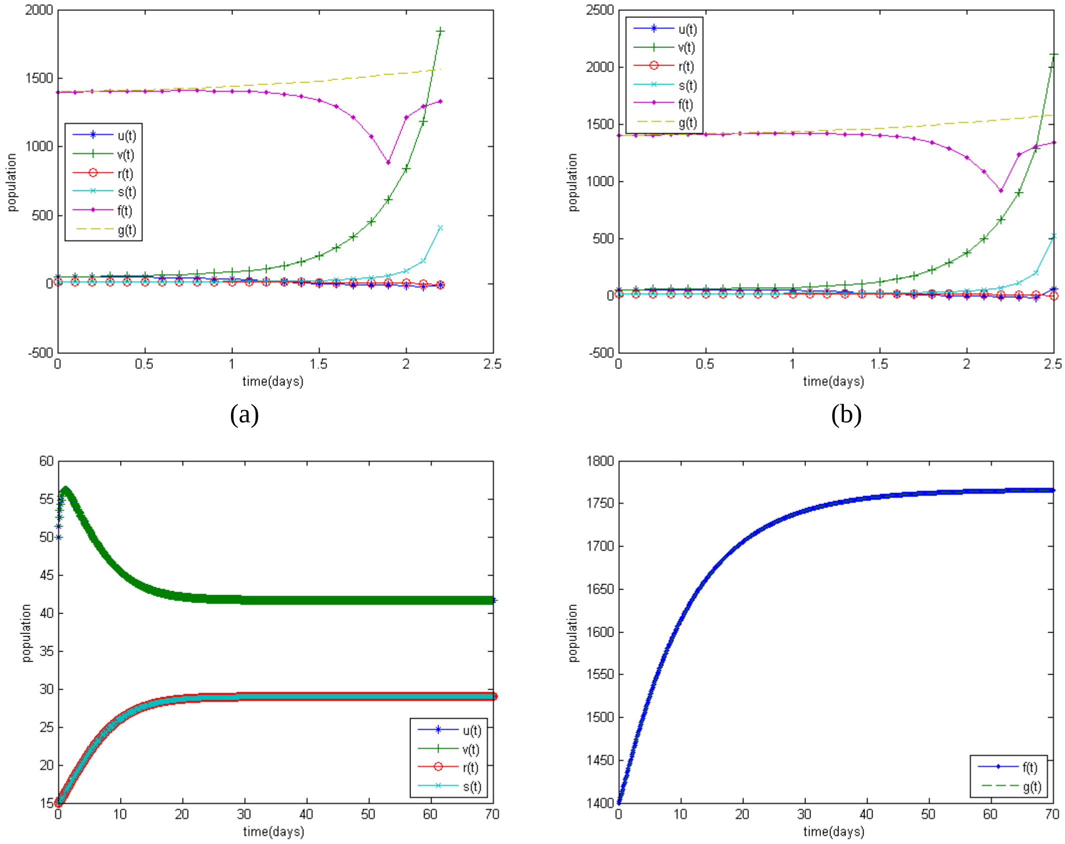
Symbol	Description	Value
$S(0)$	Initial value of suspectible tilapia population	50
$I(0)$	Initial value of infected tilapia population	15
$Y(0)$	Initial value of pelicans population	1400
r	The species growth rate of tilapia at subpopulation	3
k	Carrying capacity	75
λ	The disease transmission coefficient	0.006
p	The search rate of pelicans towards suspectible tilapia	0
m	A strictly positive constant	5.0
h	A constant related to density that dependent mortality of predator population	0.04
c	The search rate of pelicans towards infected tilapia	0.05
δ	The per capita growth rate of the pelicans	0.09

γ	The per capita death rate of infected prey	0.24

By introducing a fuzzy initial value and using the notion of the Hukuhara derivative, we have eight different type of fuzzy initial value problems as shown in Table 2. The initial value considered in this article are $u(0) = 49 + \alpha, v(0) = 51 - \alpha, r(0) = 14 + \alpha, s(0) = 16 - \alpha, f(0) = 1399 + \alpha, g(0) = 1401 - \alpha$. All cases of fuzzy initial problems have been solved using the fourth-order Runge-Kutta method. Our numerical solutions for all cases with $\alpha = 0$ and $\alpha = 0.5$ have no biological meaning since the solutions are convergent to $\pm\infty$ or there is no fuzzy solution on some intervals because $u(t) > v(t)$ or $f(t) > g(t)$. However, for $\alpha = 1$, all cases have fuzzy solutions, where in these cases all solutions in the scale of Figure 2 – 5 are the same as the solutions of the classical fourth-order Runge-Kutta method. For illustration, we plot our numerical solution for the case of (1,1,1)-differentiable¸ (2,2,1)-differentiable, (2,1,2)-differentiable¸ and (2,2,2)-differentiable in Figure 2, 3, 4, and 5.

Table 2. System (1) in every cases of Hukuhara differentiable.

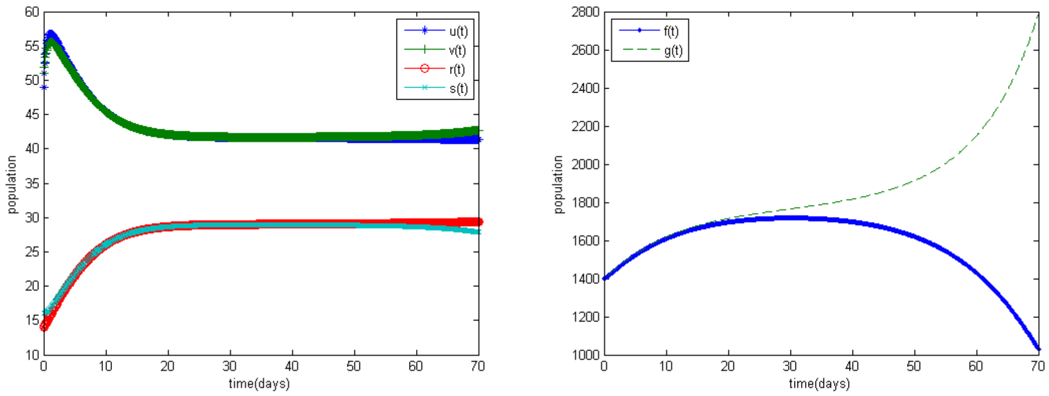
	$u'(t)$	$v'(t)$	$r'(t)$	$s'(t)$	$f'(t)$	$g'(t)$
(1,1,1)-differentiable	$ru\left(1-\frac{v+s}{k}\right)-\lambda vs-\frac{pgv}{mf+u}$	$rv\left(1-\frac{u+r}{k}\right)-\lambda ur-\frac{pfu}{mg+v}$	$\lambda ur-\frac{cgs}{mf+r}-\gamma s$	$\lambda vs-\frac{cfr}{mf+r}-\gamma r$	$\delta f\left(1-\frac{hg}{r+u}\right)$	$\delta g\left(1-\frac{hf}{s+v}\right)$
(1,1,2)-differentiable	$ru\left(1-\frac{v+s}{k}\right)-\lambda vs-\frac{pgv}{mf+u}$	$rv\left(1-\frac{u+r}{k}\right)-\lambda ur-\frac{pfu}{mg+v}$	$\lambda ur-\frac{cgs}{mf+r}-\gamma s$	$\lambda vs-\frac{cfr}{mf+r}-\gamma r$	$\delta g\left(1-\frac{hf}{s+v}\right)$	$\delta f\left(1-\frac{hg}{r+u}\right)$
(1,2,1)-differentiable	$ru\left(1-\frac{v+s}{k}\right)-\lambda vs-\frac{pgv}{mf+u}$	$rv\left(1-\frac{u+r}{k}\right)-\lambda ur-\frac{pfu}{mg+v}$	$\lambda vs-\frac{cfr}{mf+r}-\gamma r$	$\lambda ur-\frac{cgs}{mf+r}-\gamma s$	$\delta f\left(1-\frac{hg}{r+u}\right)$	$\delta g\left(1-\frac{hf}{s+v}\right)$
(2,1,1)-differentiable	$rv\left(1-\frac{u+r}{k}\right)-\lambda ur-\frac{pfu}{mg+v}$	$ru\left(1-\frac{v+s}{k}\right)-\lambda vs-\frac{pgv}{mf+u}$	$\lambda ur-\frac{cgs}{mf+r}-\gamma s$	$\lambda vs-\frac{cfr}{mf+r}-\gamma r$	$\delta f\left(1-\frac{hg}{r+u}\right)$	$\delta g\left(1-\frac{hf}{s+v}\right)$
(2,2,1)-differentiable	$rv\left(1-\frac{u+r}{k}\right)-\lambda ur-\frac{pfu}{mg+v}$	$ru\left(1-\frac{v+s}{k}\right)-\lambda vs-\frac{pgv}{mf+u}$	$\lambda vs-\frac{cfr}{mf+r}-\gamma r$	$\lambda ur-\frac{cgs}{mf+r}-\gamma s$	$\delta f\left(1-\frac{hg}{r+u}\right)$	$\delta g\left(1-\frac{hf}{s+v}\right)$
(2,1,2)-differentiable	$rv\left(1-\frac{u+r}{k}\right)-\lambda ur-\frac{pfu}{mg+v}$	$ru\left(1-\frac{v+s}{k}\right)-\lambda vs-\frac{pgv}{mf+u}$	$\lambda ur-\frac{cgs}{mf+r}-\gamma s$	$\lambda vs-\frac{cfr}{mf+r}-\gamma r$	$\delta g\left(1-\frac{hf}{s+v}\right)$	$\delta f\left(1-\frac{hg}{r+u}\right)$
(1,2,2)-differentiable	$ru\left(1-\frac{v+s}{k}\right)-\lambda vs-\frac{pgv}{mf+u}$	$rv\left(1-\frac{u+r}{k}\right)-\lambda ur-\frac{pfu}{mg+v}$	$\lambda vs-\frac{cfr}{mf+r}-\gamma r$	$\lambda ur-\frac{cgs}{mf+r}-\gamma s$	$\delta g\left(1-\frac{hf}{s+v}\right)$	$\delta f\left(1-\frac{hg}{r+u}\right)$
(2,2,2)-differentiable	$rv\left(1-\frac{u+r}{k}\right)-\lambda ur-\frac{pfu}{mg+v}$	$ru\left(1-\frac{v+s}{k}\right)-\lambda vs-\frac{pgv}{mf+u}$	$\lambda vs-\frac{cfr}{mf+r}-\gamma r$	$\lambda ur-\frac{cgs}{mf+r}-\gamma s$	$\delta g\left(1-\frac{hf}{s+v}\right)$	$\delta f\left(1-\frac{hg}{r+u}\right)$



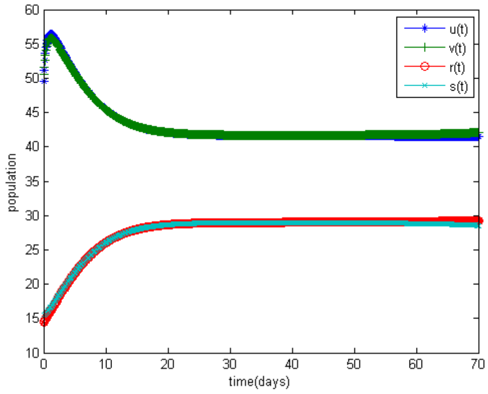
(a) (b)

(c) (d)

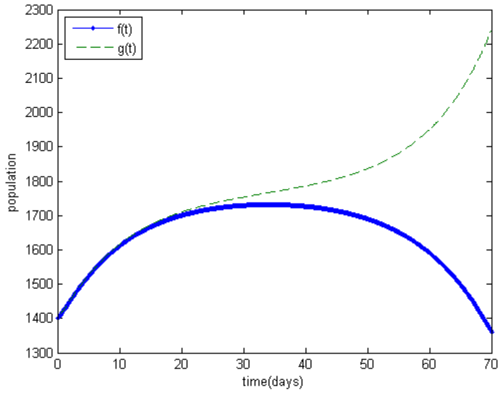
Figure 2. Numerical solution of (1,1,1)-differentiable for (a) $\alpha = 0$, (b) $\alpha = 0.5$, and (c-d) $\alpha = 1$.



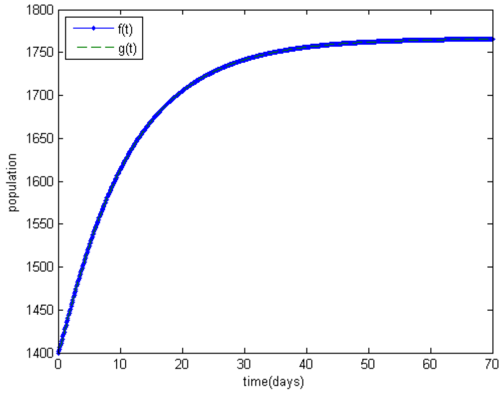
(a) (b)



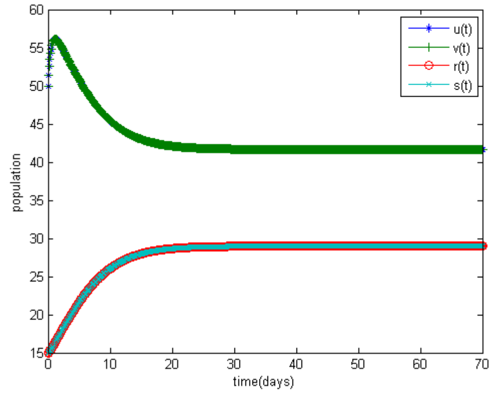
(c)



(d)

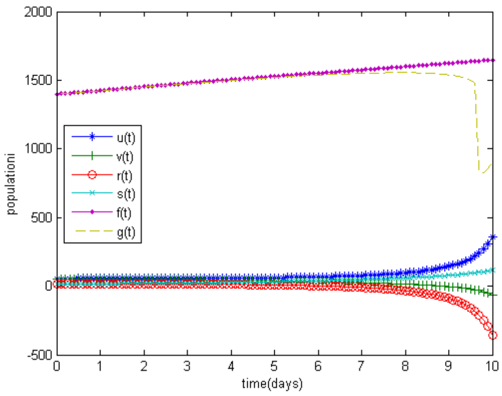


(e)

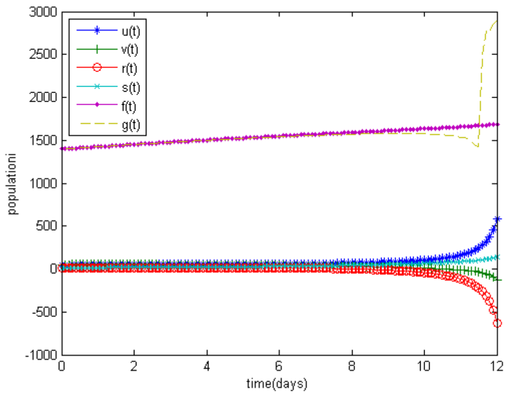


(f)

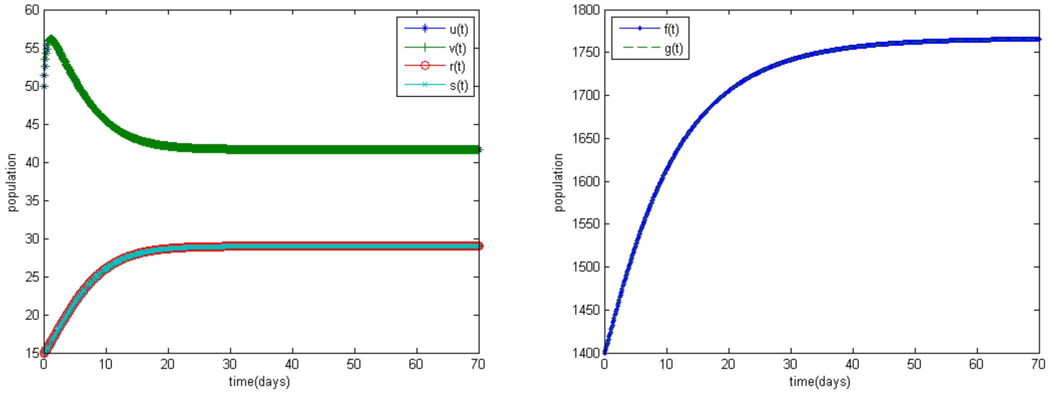
Figure 3. Numerical solution of (2,2,1)-differentiable for (a-b) $\alpha = 0$, (c-d) $\alpha = 0.5$, and (e-f) $\alpha = 1$.



(a)

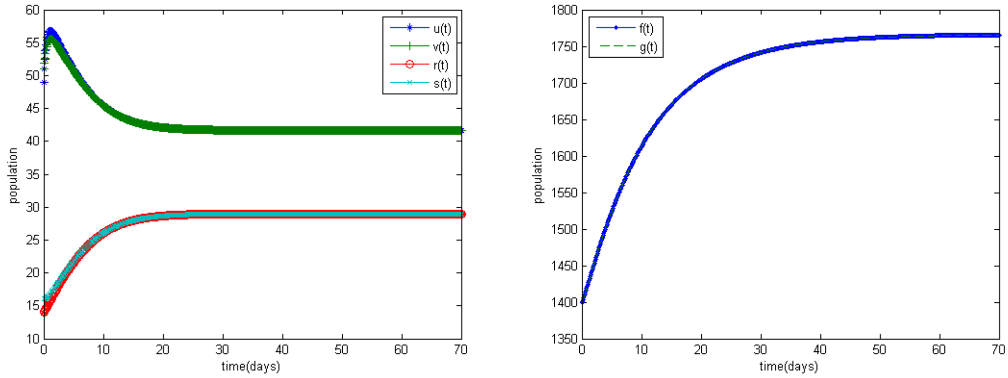


(b)

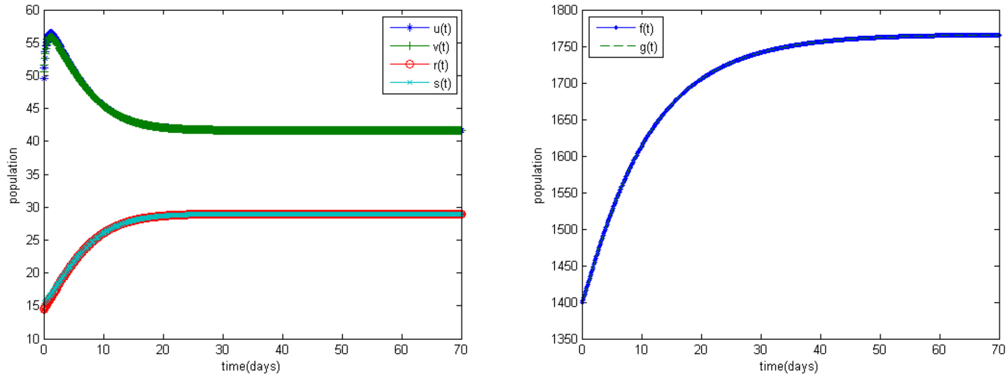


(c) (d)

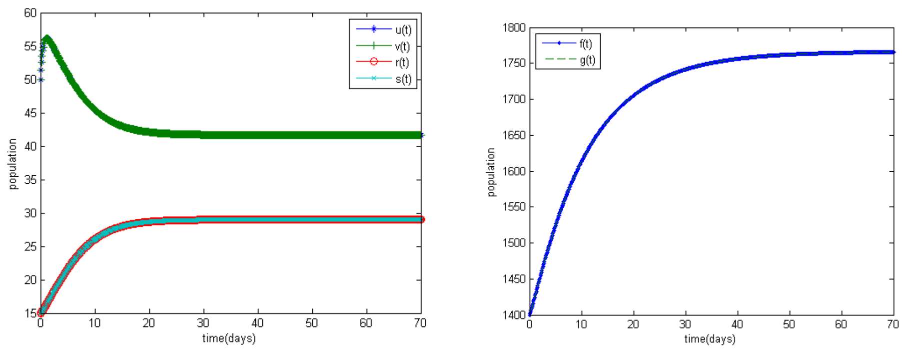
Figure 4. Numerical solution of (2,1,2)-differentiable for (a) $\alpha = 0$, (b) $\alpha = 0.5$, and (c-d) $\alpha = 1$.



(a) (b)



(c) (d)



(e) (f)

Figure 5. Numerical solution of (2,2,2)-differentiable for (a-b) $\alpha = 0$, (c-d) $\alpha = 0.5$, and (e-f) $\alpha = 1$.

Figures 2, 3, 4, and 5 show the numerical solution of u, v, r, s, f and g such that $\left[S'_{gh}(t)\right]_{\alpha} = [u'(t), v'(t)]$, $\left[I'_{gh}(t)\right]_{\alpha} = [r'(t), s'(t)]$, and $\left[Y'_{gh}(t)\right]_{\alpha} = [f'(t), g(t)]$. As observed, when $\alpha < 1$, the solutions are not convergent, and thus there is no biological compatible condition. Otherwise, At some interval, there's no fuzzy solution, like at figure (5.2) $u(t) > v(t)$. Figure (5.4) represent the best and the closest numerical solution with the Runge-Kutta fourth order method. When $\alpha = 1$, Figures (5.1), (5.2), (5.3) and (5.4) the solution is equivalent with Runge-Kutta fourth order method.

V. Conclusion

In this work, we have presented a fuzzy initial value problem of eco-epidemiological model to take into account the uncertainty. There are eight cases of Hukuhara differentiable initial value problems. However, there is only one case, namely the (2,2,2)-differentiable case, which produces fuzzy solution and at the same time the solution is reasonable biological feasible.

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