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Laplace's Theorem Based Conjecture On Elementary Integrals

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Abstract
The introduction of nonelementary integrals as standard functions like error function, exponential integral, sine and cosine integrals, logarithmic integral, etc. has stopped the interest in the study of searching new elementary and nonelementary integrals, which has stopped the development of new properties in integration in context of antiderivatives. Due to this the present generation students are not taught the basic properties of antiderivative like Laplace Theorem, Abel Theorem, Liouville Theorem, Liouville Hardy theorem, etc., which make the integration more beautiful, attractive and informative. So it is the need to introduce these concepts to mathematics learners and teachers to make them aware about these properties of integration. The aim of this paper is to find the type of such functions (integrands), which are always integrable in the sense of antiderivatives, as a conjecture having integrands a composition of exponential function, algebraic function and inverse hyperbolic functions, whose particular cases have been proved by Laplace's theorem for algebraic function as a polynomial of degree one and two only. The paper opens a new scope of research in the field of propounding conjectures and properties on elementary and nonelementary antiderivatives.
Key-Words: *Conjecture, Elementary Integral, Laplace's Theorem, Mathematica Software.*

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I. Introduction

Integration is studied in two different views: the first is *the inverse process of differentiation* and the second is as *the limit of a sum*. The first one is treated as antiderivative or the indefinite integral and the second one as the definite integral. The fundamental theorem of Calculus provides a relation between them. In general, a function f(x) is said to be integrable if there exists a function F(x) such that F'(x) = f(x) and in that case F(x) (or more generally F(x) + K) is said to be an integral of f(x) (Anton et al., 2014; Hardy, 2018; Marchisotto et al., 1994; Thompson, 2021; Yadav, 2023). But the problem starts when such F(x) does not exist. At this point, we need the concepts of elementary and nonelementary functions, because integration is a mathematical operation, which when applied on a function (elementary) need not produce only elementary functions but beyond it also, which is generally called nonelementary functions. Many such functions have been called error function, exponential integral, sine and cosine integrals, logarithmic integral, etc. (Marchisotto et al., 1994; Hardy, 2018; Nonelementary integral – Wikipedia; Yadav, 2023).

An elementary function is a single variable function which is expressed using the mathematical ordinary operations sum, difference, product, division, root and composition of finitely many polynomials, rational, trigonometric, hyperbolic, exponential, and their inverse functions. For example, $x^2 + x + 1$, $\sqrt{x^2 + 1}$, e^{2x}, $\log(3x)$, $sinx + x^2$, $\int x^2 \, sinx \, dx$, $\int x^2 \, e^{x^3} \, dx$, π, e, 7, $sinhx$, $arcsinx$, $|x|$, etc. are elementary functions. But every function is not necessarily an elementary function. For example, the indefinite integral $\int e^{-x^2} dx$ is not an elementary function. The detail concept of elementary functions were introduced by Joseph Liouville in a series of papers between 1833 to 1841 and the algebraic treatment of elementary functions was started by Joseph Fels Ritt in 1930s (Marchisotto et al., 1994; Elementary Function – Wikipedia; Cherry, 1985, 1986; Hardy, 2018; Kasper, 1980; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Yadav, 2023).

Every elementary function can always be written in closed form. A closed form expression uses a finite number of mathematical operations. It contains constants, variables, arithmetic ordinary operations addition +, subtraction -, multiplication x, division / and functions like nth root, exponent, logarithm, trigonometric, hyperbolic, inverse trigonometric, inverse hyperbolic, etc. It usually does not contain limit or integral (Marchisotto et al., 1994; Closed form expression – Wikipedia; Hardy, 2018; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Yadav, 2023). A special care must be taken while talking about elementary and nonelementary functions or closed form and non-closed form expressions. There exist many expressions which are not in closed form but can be reduced into it after simplification. For example, the expression

$$\mathrm{f}(x)=\sum_{n=0}^{\infty}\frac{x^2}{2^n}=\frac{x^2}{2^0}+\frac{x^2}{2^1}+\frac{x^2}{2^2}+\cdots+\frac{x^2}{2^n}+\cdots$$

is not in closed form because it contains infinite number of terms, however using the summation rule of a geometric series, it can be expressed as $\mathrm{f}(x)=2x^2$. If a function is written in closed form, its derivative can also be expressed in closed form. But its indefinite integral may or may not be written in closed form. The antiderivative of e^{-x^2} does not have a closed form expression and its antiderivative has been called the error function given by

$$\mathrm{erf}(x)=\frac{1}{\pi}\int_0^x e^{-t^2}dt.$$

Some well known nonelementary antiderivatives (integrals) are the elliptic integral, logarithmic integral, Gaussian integral, Fresnel integrals, Sine integral (or Dirichlet integral), Exponential integral, etc. (Closed-form expression-Wikipedia; Corliss et al., 1989; Hardy, 2018; Marchisotto et al., 1994; Nijimbere, 2017, 2018, 2020a, 2020b; Sao, 2021; Sharma et al., 2020; Singer et al., 1985; Trager, 2022; Victor, 2017; Nonelementary integral – Wikipedia; Trigonometric integral – Wikipedia; Exponential integral – Wikipedia; Error function – Wikipedia; Elliptic integral – Wikipedia; Fresnel integral – Wikipedia; Gaussian integral – Wikipedia; Yadav, 2023).

The first example which leads us beyond the domain of elementary functions is the elliptic integrals. The first reported study of such integrals was due to John Wallis in 1655. Euler also studied elliptic functions and found that they were not integrable in terms of the elementary functions. Such integrals cannot be expressed in terms of elementary functions was proved by Joseph Liouville in 1833. Although many pioneers contributed in the advancement of the subject like John Bernoulli (1702), Laplace (1812), A. M. Legendre (1825), N. H. Abel (1826, 1829), P. L. Chebyshev (1853), E. Hermite (1872), G. H. Hardy (1905), D. D. Mordoukhay Boltovskoy (1906-1910, 1913, 1937), C. Hermite (1912), A. Ostrowski (1940), Joseph F. Ritt (1916, 1948), M. Rosenlicht (1967-68), etc. but in 1833 Joseph Liouville created a framework for constructive integration by finding out when antiderivative of elementary functions are again elementary functions (Cherry, 1985, 1986; Hardy, 2018; Marchisotto et al., 1994; Kasper, 1980; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Trager, 2022; Yadav, 2023).

He propounded Liouville's First Theorem on Integration in 1833 and in 1835 he generalized this theorem for several variables and established strong Liouville theorem. He showed that the elliptic integrals of the first and second kinds have no elementary expressions. By 1841, Liouville had developed a theory of integration that settled the question of integration in finite terms for many important cases (Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Trager, 2022; Yadav, 2023). Although Liouville theorems play an important role in studying elementary and nonelementary functions in context of antiderivatives, Laplace's theorem is one of the primary property for algebraic and rational functions and the present paper is focused on its applications in propounding the conjecture.

II. Preliminary Ideas

This paper is an attempt to search some special type of elementary functions (integrals) in terms of antiderivatives in which inverse hyperbolic functions play an important role as a component in the integrand. The Laplace's theorem and its conjecture have been used to verify the results in the paper. In 1812 Laplace found that 'the integral of a rational function of x, e^x and logx is either a rational function of those functions or the sum of such a rational function and of a finite number of constant multiples of logarithms of similar functions'. Based on this fact he propounded the following well known theorem on integration:

Laplace's Theorem: A rational function has an anti-derivative and its integral is always an elementary function. It is composed of two parts: one of a rational function part and another one the transcendental or logarithmic part. For example,

$$\int\frac{2x\,dx}{1+x^2}=\ln(1+x^2)$$

$$\int\frac{(1+x^2)^2+x}{x(1+x^2)}dx=\frac{\mathrm{x}^2}{2}+\ln|\mathrm{x}|+\frac{\mathrm{i}}{2}\ln\left|\frac{\mathrm{x}+\mathrm{i}}{\mathrm{x}-\mathrm{i}}\right|.$$

He also proposed a conjecture known as Laplace's Conjecture which states that "the integral of an algebraic function need contain only those algebraic functions which are present in the integrand". This conjecture was later proved by Abel. For example,

$$\int(x^3+1)dx=\frac{x^4}{4}+x+K$$

$$\int (x^4 + 6x^3 + 4x^2 + x - 6)dx = \frac{x^5}{5} + 6\frac{x^4}{4} + 4\frac{x^3}{3} + \frac{x^2}{2} - 6x + K$$

(Hardy, 2018; Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Yadav, 2023).

III. Methodology

As stated earlier that we will use Laplace's theorem to prove that whether antiderivative of some elementary functions containing the composition of exponential function, polynomial of degree one and two, and the inverse hyperbolic functions as a component in the integrands is elementary or nonelementary.

IV. Discussion

There are many conjectures on elementary and nonelementary functions (integrals) like Bernoulli's conjecture, Laplace's conjecture, etc. Yadav et. al. (2012) propounded six conjectures on indefinite nonintegrable functions, in which they didn't consider the inverse hyperbolic functions as a component in the integrands. Based on the above conjectures, the following conjecture has been propounded:

Conjecture: An indefinite integral of the form

$$\int \frac{e^{g\{f(x)\}}}{g'\{f(x)\}} dx \quad = I \quad (\text{Let}) \qquad (i)$$

where g(x) is an inverse hyperbolic function, f(x) a polynomial of degree one and two, and $g'\{f(x)\}$ the derivative of g with respect to x, is always elementary.

Proof: We know that there are six inverse hyperbolic functions and infinite number of polynomials exist of degree greater than or equal to one. To prove it for polynomial of all degrees is impossible. We would prove the statement for some cases. This is why it has been proffered as a conjecture. In each case, we will prove the conjecture for a polynomial of degree one and two. Let us discuss them one by one in six cases and twelve sub-cases as follows:

Case-I: When $g(x) = \sinh^{-1} f(x)$ and f(x) a polynomial in x of degree one and two, from (i) we have

$$I = \int \frac{e^{\sinh^{-1}\{f(x)\}}\sqrt{1 + \{f(x)\}^2}}{f'(x)} dx$$

where f'(x) is the derivative of f(x) with respect to x. Putting $\sinh^{-1}\{f(x)\} = z$ i.e., $f(x) = \sinh z$, and $f'(x)dx = \cosh z\, dz$, we get

$$I = \frac{1}{2}\int \frac{e^z(1 + \cosh 2z)}{\{f'(x)\}^2} dz \qquad (ii)$$

which will be elementary or nonelementary depends on f(x). So let us consider two different cases of f(x):

Sub-case-I: Putting $f(x) = x + b$ a polynomial of degree one, we get $f'(x) = 1$, then from (ii) we get

$$I = \frac{1}{2}\int e^z(1 + \cosh 2z)dz = \frac{e^z}{2} + \frac{1}{2}\int e^z \cosh 2z\, dz \qquad (ii\ a)$$

For the second integral using integration by parts, we get

$$I_1 = \int e^z \cosh 2z\, dz = \frac{2}{3} e^z \sinh 2z - \frac{1}{3} e^z \cosh 2z$$

Therefore from (ii a) we get

$$I = \frac{e^z}{2} + \left[\frac{1}{3} e^z \sinh 2z - \frac{1}{6} e^z \cosh 2z\right] = \frac{e^z}{2}\left[1 + \frac{2}{3}\sinh 2z - \frac{1}{3}\cosh 2z\right]$$

Putting the value of z in it, we get

$$I = \frac{e^{\sinh^{-1}(x+b)}}{2}\left[1 + \frac{2}{3}\left\{\sinh\{2\sinh^{-1}(x + b)\} - \frac{1}{3}\cosh\{2\sin^{-1}(x + b)\right\}\right]$$

which is elementary.

Sub-case-II: If we take $f(x) = x^2 + bx + c$, where b and c are arbitrary rational real numbers, a polynomial of degree two, then $f'(x) = 2x + b$ and $\sinh^{-1} f(x) = z$ i.e., $\sinh z = x^2 + bx + c$, then from (ii) we get

$$I = \frac{1}{2}\int \frac{e^z(1 + \cosh 2z)}{(2x + b)^2} dz \qquad (iii)$$

From $\sinh z = x^2 + bx + c$, we get

$$(2x + b)^2 = 4(\sinh z + K), \text{where } K = \frac{b^2 - 4c}{4}$$

Thus from (iii) we have

$$I = \frac{1}{8}\int \frac{e^z(1 + \cosh 2z)}{(\sinh z + K)} dz \qquad (iv)$$

The simple case arise for K = 0. For this we get from (iv)

$$I = \frac{1}{8}\int \frac{e^z(1+\cosh 2z)}{\sinh z}dz = \frac{1}{8}\int \frac{1}{e^z}\left(\frac{e^{4z}+2e^{2z}+1}{e^{2z}-1}\right)e^z dz$$

Putting $e^z = w$, we get

$$I = \frac{1}{8}\int \frac{1}{w}\left(\frac{w^4+2w^2+1}{w^2-1}\right)dw$$

whose integrand is algebraic in nature and by Laplace theorem it has always an antiderivative and thus will be elementary. Its integral will be given by

$$I = \frac{1}{8}\int \frac{1}{w}\frac{(w^2+1)^2}{(w^2-1)}dw$$

Putting $w^2+1 = p$, we get

$$I = \frac{1}{16}\int \frac{p^2}{(p-1)(p-2)}dp = \frac{1}{16}\left[(w^2+1)+\log\frac{(w^2-1)^4}{w^2}\right]$$

Putting the values of w, we get

$$I = \frac{1}{16}\left[(e^{2z}+1)+\log\frac{(e^{2z}-1)^4}{e^{2z}}\right]$$

Again putting the value of z, we get

$$I = \frac{1}{16}\left[\{e^{2\sinh^{-1}(x^2+bx+c)}+1\}+\log\frac{\{e^{2\sinh^{-1}(x^2+bx+c)}-1\}^4}{e^{2\sinh^{-1}(x^2+bx+c)}}\right]$$

Hence the integral (iv) is elementary for K = 0 i.e., for $f(x) = x^2+2\sqrt{c}\,x+c$.
Let us consider that $K \neq 0$. Then we have from (iv)

$$I = \frac{1}{8}\int \frac{e^z(1+\cosh 2z)}{(\sinh z+K)}dz = \frac{1}{8}\int \frac{e^z(e^{2z}+1)^2}{4(e^{2z}-1+2Ke^z)}\frac{1}{e^{2z}}e^z dz$$

Putting $e^z = w$, we get

$$I = \frac{1}{8}\int \frac{w(w^2+1)^2}{(w^2-1+2Kw)}\frac{1}{w^2}dw = \frac{1}{8}\int \frac{(w^2+1)^2}{(w^2-1+2Kw)}\frac{1}{w}dw$$

whose integrand is algebraic in nature and by Laplace theorem it has always an antiderivative and thus it is elementary. Its integral will be given as follows

$$I = \frac{1}{8}\int \frac{(w^2+1)^2}{(w^2-1+2Kw)}\frac{1}{w}dw$$

Since K is an arbitrary constant. Let us take K = 1 for simplicity of the calculations and must be non-zero. Thus we get

$$I = \frac{1}{8}\int \frac{(w^2+1)^2}{(w^2-1+2w)}\frac{1}{w}dw\, I = \frac{1}{8}\int \frac{(w^2-1)^2+4w^2}{(w^2-1+2w)}\frac{1}{w}dw$$

$$= \frac{1}{8}\int \frac{(w^2-1)^2}{(w^2-1+2w)}\frac{1}{w}dw + \frac{1}{2}\int \frac{w}{(w^2-1+2w)}dw = \frac{1}{8}I_1 + \frac{1}{2}I_2$$

Now integrating the first integral, we get

$$I_1 = \int\left[\left(w+2-\frac{1}{w}\right)-4\right]dw + 4I_2 = \frac{w^2}{2} - \log(w) - 2w + 4I_2$$

The second integral is given by

$$I_2 = \int \frac{w\,dw}{(w^2+2w-1)} = \frac{1}{2}\log(w^2+2w-1) - \frac{1}{2\sqrt{2}}\log\frac{w+1-\sqrt{2}}{w+1+\sqrt{2}}$$

Putting the values of two integrals in I we get

$$I = \frac{w^2}{16} - \frac{\log(w)}{8} - \frac{w}{4} + \frac{1}{2}\log(w^2+2w-1) - \frac{1}{2\sqrt{2}}\log\frac{w+1-\sqrt{2}}{w+1+\sqrt{2}}$$

Putting the value of w in above integral, we get

$$= \frac{e^{2z}}{16} - \frac{\log(e^z)}{8} - \frac{e^z}{4} + \frac{1}{2}\log(e^{2z}+2e^z-1) - \frac{1}{2\sqrt{2}}\log\frac{e^z+1-\sqrt{2}}{e^z+1+\sqrt{2}}$$

Again putting the value of z in above integral, we will get the desired integral. Obviously this integral is elementary. Hence the integral (iv) is elementary for non-zero K also. Thus the integral (i) is elementary for both linear and quadratic f(x), when $g(x) = \sinh^{-1}f(x)$.

Case-II: When $g(x) = \cosh^{-1}f(x)$ and f(x) a polynomial in x of degree one or two, we get from (i)

$$I = \int \frac{e^{\cosh^{-1}\{f(x)\}}\sqrt{\{f(x)\}^2-1}}{f'(x)}dx$$

Putting $\cosh^{-1}\{f(x)\} = z$ i.e., $f(x) = \cosh z$ and $f'(x)dx = \sinh z\,dz$, we get

$$I = \frac{1}{2}\int \frac{e^z(\cosh 2z - 1)}{\{f'(x)\}^2} dz \qquad (v)$$

Sub-case-III: Taking $f(x) = x + b$ i.e., $f'(x) = 1$, from (v) we get

$$I = \frac{1}{3}e^z \sinh 2z - \frac{2}{3}e^z \cosh 2z - \frac{e^z}{2}$$

Putting the value of z in it, we get

$$I = \frac{e^{\cosh^{-1}(x+b)}}{3}\left[\sinh 2\{\cosh^{-1}(x+b)\} - 2\left\{\cosh\{2\cosh^{-1}(x+b)\} - \frac{3}{2}\sinh\{2\cosh^{-1}(x+b)\}\right\}\right]$$

which is elementary. Therefore the given integral (i) is elementary for a polynomial f(x) of degree one, when $g(x) = \cosh^{-1}\{f(x)\}$.

Sub-case-IV: Taking $f(x) = x^2 + bx + c$, where b and c are arbitrary as earlier, we have $f'(x) = 2x + b$ and $\cosh^{-1} f(x) = z$ i.e., $\cosh z = x^2 + bx + c$, then from (v) we get

$$I = \frac{1}{2}\int \frac{e^z(\cosh 2z - 1)}{(2x+b)^2} dz$$

From $\cosh z = x^2 + bx + c$, we get

$$(2x+b)^2 = 4(\cosh z + K), \text{ where } K = \frac{b^2 - 4c}{4}$$

Thus we get

$$I = \frac{1}{8}\int \frac{e^z(\cosh 2z - 1)}{(\cosh z + K)} dz \qquad (vi)$$

The most simple case arise for K = 0. For this we get from (vi)

$$I = \frac{1}{8}\int \frac{e^z(\cosh 2z - 1)}{\cosh z} dz = \frac{1}{4}\int e^z\left(\frac{e^z - e^{-z}}{e^z + e^{-z}}\right)\left(\frac{e^z - e^{-z}}{2}\right) dz = \frac{1}{8}\int e^z\left(\frac{e^{2z} - 1}{e^{2z} + 1}\right)\left(\frac{e^{2z} - 1}{e^z}\right) dz$$

Putting $e^z = w$, we get

$$I = \frac{1}{8}\int \left(\frac{w^2 - 1}{w^2 + 1}\right)\left(\frac{w^2 - 1}{w}\right) dz$$

whose integrand is algebraic and by Laplace theorem, it has always an antiderivative and thus it is elementary and its integral is given by

$$I = \frac{1}{8}\int \left(w + \frac{1}{w} - 2\frac{2w}{w^2 + 1}\right) dz$$

$$= \frac{1}{8}\left[\frac{w^2}{2} + \log w - 2\log(w^2 + 1)\right] = \frac{w^2}{16} + \frac{\log w}{8} - \frac{1}{4}\log(w^2 + 1) = \frac{e^{2z}}{16} + \frac{z}{8} - \frac{1}{4}\log(e^{2z} + 1)$$

where $z = \cosh^{-1}(x^2 + 2\sqrt{c}\,x + c)$. In above integration, we have not added constant of integration. Hence the integral (vi) is elementary for K = 0.

Let us consider that K ≠ 0. Then we have from (vi)

$$I = \frac{1}{8}\int \frac{e^z(\cosh 2z - 1)}{(\cosh z + K)} dz = \frac{1}{8}\int e^z \frac{(e^{2z} - 1)^2}{e^{2z}(e^{2z} + 2\,K\,e^z + 1\,)} e^z\, dz$$

Putting $e^z = w$, we get

$$I = \frac{1}{8}\int \frac{w\,(w^2 - 1)^2}{w^2(w^2 + 2\,K\,w + 1\,)} dw$$

whose integrand is algebraic in w and by Laplace theorem, an algebraic integrand is always elementary i.e., it has an antiderivative for different values of K. Let us take one case for K = 1, we get

$$I = \frac{1}{8}\int \frac{w\,(w^2 - 1)^2}{w^2(w^2 + 2\,w + 1\,)} dw = \frac{1}{8}\left(\frac{w^2}{2} - 2\,w + \log w\right) = \frac{w^2}{16} - \frac{w}{4} + \frac{\log w}{8}$$

Putting the value of $e^z = w$ in above integral, we get

$$I = \frac{e^{2z}}{16} - \frac{e^z}{4} + \frac{z}{8}$$

where $z = \cosh^{-1}(x^2 + 2\sqrt{c}\,x + c)$. Thus the above integral is elementary. Hence the integral (vi) is elementary for non-zero K also. Similarly we can prove it elementary for other values of K. Thus the integral (i) is elementary for both linear f(x) and quadratic f(x), when $g(x) = \cosh^{-1}\{f(x)\}$.

Case-III: When $g(x) = \tanh^{-1} f(x)$ and f(x) a polynomial in x of degree one and two, we get from (i)

$$I = \int \frac{e^{g\{f(x)\}}}{g'\{f(x)\}} dx = \int \frac{e^{\tanh^{-1}\{f(x)\}}[1 - \{f(x)\}^2]}{f'(x)} dx$$

Putting $\tanh^{-1}\{f(x)\} = z$ i.e. $f(x) = \tanh z$ and $f'(x)dx = \operatorname{sech}^2 z\, dz$, we get

$$I = \int \frac{e^z(1-\tanh^2 z)}{\{f'(x)\}^2} \operatorname{sech}^2 z\, dz \qquad \text{(vii)}$$

Sub-Case-V: Taking $f(x) = x + b$ i.e., $f'(x) = 1$, from (vii) we get

$$I = \int e^z(1-\tanh^2 z)\operatorname{sech}^2 z\, dz = \int e^z \left(\frac{2}{e^z + e^{-z}}\right)^4 dz = 16\int \frac{e^{4z}\, e^z}{(e^{2z}+1)^4}\, dz$$

Putting the value of $e^z = w$ in above integral, we get

$$I = 16\int \frac{w^4\, dw}{(w^2+1)^4}$$

whose integrand is algebraic in w and by Laplace theorem, it has an antiderivative. So it is elementary and its integral is given by

$$I = 16\int \frac{w^4\, dw}{(w^2+1)^4} = 16\int \frac{dw}{(w^2+1)^2} - 16\int \frac{2\, dw}{(w^2+1)^3} + 16\int \frac{dw}{(w^2+1)^4}$$

Putting $w = \tan\theta$ in above integrals, we get

$$I = \frac{\theta}{1} - \frac{\sin 2\theta}{4} - \frac{\sin 4\theta}{4} + \frac{\sin 6\theta}{12} + C, \text{where } \theta = \tan^{-1} w \text{ and } w = e^z.$$

Sub-Case-VI: Taking $f(x) = x^2 + bx + c$, where b and c are arbitrary constants i.e., $f'(x) = 2x + b$ and $\tanh^{-1} f(x) = z$ i.e., $\tanh z = x^2 + bx + c$, from (vii) we get

$$I = \int \frac{e^z(1-\tanh^2 z)}{\{f'(x)\}^2}\operatorname{sech}^2 z\, dz = \int \frac{e^z(1-\tanh^2 z)}{(2x+b)^2}\operatorname{sech}^2 z\, dz$$

From $\tanh z = x^2 + bx + c$, we get

$$(2x+b)^2 = 4(\tanh z + K), \text{where } K = \frac{b^2-4c}{4}$$

Thus we get

$$I = \frac{1}{4}\int \frac{e^z(1-\tanh^2 z)\operatorname{sech}^2 z}{(\tanh z + K)} dz \qquad \text{(viii)}$$

The simple case arise for K = 0. For this we get from (viii)

$$I = \frac{1}{4}\int \frac{e^z(1-\tanh^2 z)\operatorname{sech}^2 z}{\tanh z} dz = \frac{1}{4}\int \frac{e^z \operatorname{sech}^4 z}{\tanh z} dz$$

Putting $\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$ and $\operatorname{sech} z = \frac{2}{e^z+e^{-z}}$, we get

$$I = 4\int \frac{e^{5z} dz}{(e^{2z}+1)^3\,(e^{2z}-1)}$$

Putting $e^z = w$, we get

$$I = 4\int \frac{w^4 dw}{(w^2+1)^3\,(w^2-1)}$$

in which the integrand is algebraic in w and so by Laplace's theorem, it is always elementary and its integral as

$$= \frac{1}{4}\left(-\frac{2w}{(1+w^2)^2} + \frac{3w}{1+w^2} + \mathrm{ArcTan}[w] + \mathrm{Log}[1-w] - \mathrm{Log}[1+w]\right)$$

Now after putting the values of w and z, we can find its value in terms of x. Thus (viii) is elementary for K = 0. Let us consider that K ≠ 0. Then we have from (viii)

$$I = \frac{1}{4}\int \frac{e^z(1-\tanh^2 z)\operatorname{sech}^2 z}{(\tanh z + K)} dz = \frac{1}{4}\int \frac{e^z \operatorname{sech}^4 z}{\tanh z + K} dz$$

Putting $\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$ and $\operatorname{sech} z = \frac{2}{e^z+e^{-z}}$, we get

$$I = 4\int \frac{e^{5z}\,(e^{2z}+1)\, dz}{(e^{2z}+1)^4\,(e^{2z}-1+K\,e^{2z}+K)}$$

Putting $e^z = w$, we get

$$I = 4\int \frac{w^4\ dw}{(w^2+1)^3\,(w^2-1+K\,w^2+K)}$$

in which the integrand is algebraic in w and so by Laplace's theorem, it is always elementary**.** Putting different values of K, we can easily show that integral (viii) is elementary K ≠ 0 also. For example, taking K = 1, we get

$$I = 4\int \frac{w^4\ dw}{(w^2+1)^3\,(w^2-1+\ w^2+1)} = 2\int \frac{w^2\ dw}{(w^2+1)^3}$$

$$= 2\left(-\frac{w}{4(1+w^2)^2} + \frac{w}{8(1+w^2)} + \frac{\mathrm{ArcTan}[w]}{8}\right)$$

Putting the values of w and z, we can find its value in terms of x. Thus the integral (i) is elementary for both linear and quadratic f(x), when $g(x) = \tanh^{-1}\{f(x)\}$.

Case-IV: When $g(x) = \coth^{-1}f(x)$ and f(x) a polynomial in x of degree one and two, then we get from (i)

$$I = \int \frac{e^{g\{f(x)\}}}{g'\{f(x)\}}dx = \int \frac{e^{\coth^{-1}\{f(x)\}}[1-\{f(x)\}^2]}{f'(x)}dx$$

Putting $\coth^{-1}\{f(x)\} = z$ i.e., $f(x) = \coth z$ and $f'(x)dx = -\mathrm{cosech}^2\, z\, dz$, we get

$$I = -\int \frac{e^z(1-\coth^2 z)}{\{f'(x)\}^2}\mathrm{cosech}^2\, z\, dz \qquad \text{(ix)}$$

Sub-Case-VII: Taking $f(x) = x + b$ i.e., $f'(x) = 1$, from (ix) we get

$$I = -\int \frac{e^z(1-\coth^2 z)}{1}\mathrm{cosech}^2 z\, dz = 16\int \frac{e^{5z}}{(e^{2z}-1)^4}\, dz$$

Putting $e^z = w$, we get

$$I = 16\int \frac{w^4\, dw}{(w^2-1)^4}$$

whose integrand is algebraic and so by Laplace's theorem, it is elementary and its integral

$$= 16\left(-\frac{w}{6(-1+w^2)^3} - \frac{7w}{24(-1+w^2)^2} - \frac{w}{16(-1+w^2)} + \frac{1}{32}\mathrm{Log}[-1-w] - \frac{1}{32}\mathrm{Log}[-1+w]\right)$$

Sub-Case-VIII: Taking $f(x) = x^2 + bx + c$, where b and c are arbitrary constants i.e., $f'(x) = 2x + b$ and $\coth^{-1}f(x) = z$ i.e., $\coth z = x^2 + bx + c$, from (ix) we get

$$I = -\int \frac{e^z(1-\coth^2 z)}{(2x+b)^2}\mathrm{cosech}^2\, z\, dz \qquad \text{(x)}$$

From $\coth z = x^2 + bx + c$, we get

$$(2x+b)^2 = 4(\coth z + K), \text{ where } K = \frac{b^2-4c}{4}$$

Thus we get from (x) that

$$I = -\frac{1}{4}\int \frac{e^z(1-\coth^2 z)\,\mathrm{cosech}^2 z}{(\coth z + K)}dz \qquad \text{(xi)}$$

The simple case arise for K = 0. For this we get from (xi)

$$I = -\frac{1}{4}\int \frac{e^z(1-\coth^2 z)\mathrm{cosech}^2 z}{\coth z}dz = \frac{1}{4}\int \frac{e^z \mathrm{cosech}^4 z}{\coth z}dz$$

Putting the values of cosechz and cothz in terms of exponential functions, we get

$$I = 4\int \frac{e^{5z}\, dz}{(e^{2z}-1)^3(e^{2z}+1)}$$

Putting $e^z = w$, we get

$$I = 4\int \frac{w^4\, dw}{(w^2-1)^3\,(w^2+1)}$$

whose integrand is algebraic and so by Laplace's theorem, it is elementary and its integral is given by

$$= \frac{1}{8}\left(-\frac{4w}{(-1+w^2)^2} - \frac{6w}{-1+w^2} - 4\mathrm{ArcTan}[w] + \mathrm{Log}[-1-w] - \mathrm{Log}[-1+w]\right)$$

Let us consider K ≠ 0. Then we have from (xi)

$$I = -\frac{1}{4}\int \frac{e^z(1-\coth^2 z)\,\mathrm{cosech}^2 z}{(\coth z + K)}dz = \frac{1}{4}\int \frac{e^z \mathrm{cosech}^4 z}{(\coth z + K)}dz$$

Putting the values of cosechz and cothz in terms of exponential functions, we get

$$I = 4\int \frac{e^{5z}\, dz}{(e^{2z}-1)^3(e^{2z}+1+Ke^{2z}-K)}$$

Putting $e^z = w$, we get

$$I = 4\int \frac{w^4\, dw}{(w^2-1)^3\,(w^2+1+Kw^2-K)}$$

whose integrand is algebraic and so by Laplace's theorem, it is elementary and its integral for particular value of K (here = 1) is given by

$$= \frac{1}{8}\left(-\frac{2(w+w^3)}{(-1+w^2)^2} + Log[-1-w] - Log[-1+w]\right)$$

Thus the integral (i) is elementary for both linear f(x) and quadratic f(x), when $g(x) = \coth^{-1}\{f(x)\}$.

Case-V: When $g(x) = \mathrm{sech}^{-1}f(x)$ and f(x) a polynomial in x of degree one and two, we get from (i)

$$I = \int \frac{e^{g\{f(x)\}}}{g'\{f(x)\}}dx = -\int \frac{e^{\mathrm{sech}^{-1}\{f(x)\}}\left[f(x)\sqrt{1-\{f(x)\}^2}\right]}{f'(x)}dx$$

Putting $\mathrm{sech}^{-1}\{f(x)\} = z$ i.e., $f(x) = \mathrm{sech}\, z$ and $f'(x)dx = -\mathrm{sech}\, z \tanh z\, dz$, we get

$$I = \int \frac{e^z \left[\operatorname{sech} z \sqrt{1 - \{\operatorname{sech} z\}^2}\right]}{\{f'(x)\}^2} \operatorname{sech} z \tanh z \, dz \qquad \text{(xii)}$$

Sub-Case-IX: Taking $f(x) = x + b$ i.e., $f'(x) = 1$, from (xii) we get

$$I = \int e^z \operatorname{sech}^2 z \tanh^2 z \, dz$$

Putting the values of sechz and tanhz in terms of exponential functions, we get

$$I = 4 \int \frac{e^{3z} (e^{2z} - 1)^2 \, dz}{(e^{2z} + 1)^4}$$

Putting $e^z = w$, we get

$$I = 4 \int \frac{w^2 (w^2 - 1)^2 \, dw}{(w^2 + 1)^4}$$

which is elementary by Laplace's theorem and its integral is given by

$$= \frac{-w(3 + 4w^2 + 9w^4) + 3(1 + w^2)^3 ArcTan[w]}{3(1 + w^2)^3}$$

Sub-Case-X: Taking $f(x) = x^2 + bx + c$, where b and c are arbitrary constants i.e., $f'(x) = 2x + b$ and $\operatorname{sech}^{-1} f(x) = z$ i.e., $\operatorname{sech} z = x^2 + bx + c$, from (xii) we get

$$I = \int \frac{e^z \left[\operatorname{sech} z \sqrt{1 - \{\operatorname{sech} z\}^2}\right]}{(2x + b)^2} \operatorname{sech} z \tanh z \, dz$$

From $\operatorname{sech} z = x^2 + bx + c$, we get

$$(2x + b)^2 = 4(\operatorname{sech} z + K), \text{where } K = \frac{b^2 - 4c}{4}$$

Thus we get

$$I = \frac{1}{4} \int \frac{e^z \left[\operatorname{sech} z \sqrt{1 - \{\operatorname{sech} z\}^2}\right]}{(\operatorname{sech} z + K)} \operatorname{sech} z \tanh z \, dz \qquad \text{(xiii)}$$

The simple case arise for K = 0. For this we get from (xiii)

$$I = \frac{1}{4} \int e^z \tanh^2 z \operatorname{sech} z \, dz$$

Putting the values of sechz and tanhz in terms of exponential functions, we get

$$I = \frac{1}{2} \int \frac{e^{2z} (e^{2z} - 1)^2 \, dz}{(e^{2z} + 1)^3}$$

Putting $e^z = w$, we get

$$I = \frac{1}{2} \int \frac{w(w^2 - 1)^2 \, dw}{(w^2 + 1)^3}$$

which is elementary by Laplace's theorem and its integral is given by

$$= \frac{1}{2} \left(\frac{1 + 2w^2}{(1 + w^2)^2} + \frac{1}{2} \mathrm{Log}[1 + w^2] \right)$$

Let us consider that K ≠ 0. Then we have from (xiii)

$$I = \frac{1}{4} \int \frac{e^z \operatorname{sech}^2 z \tanh^2 z}{(\operatorname{sech} z + K)} dz$$

Putting the values of sechz and tanhz in terms of exponential functions, we get

$$I = \int \frac{e^{3z} (e^{2z} - 1)^2 \, dz}{(e^{2z} + 1)^3 (2e^z + Ke^{2z} + K)}$$

Putting $e^z = w$, we get

$$I = \int \frac{w^2 (w^2 - 1)^2 \, dw}{(w^2 + 1)^3 (2w + Kw^2 + K)}$$

which is elementary by Laplace's theorem and its integral for particular values K = 1 (may be considered different rational number) is given by

$$= -\frac{-1 + w - 2w^2 + w^3 - (1 + w^2)^2 \mathrm{ArcTan}[w]}{2(1 + w^2)^2}$$

Thus the integral (i) is elementary for both linear f(x) and quadratic f(x), when $g(x) = \sec^{-1}\{f(x)\}$.

Case-VI: When $g(x) = \operatorname{cosech}^{-1} f(x)$ and f(x) a polynomial in x of degree one and two, we have from (i)

$$I = -\int \frac{e^{\operatorname{cosech}^{-1}\{f(x)\}} \left[f(x)\sqrt{\{f(x)\}^2 + 1}\right]}{f'(x)} dx$$

Putting $\operatorname{cosech}^{-1}\{f(x)\} = z$ i.e., $f(x) = \operatorname{cosech} z$ and $f'(x)dx = -\operatorname{cosech} z \coth z \, dz$, we get

$$I = -\int \frac{e^z \left[\operatorname{cosech} z \sqrt{\{\operatorname{cosech} z\}^2 + 1}\right]}{\{f'(x)\}^2} \operatorname{cosech} z \coth z \, dz \qquad \text{(xiv)}$$

Sub-Case-XI: Taking $f(x) = x + b$ i.e., $f'(x) = 1$, from (xiv) we get

$$I = -\int e^z \operatorname{cosech}^2 z \coth^2 z \, dz$$

Putting the values of cosechz and cothz in terms of exponential functions, we get

$$I = -4\int \frac{e^{3z}\,(e^{2z}+1)^2\,dz}{(e^{2z}-1)^4}$$

Putting $e^z = w$, we get

$$I = -4\int \frac{w^2(w^2+1)^2\,dw}{(w^2-1)^4}$$

which is elementary by Laplace's theorem and its integral is given by

$$= -4\left(-\frac{2w}{3(-1+w^2)^3} - \frac{7w}{6(-1+w^2)^2} - \frac{3w}{4(-1+w^2)} + \frac{1}{8}\mathrm{Log}[1-w] - \frac{1}{8}\mathrm{Log}[1+w]\right)$$

Sub-Case-XII: Taking $f(x) = x^2 + bx + c$, where b and c are arbitrary constants i.e., $f'(x) = 2x + b$ and $\operatorname{cosech}^{-1} f(x) = z$ i.e., $\operatorname{cosech} z = x^2 + bx + c$, from (xiv) we get

$$I = -\int \frac{e^z \left[\operatorname{cosech} z \sqrt{\{\operatorname{cosech} z\}^2 + 1}\right]}{(2x+b)^2} \operatorname{cosech} z \coth z \, dz$$

From $\operatorname{cosech} z = x^2 + bx + c$, we get

$$(2x+b)^2 = 4(\operatorname{cosech} z + K), \text{where } K = \frac{b^2-4c}{4}$$

Thus we get

$$I = -\frac{1}{4}\int \frac{e^z \left[\operatorname{cosech} z \sqrt{\{\operatorname{cosech} z\}^2 + 1}\right]}{(\operatorname{cosech} z + K)} \operatorname{cosech} z \coth z \, dz \qquad \text{(xv)}$$

The simple case arise for K = 0. For this we get from (xv)

$$I = -\frac{1}{4}\int e^z \operatorname{cosech} z \coth^2 z \, dz$$

Putting the values of cosechz and cothz in terms of exponential functions, we get

$$I = -\frac{1}{2}\int \frac{e^{2z}\,(e^{2z}+1)^2\,dz}{(e^{2z}-1)^3}$$

Putting $e^z = w$, we get

$$I = -\frac{1}{2}\int \frac{w(w^2+1)^2\,dw}{(w^2-1)^3}$$

which is elementary by Laplace's theorem and its integral is given by

$$= -\frac{1-2w^2}{(-1+w^2)^2} - \frac{1}{2}\mathrm{Log}[-1+w^2]$$

Let us consider that $K \neq 0$. Then we have from (xv) that

$$I = -\frac{1}{4}\int \frac{e^z \operatorname{cosech}^2 z \coth^2 z}{(\operatorname{cosech} z + K)}\,dz$$

Putting the values of cosechz and cothz in terms of exponential functions, we get

$$I = -\int \frac{e^{3z}\,(e^{2z}+1)^2\,dz}{(e^{2z}-1)^3(2e^z + Ke^{2z} - K)}$$

Putting $e^z = w$, we get

$$I = -\int \frac{w^2(w^2+1)^2\,dw}{(w^2-1)^3(2w + Kw^2 - K)}$$

which is elementary by Laplace's theorem and its integral for particular value K = 1 is given by

$$= \frac{1}{4}\Big(\frac{2}{(-1+w^2)^2} - \frac{2(-2+w)}{-1+w^2} - 3\mathrm{Log}[-1-w] - (-2+\sqrt{2})\mathrm{Log}[-1+\sqrt{2}-w] - \mathrm{Log}[-1+w] + (2 + \sqrt{2})\mathrm{Log}[1+\sqrt{2}+w]\Big)$$

Thus the integral (i) is elementary for both linear and quadratic f(x), when $g(x) = \operatorname{cosec}^{-1}\{f(x)\}$. Similarly we can prove that the indefinite integral (i) is elementary or nonelementary for higher degree polynomial f(x).

V. Conclusion

From above discussion we conclude that the integral (i)

$$\int \frac{e^{g\{f(x)\}}}{g'\{f(x)\}}dx$$

where g(x) is an inverse hyperbolic function, f(x) a polynomial of degree one and two, and $g'\{f(x)\}$ a derivative of g with respect to x, is always elementary.

VI. Future Scope of Research

The indefinite integral given by (i) has been discussed for only two cases of the polynomial of linear and quadratic nature. In all cases, K has been considered for particular value K = 1 and 0 only in quadratic polynomial. A big scope is available for research for higher degree and its special cases polynomials as well as for different rational and irrationals K in quadratic polynomials. This is why that the integral (i) has been named a conjecture and not a theorem or a property.

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