IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661,p-ISSN: 2278-8727, Volume 20, Issue 1, Ser. 3 (Jan. – Feb. 2024), PP 01-09
www.iosrjournals.org

Review of the Conjecture on Nonelementary Integrals Using Computer Software Mathematica

Dharmendra Kumar Yadav[1*] & Manoj Kumar Chaudhary[2]
[1*]*University Department of Mathematics, Lalit Narayan Mithila University, Darbhanga, Bihar*
[2]*University Department of Mathematics, Lalit Narayan Mithila University, Darbhanga, Bihar*

Abstract
The article is a review work of the strong Liouville's theorem based conjecture on nonelementary integrals using computer software Mathematica. The fast growing applications of computer software has opened a new scope of research in Mathematics and its allied sciences. It is being used in justifying theorems, conjectures, properties, examples and applications in many fields of sciences, arts and commerce. Mathematica provides the best computational techniques, which works like a calculator and solve many problems of mathematics in a single step. Its direct use makes the work easier and time saving. In the present paper, the proof of the conjecture on nonelementary integrals has been justified by applying Mathematica techniques of integration. The direct integration using Mathematica has explored some more cases of elementary integrals in the previous study. Due to the lack of concepts of elementary and nonelementary functions in Mathematica software library, the property 'a function as an output in Mathematica in terms hypergeometric functions is a nonelementary function' has been used as a standard method to classify the integrals as elementary and nonelementary. The paper opens a new scope of research for higher degree polynomials and for the computer programmer experts to explore new ideas of elementary and nonelementary functions in programming languages of Mathematica and other computer software.

Key-words: *Conjecture, Elementary Integral, Elementary Function, Nonelementary Integral, Nonelementary Function, Hypergeometric Function, Mathematica.*

Date of Submission: 30-01-2024 Date of Acceptance: 10-02-2024

I. Introduction

Computer software Mathematica is a computational technique system developed by Wolfram Research, the software's creator. It is a platform for computing that provides a wide range of functionalities for symbolic computations, numerical computations, data visualization, and programming. It is used in various fields like in mathematics, physics, engineering, computer science, finance, computer science, and many other fields (Abbott, 1997; Abell et al., 2021; Hayes, 1990; Wikipedia contributors, 2023; Wolfram, 1999).

It is a comprehensive numerical and symbolic computational package with extensive associated graphical capabilities and a programming language with an interactive document or notebook interface. It is the well known and well used technical computing package in the market (Hilbe, 2006; Wikipedia contributors, 2023; Wolfram, 1999). It is widely used in academia, research, and industry for a variety of purposes, including mathematical research, scientific simulations, data analysis, and educational purposes. It has a broad user base across different disciplines due to its versatility and powerful computational capabilities. It is known for its versatility, ease of use, and powerful computational capabilities, making it a popular tool for a wide range of technical and scientific applications (Fitelson, 1998; Wikipedia contributors, 2023; Wolfram, 1999).

Using it we can also manipulate and solve mathematical expressions and equations symbolically (Hayes, 1990). It provides extensive numerical capabilities for tasks such as solving equations, numerical integration, optimization, and linear algebra (Hilbe, 2006). It enables users to import, manipulate, and visualize data in various formats. It supports a wide range of plotting and visualization functions. It has its own programming language called Wolfram Language (Paul, 2018; Rose et a., 2002; Stroyan, 2014). Users can write scripts, functions, and programs to perform complex computations and automate tasks. It includes a vast collection of built in mathematical and scientific functions, making it a comprehensive tool for various technical applications (Trott, 2007). It uses a notebook interface, allowing users to create interactive documents that combine code, text, graphics, and other elements. It supports interactive manipulation of variables and parameters, making it useful for exploring mathematical concepts and models (Maeder, 1991; Wagon et al., 1999). It can be integrated with other technologies and programming languages. It also supports the creation of interactive interfaces and applications (Hayes, 1990; Hilbe, 2006; Paul, 2018; Rose et a., 2002; Stroyan, 2014;

Trott, 2007). In the present paper we are using the computer software Mathematica to justify a conjecture on nonelementary integrals based on strong Liouville's theorem propounded by Chaudhary & Yadav (2024).

II. Preliminary Ideas

We know that Mathematica excels in handling symbolic and numerical integration, providing a range of functions to compute definite and indefinite integrals and work with both elementary and complex functions. We can compute symbolic integrals using the `Integrate` function by using the technique: Integrate [f(x), x], which integrates f(x) with respect to x. It doesn't provide the outcomes in terms of elementary and nonelementary functions (Elementary function – Wikipedia; Hardy, 2018; Victor, 2017; Nonelementary function – Wikipedia; Yadav, 2023) but in terms of traditional and special functions i.e., in terms of components of elementary functions like polynomial, trigonometric functions, inverse trigonometric functions, hyperbolic functions, inverse hyperbolic functions, exponential functions, logarithmic functions, etc., which needs some more mathematical information to classify the outcomes. Before we start discussion on review for justification of all elementary and nonelementary integrals discussed for the conjecture using Mathematica, let us introduce hypergeometric function, which would be used in output in finding indefinite integrals of the integrands in the paper, which decide the final result.

Hypergeometric Function: It is denoted and defined by

$$2F1(\propto_1, \propto_2; \beta; x) = \sum_{r=0}^{\infty} \frac{(\propto_1)_r . (\propto_2)_r}{(\beta)_r} . \frac{x^r}{r!}$$

There are some elementary functions which can be expressed in terms of hypergeometric functions. But all hypergeometric functions cannot be expressed in terms of elementary functions. Those hypergeometric functions which are not expressible in term of elementary function or in closed form can undoubtedly be called as nonelementary functions (Closed-form expression-Wikipedia; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia). For example, following hypergeometric functions are nonelementary functions:

$$2\mathrm{F1}\left[\frac{1}{2} - \frac{i}{2}, 1, \frac{3}{2} - \frac{i}{2}, -e^{2ix}\right], 2\mathrm{F1}\left[-\frac{i}{2}, 1, 1 - \frac{i}{2}, -e^{2ix}\right], 2\mathrm{F1}\left[1, 1 - \frac{i}{2}, 2 - \frac{i}{2}, e^{2ix}\right], etc.$$

Based on the above facts, we shall classify the outcome in terms of elementary and nonelementary functions (integrals).

Conjecture: Yadav & Sen (2012) had propounded six conjectures on nonelementary integrals in which they didn't take inverse trigonometric functions as a component in the integrands, while considering antiderivative of special type of functions. Continuing on their based hypothesis and taking advantage of the exceptional cases, Chaudhary & Yadav (2024) have propounded the conjecture on nonelementary integrals based on strong Liouville's theorem (Marchisotto et al., 1994; Risch, 2022; Nijimbere, 2020; Ritt, 2022; Yadav, 2023), which states that "an antiderivative

$$\int \frac{\mathrm{e}^{\mathrm{g}\{\mathrm{f}(\mathrm{x})\}}}{\mathrm{g}'\{\mathrm{f}(\mathrm{x})\}} \mathrm{dx} \qquad (1)$$

where g(x) is an inverse trigonometric function, f(x) a complete non-perfect square polynomial of degree two, and $\mathrm{g}'\{\mathrm{f}(\mathrm{x})\}$ a derivative of g with respect to x, is always nonelementary.

III. Methodology

As far as the methodology is concerned, we shall treat those integrals as elementary whose antiderivative is expressible in terms of elementary functions and those whose integrals are expressed in terms of Hypergeometric functions and are not expressible in terms of elementary functions would be treated as nonelementary integrals (Closed-form expression-Wikipedia; Hypergeometric function – Wikipedia; Sao, 2021; Sharma, et al., 2020). In general Mathematica expresses the integrals in terms of either elementary or in terms of special functions or hypergeometric functions. If the result is coming out in terms of hypergeometric functions, it means it is not elementary but nonelementary functions. In the paper wherever we have mentioned the outcome as elementary or nonelementary functions, means that their corresponding (integrals of the) integrands are elementary or nonelementary integrals.

IV. Discussion

Chaudhary & Yadav (2024) concluded that the integral (1) is elementary for some cases like when $\mathrm{f(x)} = \mathrm{x} + \mathrm{b}$ and $\mathrm{g(x)} = \sin^{-1}\mathrm{f(x)}$; $\mathrm{f(x)} = \mathrm{x} + \mathrm{b}$ and $\mathrm{g(x)} = \cos^{-1}\mathrm{f(x)}$; $\mathrm{f(x)} = (\mathrm{x} + \sqrt{\mathrm{c}})^2$ and $\mathrm{g(x)} = \sec^{-1}\mathrm{f(x)}$; and when $\mathrm{f(x)} = (\mathrm{x} + \sqrt{\mathrm{c}})^2$ and $\mathrm{g(x)} = \mathrm{cosec}^{-1}\mathrm{f(x)}$. They also obtain some characters of the conjecture that it gives the same result for pair wise inverse trigonometric functions for the pairs: $\sin^{-1}\mathrm{f(x)}$ and

$\cos^{-1}f(x)$, $\tan^{-1}f(x)$ and $\cot^{-1}f(x)$, and $\sec^{-1}f(x)$ and $\mathrm{cosec}^{-1}f(x)$. In this section we shall review and justify them one by one using Mathematica. Previously they have discussed the conjecture in six cases and twelve sub-cases but we shall review and justify the conjecture in six cases only. For this we suppose that the integral (1) is denoted by I, then we have:

Case-I: For $g(x) = \sin^{-1}f(x)$, $f(x) = x + b$ and $\sin^{-1}\{f(x)\} = z$, the integral (1) becomes

$$I = \int e^z \cos^2 z\, dz$$

Applying the Mathematica technique to integrate it, we get

In[1]: $\text{Integrate}[\text{Exp}[z] * \text{Cos}[z] * \text{Cos}[z], z]$

Out[1]: $\frac{1}{10}e^z(5 + \text{Cos}[2z] + 2\text{Sin}[2z])$

which is an elementary function i.e., the integral of input [1] is elementary. In above code, In[1] denotes input [1] and Out[1] denotes the outcome[1] i.e. antiderivative of input function in [1]. Similarly In[i] will denote input [i] and Out[i] will denote outcome[i] in further discussion.

For $g(x) = \sin^{-1}f(x)$, $f(x) = x^2 + bx + c$, $\sin^{-1}\{f(x)\} = z$, and $K = \frac{b^2-4c}{4} = 0$, (1) becomes

$$I = \frac{1}{4}\int e^z \cot z \cos z\, dz$$

Using Mathematica technique and omitting the coefficient in calculation and adjusting it in the final result, we find its integral as

In[2]: $\text{Integrate}[\text{Exp}[z] * \text{Cos}[z] * \text{Cot}[z], z]$

Out[2]: $\frac{1}{8}e^z(\text{Cos}[z] - (2 + 2i)e^{iz}\text{Hypergeometric2F1}[\frac{1}{2} - \frac{i}{2}, 1, \frac{3}{2} - \frac{i}{2}, e^{2iz}] - \text{Sin}[z])$

which is in terms of hypergeometric function and so it is nonelementary function i.e., the integral of input [2] is nonelementary.

For $g(x) = \sin^{-1}f(x)$, $f(x) = x^2 + bx + c$, $\sin^{-1}\{f(x)\} = z$, $K = \frac{b^2-4c}{4} \neq 0$, and the integral (1) becomes

$$I = \frac{1}{4}\int \frac{e^z \cos^2 z}{\sin z + K} dz \qquad (2)$$

Applying the Mathematica technique to integrate it, we get

$$\text{In}[3]: \text{Integrate}[\frac{1}{4} * \frac{\text{Exp}[z] * \text{Cos}[z] * \text{Cos}[z]}{\text{Sin}[z] + K}, z]$$

$$\text{Out}[3]: \frac{1}{4}\int \frac{e^z\text{Cos}[z]^2}{K + \text{Sin}[z]} dz$$

which doesn't give any satisfactory result due to arbitrary constant K. Taking some particular value of K as -1, 1, -2, 2, we get that the Mathematica technique gives result for -1 and 1 but no satisfactory result is found for another values of K. Their results are as follows:

$$\text{In}[4]: \text{Integrate}[\frac{1}{4} * \frac{\text{Exp}[z] * \text{Cos}[z] * \text{Cos}[z]}{\text{Sin}[z] - 1}, z]$$

$$\text{Out}[4]: \frac{1}{8}e^z(-2 + \text{Cos}[z] - \text{Sin}[z])$$

$$\text{In}[5]: \text{Integrate}[\frac{1}{4} * \frac{\text{Exp}[z] * \text{Cos}[z] * \text{Cos}[z]}{\text{Sin}[z] + 1}, z]$$

$$\text{Out}[5]: \frac{1}{8}e^z(2 + \text{Cos}[z] - \text{Sin}[z])$$

i.e., the integral (2) in elementary for K = -1, 1 and nonelementary for other values of K. Thus for $g(x) = \sin^{-1}f(x)$, the integral (1) is elementary for all $f(x) = x + b$. It is also elementary for $f(x) = x^2 + bx + c$, if $K = \frac{b^2-4c}{4} = -1, 1$ i.e., it is nonelementary for $f(x) = x^2 + bx + c$, for all K except -1 and 1. Chaudhary & Yadav (2024) didn't consider this special cases for K = -1 and 1 in their study for which the integral (1) is elementary.

Case-II: When $g(x) = \cos^{-1}f(x)$, $\cos^{-1}\{f(x)\} = z$, and $f(x) = x + b$, we get from (1)

$$I = -\frac{1}{2}\int e^z(1 - \cos 2z)dz$$

Using Mathematica technique, we get

$$In[6]: \text{Integrate}[\frac{-1}{2} * \text{Exp}[z] * (1 - \text{Cos}[2 * z]), z]$$

$$\text{Out[6]:}\frac{1}{10}e^{z}(-5+\text{Cos}[2z]+2\text{Sin}[2z])$$

which is elementary i.e., the integral of input [6] is elementary.

For $\mathrm{f(x)}=x^2+\mathrm{bx}+\mathrm{c}$, where b and c are arbitrary and $\mathrm{K}=\frac{\mathrm{b^2-4c}}{4}$, we have

$$\mathrm{I}=-\frac{1}{8}\int\frac{\mathrm{e^z(1-\cos 2z)}}{(\cos\mathrm{z}+\mathrm{K})}\mathrm{dz} \qquad (3)$$

The simple case arises for K = 0 and for this we get from (3)

$$\mathrm{I}=-\frac{1}{4}\int\frac{\mathrm{e^z\sin^2 z}}{\cos\mathrm{z}}\mathrm{dz}$$

Integrating using Mathematica technique without considering coefficient and negative sign, we get

$$\text{In[7]: Integrate[Exp}[x]*\text{Sin}[x]*\text{Tan}[x],x]$$

$$\text{Out[7]:}-\frac{1}{2}e^{x}(\text{Cos}[x]-(2-2i)e^{ix}\text{Hypergeometric2F1}[\frac{1}{2}-\frac{i}{2},1,\frac{3}{2}-\frac{i}{2},-e^{2ix}]+\text{Sin}[x])$$

which is nonelementary i.e., the integral of input [7] is nonelementary.

Let us consider that K ≠ 0. Then we have from (3)

$$\mathrm{I}=-\frac{1}{4}\int\frac{\mathrm{e^z\sin^2 z}}{\cos\mathrm{z}+\mathrm{K}}\mathrm{dz}$$

Considering different non-zeros values of K like -1, 1, -2, 2 etc. and using Mathematica technique to integrate without considering the negative sign and coefficient factor, we get

$$\text{In[8]: Integrate}[\frac{\text{Exp}[z]*\text{Sin}[z]*\text{Sin}[z]}{\text{Cos}[z]-1},z]$$

$$Out[8]:-\frac{1}{2}e^{z}(2+\text{Cos}[z]+\text{Sin}[z])$$

$$\text{In[9]: Integrate}[\frac{\text{Exp}[z]*\text{Sin}[z]*\text{Sin}[z]}{\text{Cos}[z]+1},z]$$

$$Out[9]:-\frac{1}{2}e^{z}(-2+\text{Cos}[z]+\text{Sin}[z])$$

But for K = -2, 2, … etc, we don't get any satisfactory result using Mathematica. Hence the integral (3) is elementary for K = -1, 1 and nonelementary for another values. Thus for $\mathrm{g(x)}=\cos^{-1}\mathrm{f(x)}$, the integral (1) is elementary for all $\mathrm{f(x)}=\mathrm{x}+\mathrm{b}$. It is also elementary for $\mathrm{f(x)}=\mathrm{x^2}+\mathrm{bx}+\mathrm{c}$, if $\mathrm{K}=\frac{\mathrm{b^2-4c}}{4}=-1,1$ i.e., it is nonelementary for $\mathrm{f(x)}=\mathrm{x^2}+\mathrm{bx}+\mathrm{c}$, for all K except -1 and 1. Chaudhary & Yadav (2024) didn't consider this special cases for K = -1 and 1 also in their study for which the integral (1) is elementary.

Case-III: When $\mathrm{g(x)}=\tan^{-1}\mathrm{f(x)}$ and $\tan^{-1}\{\mathrm{f(x)}\}=\mathrm{z}$, we get from (1)

$$\mathrm{I}=\int\frac{\mathrm{e^z(1+\tan^2 z)}}{\{\mathrm{f'(x)}\}^2}\sec^2\mathrm{z\,dz} \qquad (4)$$

For $\mathrm{f(x)}=\mathrm{x}+\mathrm{b}$, we get

$$\mathrm{I}=\int\mathrm{e^z\sec^4 z\,dz}$$

Using Mathematica, we get its integral as

$$\text{In[10]: Integrate[Exp}[z]*\text{Sec}[z]*\text{Sec}[z]*\text{Sec}[z]*\text{Sec}[z],z]$$

$$\text{Out[10]:}\frac{1}{6}e^{z}(-5i\text{Hypergeometric2F1}[-\frac{i}{2},1,1-\frac{i}{2},-e^{2iz}]+(2+i)e^{2iz}\text{Hypergeometric2F1}[1,1-\frac{i}{2},2-\frac{i}{2},-e^{2iz}]-\text{Sec}[z]^2+5\text{Tan}[z]+2\text{Sec}[z]^2\text{Tan}[z])$$

which is nonelementary i.e., the integral of input [10] is nonelementary.

For $\mathrm{f(x)}=x^2+\mathrm{bx}+\mathrm{c}$, where b and c are arbitrary constants and $\mathrm{K}=\frac{\mathrm{b^2-4c}}{4}$, from (4) we get

$$\mathrm{I}=\frac{1}{4}\int\frac{\mathrm{e^z\sec^4 z}}{(\tan\mathrm{z}+\mathrm{K})}\mathrm{dz} \qquad (5)$$

The simple case arises for K = 0 and for this we get from (5)

$$\mathrm{I}=\frac{1}{4}\int\frac{\mathrm{e^z\sec^4 z}}{\tan\mathrm{z}}\mathrm{dz}$$

Using Mathematica and without considering coefficient, we get its integral

$$\text{In[11]: Integrate[Exp}[z]*\frac{\text{Sec}[z]*\text{Sec}[z]*\text{Sec}[z]*\text{Sec}[z]}{\text{Tan}[z]},x]$$

$$\text{Out}[11]: (\frac{1}{5}+\frac{i}{10})e^{z}((3+6i)\text{Hypergeometric2F1}[-\frac{i}{2},1,1-\frac{i}{2},-e^{2ix}]-(2+4i)\text{Hypergeometric2F1}[-\frac{i}{2},1,1-\frac{i}{2},e^{2iz}]-3e^{2iz}\text{Hypergeometric2F1}[1,1-\frac{i}{2},2-\frac{i}{2},-e^{2iz}]-2e^{2iz}\text{Hypergeometric2F1}[1,1-\frac{i}{2},2-\frac{i}{2},e^{2iz}]+(2-i)\text{Sec}[z]^{2}-(2-i)\text{Tan}[z])$$

which is nonelementary i.e., the integral of input [11] is nonelementary.

Now let us consider some non-zero values of K like -1, 1, -2, 2, …, we have from (5) using Mathematica

$$\text{In}[12]: \text{Integrate}[\frac{\text{Exp}[z]*\text{Sec}[z]*\text{Sec}[z]*\text{Sec}[z]*\text{Sec}[z]}{\text{Tan}[z]-1},z]$$

$$\text{Out}[12]: \frac{1}{-1+\text{Tan}[z]}(\frac{1}{5}+\frac{i}{10})\text{Sec}[z](\text{Cos}[z]-\text{Sin}[z])((8-4i)e^{z}-(12+4i)\text{Cosh}[z]-(3+6i)e^{z}\text{Hypergeometric2F1}[-\frac{i}{2},1,1-\frac{i}{2},-e^{2iz}]+3e^{(1+2i)z}\text{Hypergeometric2F1}[1,1-\frac{i}{2},2-\frac{i}{2},-e^{2iz}]-(2-i)e^{z}\text{Sec}[z]^{2}-(12+4i)\text{Sinh}[z]+(8+16i)\text{Hypergeometric2F1}[-\frac{i}{2},1,1-\frac{i}{2},-i\text{Cos}[2z]+\text{Sin}[2z]](\text{Cosh}[z]+\text{Sinh}[z])-(2-i)e^{z}\text{Tan}[z])$$

$$\text{In}[13]: \text{Integrate}[\frac{\text{Exp}[z]*\text{Sec}[z]*\text{Sec}[z]*\text{Sec}[z]*\text{Sec}[z]}{\text{Tan}[z]+1},z]$$

$$\text{Out}[13]: \frac{1}{1+\text{Tan}[z]}(\frac{1}{5}+\frac{i}{10})\text{Sec}[z](\text{Cos}[z]+\text{Sin}[z])((8-4i)e^{z}-(4-12i)\text{Cosh}[z]+(7+14i)e^{z}\text{Hypergeometric2F1}[-\frac{i}{2},1,1-\frac{i}{2},-e^{2iz}]-7e^{(1+2i)z}\text{Hypergeometric2F1}[1,1-\frac{i}{2},2-\frac{i}{2},-e^{2iz}]+(2-i)e^{z}\text{Sec}[z]^{2}-(4-12i)\text{Sinh}[z]-(8+16i)\text{Hypergeometric2F1}[-\frac{i}{2},1,1-\frac{i}{2},i\text{Cos}[2z]-\text{Sin}[2z]](\text{Cosh}[z]+\text{Sinh}[z])-(6-3i)e^{z}\text{Tan}[z])$$

$$\text{In}[14]: \text{Integrate}[\frac{\text{Exp}[z]*\text{Sec}[z]*\text{Sec}[z]*\text{Sec}[z]*\text{Sec}[z]}{\text{Tan}[z]-2},z]$$

$$Out[14]: (\frac{1}{5}+\frac{i}{10})e^{z}((10+20i)+(7+14i)\text{Hypergeometric2F1}[-\frac{i}{2},1,1-\frac{i}{2},-e^{2iz}]-(20+40i)\text{Hypergeometric2F1}[-\frac{i}{2},1,1-\frac{i}{2},(-\frac{3}{5}-\frac{4i}{5})e^{2iz}]-7e^{2iz}\text{Hypergeometric2F1}[1,1-\frac{i}{2},2-\frac{i}{2},-e^{2iz}]+(2-i)\text{Sec}[z]^{2}+(6-3i)\text{Tan}[z])$$

all are nonelementary. Thus the integral (1) is nonelementary for linear f(x) as well as for quadratic f(x), when $g(x) = \tan^{-1}\{f(x)\}$.

Case-IV: When $g(x) = \cot^{-1}f(x)$ and $\cot^{-1}\{f(x)\} = z$, we get

$$I = -\int \frac{e^{z}(1+\cot^{2}z)}{\{f'(x)\}^{2}}\operatorname{cosec}^{2} z\, dz \qquad (6)$$

For $f(x) = x + b$, from (6) we get

$$I = -\int e^{z}\operatorname{cosec}^{4}z\, dz$$

Using Mathematica and ignoring negative sign of the above integral, we get its antiderivative as

$$\text{In}[15]: \text{Integrate}[\text{Exp}[z]*\text{Csc}[z]*\text{Csc}[z]*\text{Csc}[z]*\text{Csc}[z],z]$$

$$\text{Out}[15]: -\frac{1}{6}e^{z}(5\text{Cot}[z]+\text{Csc}[z]^{2}+2\text{Cot}[z]\text{Csc}[z]^{2}+5i\text{Hypergeometric2F1}[-\frac{i}{2},1,1-\frac{i}{2},e^{2iz}]+(2+i)e^{2iz}\text{Hypergeometric2F1}[1,1-\frac{i}{2},2-\frac{i}{2},e^{2iz}])$$

which is nonelementary i.e., the integral of input [15] is nonelementary.

For $f(x) = x^2 + bx + c$, where b and c are arbitrary constants and $K = \frac{b^2-4c}{4}$, from (6) we get

$$\mathrm{I} = -\frac{1}{4}\int \frac{\mathrm{e^z cosec^4 z}}{(\cot \mathrm{z} + \mathrm{K})}\mathrm{dz} \qquad (7)$$

Again the simple case arises for K = 0 and for this we get from (7)

$$\mathrm{I} = -\frac{1}{4}\int \frac{\mathrm{e^z cosec^4 z}}{\cot \mathrm{z}}\mathrm{dz}$$

Ignoring the negative sign and coefficient, and then suing Mathematica, we get

$$\mathrm{In}[16]\text{: Integrate}[\mathrm{Exp}[z] * \frac{\mathrm{Csc}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z]}{\mathrm{Cot}[z]}, z]$$

$$\mathrm{Out}[16]\text{: }(\frac{1}{5} + \frac{i}{10})e^z((-2+i)\mathrm{Cot}[z] - (2-i)\mathrm{Csc}[z]^2 + (2+4i)\mathrm{Hypergeometric2F1}[-\frac{i}{2}, 1{,}1-\frac{i}{2}, -e^{2iz}] - (3+6i)\mathrm{Hypergeometric2F1}[-\frac{i}{2}, 1{,}1-\frac{i}{2}, e^{2iz}] - 2e^{2iz}\mathrm{Hypergeometric2F1}[1{,}1-\frac{i}{2}, 2-\frac{i}{2}, -e^{2iz}] - 3e^{2iz}\mathrm{Hypergeometric2F1}[1{,}1-\frac{i}{2}, 2-\frac{i}{2}, e^{2iz}])$$

which is nonelementary i.e., the integral of input [16] is nonelementary.

Now let us consider non-zeros values of K like -1, 1, 2, etc. Then, we have from (7) using Mathematica without considering negative sign and coefficient of the integral

$$\mathrm{In}[17]\text{: Integrate}[\frac{\mathrm{Exp}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z]}{\mathrm{Cot}[z] - 1}, z]$$

$$\mathrm{Out}[17]\text{: }\frac{1}{-1+\mathrm{Cot}[z]}\left(\frac{1}{5}+\frac{i}{10}\right)\mathrm{Csc}[z](\mathrm{Cos}[z]-\mathrm{Sin}[z])\Bigg((8-4i)e^z - (12+4i)\mathrm{Cosh}[z] - (6-3i)e^z\mathrm{Cot}[z] - (2-i)e^z\mathrm{Csc}[z]^2 - (7+14i)e^z\mathrm{Hypergeometric2F1}\left[-\frac{i}{2}, 1{,}1-\frac{i}{2}, e^{2iz}\right] - 7e^{(1+2i)z}\mathrm{Hypergeometric2F1}\left[1{,}1-\frac{i}{2}, 2-\frac{i}{2}, e^{2iz}\right] - (12+4i)\mathrm{Sinh}[z] + (8+16i)\mathrm{Hypergeometric2F1}\left[-\frac{i}{2}, 1{,}1-\frac{i}{2}, -i\mathrm{Cos}[2z]+\mathrm{Sin}[2z]\right](\mathrm{Cosh}[z] + \mathrm{Sinh}[z])\Bigg)$$

$$\mathrm{In}[18]\text{: Integrate}\left[\frac{\mathrm{Exp}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z]}{\mathrm{Cot}[z] + 1}, z\right]$$

$$\mathrm{Out}[18]\text{: }\frac{1}{1+\mathrm{Cot}[z]}\left(\frac{1}{5}+\frac{i}{10}\right)\mathrm{Csc}[z](\mathrm{Cos}[z]+\mathrm{Sin}[z])\Bigg((-8+4i)e^z + (4-12i)\mathrm{Cosh}[z] + (2-i)e^z\mathrm{Cot}[z] - (2-i)e^z\mathrm{Csc}[z]^2 - (3+6i)e^z\mathrm{Hypergeometric2F1}\left[-\frac{i}{2}, 1{,}1-\frac{i}{2}, e^{2iz}\right] - 3e^{(1+2i)z}\mathrm{Hypergeometric2F1}\left[1{,}1-\frac{i}{2}, 2-\frac{i}{2}, e^{2iz}\right] + (4-12i)\mathrm{Sinh}[z] + (8+16i)\mathrm{Hypergeometric2F1}\left[-\frac{i}{2}, 1{,}1-\frac{i}{2}, i\mathrm{Cos}[2z]-\mathrm{Sin}[2z]\right](\mathrm{Cosh}[z] + \mathrm{Sinh}[z])\Bigg)$$

$$\mathrm{In}[19]\text{: Integrate}[\frac{\mathrm{Exp}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z]}{\mathrm{Cot}[z] + 2}, z]$$

$$\mathrm{Out}[19]\text{: }\frac{1}{2+\mathrm{Cot}[z]}(\frac{1}{5}+\frac{i}{10})e^z\mathrm{Csc}[z]((-10-20i) + (6-3i)\mathrm{Cot}[z] - (2-i)\mathrm{Csc}[z]^2 + (20+40i)\mathrm{Hypergeometric2F1}[-\frac{i}{2}, 1{,}1-\frac{i}{2}, (\frac{3}{5}+\frac{4i}{5})e^{2iz}] - (7+14i)\mathrm{Hypergeometric2F1}[-\frac{i}{2}, 1{,}1-\frac{i}{2}, e^{2iz}] - 7e^{2iz}\mathrm{Hypergeometric2F1}[1{,}1-\frac{i}{2}, 2-\frac{i}{2}, e^{2iz}])(\mathrm{Cos}[z] + 2\mathrm{Sin}[z])$$

all are nonelementary. Thus the integral (1) is nonelementary for linear f(x) as well as for quadratic f(x), when $\mathrm{g(x)} = \cot^{-1}\{\mathrm{f(x)}\}$.

Case-V: When $\mathrm{g(x)} = \sec^{-1}\mathrm{f(x)}$ and $\sec^{-1}\{\mathrm{f(x)}\} = \mathrm{z}$, we get from (1)

$$\mathrm{I} = \int \frac{\mathrm{e^z}\left[\sec \mathrm{z}\sqrt{\{\sec \mathrm{z}\}^2 - 1}\right]}{\{\mathrm{f'(x)}\}^2} \sec \mathrm{z} \tan \mathrm{z}\, \mathrm{dz} \qquad (8)$$

For $\mathrm{f(x)} = \mathrm{x} + \mathrm{b}$, from (8) we get

$$\mathrm{I} = \int \mathrm{e^z} \sec^2 \mathrm{z} \tan^2 \mathrm{z}\, \mathrm{dz}$$

Using Mathematica we get its integral as

$$\mathrm{In}[20]\mathrm{: Integrate}[\mathrm{Exp}[x] * \mathrm{Sec}[x] * \mathrm{Sec}[x] * \mathrm{Tan}[x] * \mathrm{Tan}[x], x]$$

$$\mathrm{Out}[20]\mathrm{:} (\frac{1}{15} + \frac{i}{30})e^x((1+2i)\mathrm{Hypergeometric2F1}[-\frac{i}{2}, 1, 1-\frac{i}{2}, -e^{2ix}] - e^{2ix}\mathrm{Hypergeometric2F1}[1, 1-\frac{i}{2}, 2-\frac{i}{2}, -e^{2ix}] + (2-i)(-\mathrm{Tan}[x] + \mathrm{Sec}[x]^2(-1+2\mathrm{Tan}[x])))$$

which is nonelementary i.e., the integral of input [20] is nonelementary.

For $\mathrm{f(x)} = x^2 + \mathrm{bx} + \mathrm{c}$, where b and c are arbitrary constants and $\mathrm{K} = \frac{\mathrm{b}^2 - 4\mathrm{c}}{4}$, from (8) we get

$$\mathrm{I} = \frac{1}{4}\int \frac{\mathrm{e^z}\left[\sec \mathrm{z}\sqrt{\{\sec \mathrm{z}\}^2 - 1}\right]}{(\sec \mathrm{z} + \mathrm{K})} \sec \mathrm{z} \tan \mathrm{z}\, \mathrm{dz} \qquad (9)$$

Again the simple case arise for K = 0 and for this we get

$$\mathrm{I} = \frac{1}{4}\int \mathrm{e^z} \tan^2 \mathrm{z} \sec \mathrm{z}\, \mathrm{dz}$$

Integrating using Mathematica, we get

$$\mathrm{In}[21]\mathrm{: Integrate}[\mathrm{Exp}[z] * \mathrm{Sec}[z] * \mathrm{Tan}[z] * \mathrm{Tan}[z], z]$$

$$\mathrm{Out}[21]\mathrm{:} \frac{1}{2}e^z\mathrm{Sec}[z](-1 + \mathrm{Tan}[z])$$

which is elementary i.e., the integral of input [21] is elementary.

Now let us consider that K ≠ 0. Taking its values as -1, 1, 2, …, we have from (9)

$$\mathrm{In}[22]\mathrm{: Integrate}[\frac{\mathrm{Exp}[z] * \mathrm{Sec}[z] * \mathrm{Sec}[z] * \mathrm{Tan}[z] * \mathrm{Tan}[z]}{\mathrm{Sec}[z] - 1}, z]$$

$$\mathrm{Out}[22]\mathrm{:} \frac{1}{(1+e^{2iz})^2}(\frac{3}{5} - \frac{i}{5})e^{(1+2i)z}((2-i)((1-i) + (2-i)e^{iz} + (1-i)e^{2iz} + e^{3iz}) + (1+i)(1+e^{2iz})^2\mathrm{Hypergeometric2F1}[1, 2-i, 3-i, ie^{iz}])$$

$$\mathrm{In}[23]\mathrm{: Integrate}[\frac{\mathrm{Exp}[z] * \mathrm{Sec}[z] * \mathrm{Sec}[z] * \mathrm{Tan}[z] * \mathrm{Tan}[z]}{\mathrm{Sec}[z] + 1}, z]$$

$$\mathrm{Out}[23]\mathrm{:} \frac{1}{(1+e^{2iz})^2}(\frac{1}{5} + \frac{3i}{5})e^{(1+2i)z}((3+i) - (4+3i)e^{iz} + (3+i)e^{2iz} - (1+2i)e^{3iz} - (1-i)(1+e^{2iz})^2\mathrm{Hypergeometric2F1}[1, 2-i, 3-i, -ie^{iz}])$$

which are all nonelementary but Mathematica doesn't give any satisfactory result for K = 2 and others. Thus the integral (1) is nonelementary for linear and quadratic f(x), when $\mathrm{g(x)} = \sec^{-1}\{\mathrm{f(x)}\}$, and K is non-zero. For K = 0 it is elementary.

Case-VI: When $\mathrm{g(x)} = \mathrm{cosec}^{-1}\mathrm{f(x)}$ and $co\mathrm{sec}^{-1}\{\mathrm{f(x)}\} = \mathrm{z}$, we get

$$\mathrm{I} = -\int \frac{\mathrm{e^z}\left[\mathrm{cosec}\, \mathrm{z}\sqrt{\{\mathrm{cosec}\, \mathrm{z}\}^2 - 1}\right]}{\{\mathrm{f'(x)}\}^2} \mathrm{cosec}\, \mathrm{z} \cot \mathrm{z}\, \mathrm{dz} \qquad (10)$$

For $\mathrm{f(x)} = \mathrm{x} + \mathrm{b}$, from (10) we get

$$\mathrm{I} = -\int \mathrm{e^z}\, \mathrm{cosec}^2 \mathrm{z} \cot^2 \mathrm{z}\, \mathrm{dz}$$

$$\mathrm{In}[24]\mathrm{: Integrate}[\mathrm{Exp}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z] * \mathrm{Cot}[z] * \mathrm{Cot}[z], z]$$

$$Out[24]\mathrm{:} (\frac{1}{15} + \frac{i}{30})e^z((-2+i)(\mathrm{Csc}[z]^2 + \mathrm{Cot}[z](-1 + 2\mathrm{Csc}[z]^2)) + (1+2i)\mathrm{Hypergeometric2F1}[-\frac{i}{2}, 1, 1-\frac{i}{2}, e^{2iz}] + e^{2iz}\mathrm{Hypergeometric2F1}[1, 1-\frac{i}{2}, 2-\frac{i}{2}, e^{2iz}])$$

which is nonelementary i.e., the integral of input [24] is nonelementary.

For $\mathrm{f(x)} = x^2 + \mathrm{bx} + \mathrm{c}$, where b and c are arbitrary constants and $\mathrm{K} = \frac{\mathrm{b}^2 - 4\mathrm{c}}{4}$, we have

$$\mathrm{I} = -\frac{1}{4}\int \frac{\mathrm{e^z}\left[\mathrm{cosec}\, \mathrm{z}\sqrt{\{\mathrm{cosec}\, \mathrm{z}\}^2 - 1}\right]}{(\mathrm{cosec}\, \mathrm{z} + \mathrm{K})} \mathrm{cosec}\, \mathrm{z} \cot \mathrm{z}\, \mathrm{dz} \qquad (11)$$

Once again the simple case arises for K = 0, and for this we get from (11)

$$\mathrm{I} = -\frac{1}{4}\int \mathrm{e^z \, cosec\, z \cot^2 z\, dz}$$

Using Mathematica, we get

$$\mathrm{In}[25]\!: \mathrm{Integrate}[\mathrm{Exp}[z] * \mathrm{Csc}[z] * \mathrm{Cot}[z] * \mathrm{Cot}[z], z]$$

$$Out[25]\!: -\frac{1}{2}e^z(1 + \mathrm{Cot}[z])\mathrm{Csc}[z]$$

which is elementary i.e., the integral of input [25] is elementary.
Now let us consider that K ≠ 0, then from (11) we have

$$\mathrm{I} = -\frac{1}{4}\int \frac{\mathrm{e^z\, cosec^2 z \cot^2 z}}{(\mathrm{cosec\, z} + \mathrm{K})}\mathrm{dz} \qquad (12)$$

Taking value of K as -1, 1, 2, etc. and using Mathematica by ignoring negative sign and coefficient, we get from (12) that

$$\mathrm{In}[26]\!: \mathrm{Integrate}[\frac{\mathrm{Exp}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z] * \mathrm{Cot}[z] * \mathrm{Cot}[z]}{\mathrm{Csc}[z] - 1}, z]$$

$$Out[26]\!: -\frac{1}{2}e^z(2\mathrm{Cot}[z] + \mathrm{Csc}[z] + \mathrm{Cot}[z]\mathrm{Csc}[z] + 2i\mathrm{Hypergeometric2F1}[-i,1,1-i,e^{iz}] + (1 + i)e^{iz}\mathrm{Hypergeometric2F1}[1,1-i,2-i,e^{iz}])$$

$$\mathrm{In}[27]\!: \mathrm{Integrate}[\frac{\mathrm{Exp}[z] * \mathrm{Csc}[z] * \mathrm{Csc}[z] * \mathrm{Cot}[z] * \mathrm{Cot}[z]}{\mathrm{Csc}[z] + 1}, z]$$

$$\mathrm{Out}[27]\!: \frac{1}{8}e^z(8i\mathrm{Hypergeometric2F1}[-i,1,1-i,-e^{iz}] - 4(\mathrm{Cot}[z](-2 + \mathrm{Csc}[z]) + \mathrm{Csc}[z] + (1 + i)e^{iz}\mathrm{Hypergeometric2F1}[1,1-i,2-i,-e^{iz}]))$$

which are all nonelementary, where as Mathematica doesn't give any result for K = 2 and another values. Therefore it is nonelementary. Thus the integral (1) is nonelementary for linear and quadratic f(x), when $\mathrm{g(x)} = \mathrm{cosec}^{-1}\{\mathrm{f(x)}\}$. It is elementary for K = 0.

V. Conclusion

From above discussion, we conclude that Chaudhary & Yadav (2024) skipped two special cases of elementary integrals for $\mathrm{g(x)} = \sin^{-1}\{\mathrm{f(x)}\}$, $\mathrm{g(x)} = \cos^{-1}\{\mathrm{f(x)}\}, f(x) = x^2 + bx + c$ and $\mathrm{K} = \frac{\mathrm{b^2-4c}}{4} = -1{,}1$. Therefore including these cases in the conclusion of their conjecture, we summarize the result as follows: the indefinite integral

$$\int \frac{\mathrm{e^{g\{f(x)\}}}}{\mathrm{g'\{f(x)\}}}\mathrm{dx}$$

where g(x) is an inverse trigonometric function, f(x) a polynomial of degree one and two, and $\mathrm{g'\{f(x)\}}$ a derivative of g with respect to x, is elementary and nonelementary integral for the cases given below in the table-1:

Table-1

Inverse Trigonometric Functions g(x)	Polynomial f(x)	Elementary / Nonelementary
$\sin^{-1}\mathrm{f(x)}$	$\mathrm{x + b}$	Elementary
	$\mathrm{x^2 + bx + c}$ for $\mathrm{K} = \frac{\mathrm{b^2-4c}}{4} = 0$	Nonelementary
	$\mathrm{x^2 + bx + c}$ for $\mathrm{K} = \frac{\mathrm{b^2-4c}}{4} = -1, 1$	Elementary
	$\mathrm{x^2 + bx + c}$ for $\mathrm{K} = \frac{\mathrm{b^2-4c}}{4} \neq -1, 1$	Nonelementary
$\cos^{-1}\mathrm{f(x)}$	$\mathrm{x + b}$	Elementary
	$\mathrm{x^2 + bx + c}$ for $\mathrm{K} = \frac{\mathrm{b^2-4c}}{4} = 0$	Nonelementary
	$\mathrm{x^2 + bx + c}$ for $\mathrm{K} = \frac{\mathrm{b^2-4c}}{4} = -1, 1$	Elementary
	$\mathrm{x^2 + bx + c}$ for $\mathrm{K} = \frac{\mathrm{b^2-4c}}{4} \neq -1, 1$	Nonelementary
$\tan^{-1}\mathrm{f(x)}$	$\mathrm{x + b}$	Nonelementary
	$\mathrm{x^2 + bx + c}$	Nonelementary
$\cot^{-1}\mathrm{f(x)}$	$\mathrm{x + b}$	Nonelementary
	$\mathrm{x^2 + bx + c}$	Nonelementary
$\sec^{-1}\mathrm{f(x)}$	$\mathrm{x + b}$	Nonelementary

	$x^2 + bx + c$ for $K = \frac{b^2-4c}{4} = 0$	Elementary
	$x^2 + bx + c$ for $K = \frac{b^2-4c}{4} \neq 0$	Nonelementary
$\mathrm{cosec}^{-1}f(x)$	$x + b$	Nonelementary
	$x^2 + bx + c$ for $K = \frac{b^2-4c}{4} = 0$	Elementary
	$x^2 + bx + c$ for $K = \frac{b^2-4c}{4} \neq 0$	Nonelementary

The present study shows that the research done using computer software Mathematica is more reliable than using the traditional theoretical methods and theorems of integrations like strong Liouville's theorem. Although this theorem is a powerful technique to decide the elementary and nonelementary integrals but it seeks more time and many special cases to explore to reach at the final result.

VI. Future Scope of Research

The indefinite integral of the proffered conjecture has been discussed for only two cases of the polynomial of linear and quadratic nature. A big scope is available for research for higher degree polynomials and its special cases. There is a possibility that they might be nonelementary for higher degree polynomial f(x) because as the degree of the polynomial increases, the integrand becomes more complex than the previous one. This is the reason that the proffered indefinite integral has been called a conjecture and not a theorem or property, which is the big limitation of the present work. A lot of scope is also available for the computer software experts and researchers as there is no concept available in Mathematica for elementary and nonelementary functions. These can be also induced in other computer software.

References

[1]. Abbott, P. (1997). Teaching Mathematics Using Mathematica. In Proceedings Of The 2nd Asian Technology Conference In Mathematics (Pp. 24-40).

[2]. Abell, M. L., & Braselton, J. P. (2021). Mathematica By Example. Academic Press.

[3]. Chaudhary, M. K. & Yadav, D. K. (2024). Strong's Liouville Theorem Based Conjecture On Nonelementary Integrals, IOSR Journal Of Mathematics (IOSR-JM), 20 (1), Ser.1 (Jan. – Feb.), 35-44.

[4]. Fitelson, B. (1998). Using Mathematica To Understand The Computer Proof Of The Robbins Conjecture. Mathematica In Education And Research, 7, 17-26.

[5]. Hardy, G. H. (2018). The Integration Of Functions Of A Single Variable, Hawk Press, 1-12.

[6]. Hayes, B. (1990). Thoughts On Mathematica. Pixel, 1, 28-34.

[7]. Hilbe, J. M. (2006). Mathematica 5.2: A Review. The American Statistician, 60(2), 176-186.

[8]. Marchisotto, E. A., & Zakeri, G. A. (1994). An Invitation To Integration In Finite Terms. The College Mathematics Journal, 25(4), 295-308.

[9]. Maeder, R. (1991). Programming In Mathematica. Addison-Wesley Longman Publishing Co., Inc..

[10]. Nijimbere, V. (2020). Evaluation Of Some Non-Elementary Integrals Involving The Generalized Hypergeometric Function With Some Applications. Arxiv Preprintarxiv:2003.07403.

[11]. Paul, C. A. (2018). Mathematica. In Revival: The Handbook Of Software For Engineers And Scientists (1995) (Pp. 926-962). CRC Press.

[12]. Risch, R. H. (2022). On The Integration Of Elementary Functions Which Are Built Up Using Algebraic Operations. In Integration In Finite Terms: Fundamental Sources (Pp. 200-216). Cham: Springer International Publishing.

[13]. Ritt, J. F. (2022). Integration In Finite Terms Liouville's Theory Of Elementary Methods. In Integration In Finite Terms: Fundamental Sources (Pp. 31-134). Cham: Springer International Publishing.

[14]. Rose, C., & Smith, M. D. (2002). Mathstatica: Mathematical Statistics With Mathematica. In Compstat: Proceedings In Computational Statistics (Pp. 437-442). Physica-Verlag HD.

[15]. Sao, G. S. (2021). Special Functions, 3rd Revised Edition, Shree Shiksha Sahitya Prakashan, Meerut, Pp. 1-3, 40-45.

[16]. Sharma, J. N. & Gupta, R. K. (2020). Special Functions, Krishna Prakashan Media (P) Ltd., Meerut, 34th Edition, Pp. 70-72.

[17]. Stroyan, K. D. (2014). Calculus Using Mathematica. Academic Press.

[18]. Trott, M. (2007). The Mathematica Guidebook For Symbolics. Springer Science & Business Media.

[19]. Victor, N. (2017). EVALUATION OF THE NON-ELEMENTARY INTEGRAL∫ Eλx^ Adx, A≥ 2, AND OTHER RELATED INTEGRALS. Ural Mathematical Journal, 3(2 (5)), 130-142.

[20]. Wagon, S., & Wagon, S. (1999). Mathematica In Action. Springer Science & Business Media.

[21]. Wikipedia Contributors. (2023, April 5). Conjecture. In Wikipedia, The Free Encyclopedia. Retrieved 16:48, April 15, 2023, From Https://En.Wikipedia.Org/W/Index.Php?Title=Conjecture&Oldid=1148406309

[22]. Wikipedia Contributors. (2023, January 1). Elementary Function. In Wikipedia, The Free Encyclopedia. Retrieved 12:05, April 10, 2023, From Https://En.Wikipedia.Org/W/Index.Php?Title=Elementary_Function&Oldid=1130910903

[23]. Wikipedia Contributors. (2023, March 29). Closed-Form Expression. In Wikipedia, The Free Encyclopedia. Retrieved 12:48, April 10, 2023, From Https://En.Wikipedia.Org/W/Index.Php?Title=Closed-Form_Expression&Oldid=1147215178

[24]. Wikipedia Contributors. (2022, August 13). Nonelementary Integral. In Wikipedia, The Free Encyclopedia. Retrieved 13:20, April 10, 2023, From Https://En.Wikipedia.Org/W/Index.Php?Title=Nonelementary_Integral&Oldid=1104142356

[25]. Wikipedia Contributors. (2023, December 22). Wolfram Mathematica. In Wikipedia, The Free Encyclopedia. Retrieved 11:08, January 24, 2024, From Https://En.Wikipedia.Org/W/Index.Php?Title=Wolfram_Mathematica&Oldid=1191274595

[26]. Wolfram, S. (1999). The MATHEMATICA® Book, Version 4. Cambridge University Press.

[27]. Yadav, D. K. & Sen, D. K. (2012). Ph.D. Thesis: A Study Of Indefinite Nonintegrable Functions, (Vinoba Bhave University, Hazaribag, Jharkhand, India). GRIN Verlag, Open Publishing Gmbh, Germany.

[28]. Yadav, D. K. (2023). Nonelementary Integrals: Indefinite Nonintegrable Functions, Notion Press, India, 36-54.