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Further Discussion On Wash Criteria In Analytic Hierarchy Process

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Abstract:
This paper is a response of three articles: Saaty and Vargas, [published in International Journal of Management and Decision Making, 2006, 180], Liberatore and Nydick [published in Computers & Operations Research, 2004, 889], and Finan and Hurley [published in Computers & Operations Research, 2002, 1025]. Finan and Hurley (2002) claimed that their paper forms a serious challenge to Analytic Hierarchy Process. Saaty and Vargas (2006), and Liberatore and Nydick (2004) both pointed out that after wash criterion being deleted, then the weights for the upper level criteria should be re-evaluated. We provide a patch work for re-evaluation to show that the four level hierarchy decision problem in Finan and Hurley (2002) will not cause rank reversal. Our findings will help researcher to apply analytic hierarchy process with confidence.
***Key Word**: Analytic Hierarchy Process; Wash Criteria.*

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I. Introduction

Finan and Hurley [1] proposed that (a) a theorem for three level hierarchy decision problems that deleting a wash criterion will not change the order of alternatives, (b) an example of four level hierarchy decision problem that deleting a wash criterion will change the order of alternatives, and (c) every hierarchy decision problem can be compressed into three level. Finan and Hurley [1] claimed that their findings provide a serious challenge to analytic hierarchy process.

Saaty and Vargas [4], and Liberatore and Nydick [2] tried to defend analytic hierarchy process by arguing that when a wash criterion with a relative high weight is deleted then the relative weight for upper level criteria will be re-evaluated so the rank reversal phenomenon may be avoid.

There are two principles:

(a) The relative weights of the upper level have to change after the lower level wash criterion is deleted, as proposed by Liberatore and Nydick [2], and Saaty and Vargas [4].

(b) The relative weights of the upper level will not change after the lower level wash criterion is deleted as proposed by Finan and Hurley [1] and Lin et al. [3].

The purpose of this article is provided a theoretical verification to prove that after re-evaluating the weights for upper level criteria then the rank reversal phenomenon will not happen such that the four level hierarchy decision problem in Finan and Hurley [1] did not provide any challenge to analytic hierarchy process.

II. Review of Previous Results

Let us recall the four level hierarchy decision problem in Finan and Hurley [1] with the following four conditions:

(1) the top level: goal,
(2) the second level: criterion, J and J',
(3) the third level: sub-criterion, J_0, J_1, J_2, J_1' and J_2', and
(4) the fourth level: alternatives, A_1 and A_2.

Table 1. Example with wash criterion J_0

	J			J'	
	0.55			0.45	
	J_0	J_1	J_2	J_1'	J_2'
	0.6	0.2	0.2	0.5	0.5

A_1	0.5	0.8	0.4	0.2	0.6
A_2	0.5	0.2	0.6	0.8	0.6

We recall the aggregation approach by Finan and Hurley [1]. With wash criteria, J_0, the weight of A_1 and A_2 are denoted as $w\left(J_0, A_1\right)$ and $w\left(J_0, A_2\right)$, respectively, then

$w\left(J_0, A_1\right) = 0.477$, (2.1)

and

$w\left(J_0, A_2\right) = 0.523$, (2.2)

where

$w\left(J_0, A_1\right) =$ 0.55(0.6)0.5+0.55(0.2)0.8 +0.55(0.2)0.4+0.45(0.5)0.2+0.45(0.5)0.6. (2.3)

After wash criteria, J_0, is deleted, Finan and Hurley [1] only changed the relative weight of J_1 and J_2 from 0.2, and 0.2 to

$\frac{0.2}{0.2+0.2} = 0.5$, (2.4)

and

$\frac{0.2}{0.2+0.2} = 0.5$, (2.5)

then the weights of A_1 and A_2, without J_0, are denoted as $w(\not{J}_0, A_1)$ and $w(\not{J}_0, A_2)$, respectively. Finan and Hurley [1] derived that

$w(\not{J}_0, A_1) = 0.51$, (2.6)

and

$w(\not{J}_0, A_2) = 0.49$, (2.7)

where

$w(\not{J}_0, A_1) =$ 0.55(0.5)0.8 +0.55(0.5) 0.4+0.45(0.5)0.2+0.45(0.5)0.6. (2.8)

Table 2. Example without wash criterion J_0, by Finan and Hurley [1]

	J		J'	
	0.55		0.45	
	J_1	J_2	J_1'	J_2'
	0.5	0.5	0.5	0.5
A_1	0.8	0.4	0.2	0.6
A_2	0.2	0.6	0.8	0.4

Finan and Hurley [1] mentioned that they constructed a four level hierarchy decision problem with wash criteria, J_0, $A_1 \prec A_2$ and then without wash criteria, J_0, $A_1 \succ A_2$ so there is a rank reversal. Liberatore and Nydick [2] claimed that deleting wash criterion J_0, then criteria J may lose its weight up to 60%, so the relative weight between J and J' should be changed. Hence, with the new weights of J and J', the rank reversal may or may not happen. However, Liberatore and Nydick [2] did not tell us how to change. Saaty and Vargas [4] also expressed the similar comments for this four level hierarchy decision problem of Finan and Hurley [1].

III. Our Improvement

We will abstractly consider this four level hierarchy decision problem, under the condition $A_1 \prec A_2$. With wash criterion J_0, the weight of A_1, say $w\left(J_0, A_1\right)$, is computed as

$$w\left(J_0, A_1\right) = a(b)0.5 + a(c)e + a(1-b-c)f + (1-a)dg + (1-a)(1-d)h < 0.5. \qquad (3.1)$$

Table 3. Example with J_0 and abstract weights

	J			J'	
	a			$1-a$	
	J_0	J_1	J_2	J_1'	J_2'
	b	c	$1-b-c$	d	$1-d$
A_1	0.5	e	f	g	h
A_2	0.5	$1-e$	$1-f$	$1-g$	$1-h$

When the wash criterion J_0 is deleted then the relative weights of sub-criteria J_1 and J_2 are evaluated from c and $1-b-c$ into $\frac{c}{1-b}$ and $\frac{1-b-c}{1-b}$ that is consistent with Finan and Hurley [1]. Moreover, according to the sub-criterion J_0 being deleted, we can assume that the criteria J will lose some weights proportional to the weight of J_0. Therefore, the weights of J and J' are changed from a and $1-a$ into $a\left(1-b\right)$ and $1-a$, and then we normalize them as $\frac{a(1-b)}{1-ab}$, and $\frac{1-a}{1-ab}$.

Table 4. Example without J_0 and abstract weights

	J		J'	
	$\frac{a(1-b)}{1-ab}$		$\frac{1-a}{1-ab}$	
	J_1	J_2	J_1'	J_2'
	$\frac{c}{1-b}$	$\frac{1-b-c}{1-b}$	d	$1-d$
A_1	e	f	g	h
A_2	$1-e$	$1-f$	$1-g$	$1-h$

Without wash criterion J_0, we find the weight of A_1, say $w(\not{J}_0, A_1)$, is computed as

$$w(\not{J}_0, A_1) = \frac{a(1-b)}{1-ab}\left(\frac{c}{1-b}\right)e + \frac{a(1-b)}{1-ab}\left(\frac{1-b-c}{1-b}\right)f + \left(\frac{1-a}{1-ab}\right)dg + \left(\frac{1-a}{1-ab}\right)(1-d)h. \qquad (3.2)$$

In the following, under the condition $A_1 \prec A_2$ that is $w\left(J_0, A_1\right) < 0.5$ we will prove that $w(\not{J}_0, A_1)$ $< w\left(J_0, A_1\right)$. From Equations (3.1) and (3.2), we know that

$$\left(1-ab\right)w\left(\not{J}_0, A_1\right) = w\left(J_o, A_1\right) - a(b)0.5 < 0.5(1-ab). \qquad (3.3)$$

To simplify the expression, we assume that

$$\Delta = a(c)e + a(1-b-c)f + (1-a)dg + (1-a)(1-d)h, \qquad (3.4)$$

then we imply that

$$\Delta = w\left(J_o, A_1\right) - a(b)0.5\,. \qquad (3.5)$$

We know that $w(\not{J}_0, A_1) < w\left(J_0, A_1\right)$ if and only if $\frac{\Delta}{1-ab} < a(b)0.5 + \Delta$, that is

$$\Delta < 0.5\left(1-ab\right) \qquad (3.6)$$

From Equations (3.4) and (3.5), it yields that Equation (3.6) is valid so we know that $w(\not{J}_0, A_1) < w\left(J_0, A_1\right)$. We summarize our findings in the next theorem.

Theorem 1. If $w\left(J_0, A_1\right) < 0.5$, then $w(\not{J}_0, A_1) < w\left(J_0, A_1\right)$. Moreover, the rank reversal phenomenon will not happen.

At last, we consider the four level hierarchy decision problem of Finan and Hurley [1]. Without wash criterion J_0, then

$$w(\not{J}_0, A_1) = 0.4656716 < w\left(J_0, A_1\right) = 0.477\,, \qquad (3.7)$$

As we showed in Theorem 1. On the other hand, with wash criterion J_0,

$$w\left(J_0, A_1\right) = 0.477 < w\left(J_0, A_2\right) = 0.523 \qquad (3.8)$$

to imply that $A_1 \prec A_2$, and then without wash criterion J_0,

$$w\left(\not{J}_0, A_1\right) = 0.4656716 < w\left(\not{J}_0, A_2\right) = 0.5343284\,, \qquad (3.9)$$

to imply that $A_1 \prec A_2$, so there is no rank reversal. After we revise weights of criteria in upper level then the rank reversal will not happen. Hence, our findings will help researchers to clear the ambiguity that may be aroused by the four level hierarchy decision problem in Finan and Hurley [1].

In the following, we will develop a general theorem for wash criterion. We consider a four level hierarchy decision problem with (1) the top level: goal, (2) the second level: criterion $C_1,\ldots,C_N$, (3) the third level: sub-criterion, for C_1, there are sub-criterion $SC_{1,0}, SC_{1,1},\ldots,SC_{1,n_1}$ where $SC_{1,0}$ is a wash criterion and for C_j, with $2 \le j \le N$, there are sub-criterion $SC_{j,1},\ldots,SC_{j,n_j}$, and (4) the fourth (bottom) level: alternatives, $A_1,\ldots,A_M$. The weights for criterion are a_j, for $1 \le j \le N$, for sub-criterion $SC_{j,k}$ are $b_{j,k}$ for $j=1$, $0 \le k \le n_1$, and $2 \le j \le N$, $1 \le k \le n_j$, for alternative A_i corresponding to sub-criterion $SC_{j,k}$ are $\alpha_{j,k,i}$, with $\alpha_{1,0,i} = \frac{1}{M}$ for $1 \le i \le M$.

With wash criterion $SC_{1,0}$, the weight of A_i is computed as

$$w\left(SC_{1,0}, A_i\right) = a_1 b_{1,0}\alpha_{1,0,i} + \sum_{k=1}^{n_1} a_1 b_{1,k}\alpha_{1,k,i} + \sum_{j=2}^{N}\sum_{k=1}^{n_j} a_j b_{j,k}\alpha_{j,k,i}\,. \qquad (3.10)$$

After the wash criterion $SC_{1,0}$ is deleted, the related weight of A_1 and A_j for $2 \le j \le N$, is changed from a_1 and a_j for $2 \le j \le N$ into $a_1\left(1-b_{1,0}\right)$ and a_j for $2 \le j \le N$ and then they are normalized to $\frac{a_1\left(1-b_{1,0}\right)}{1-a_1 b_{1,0}}$ and $\frac{a_j}{1-a_1 b_{1,0}}$ for $2 \le j \le N$. For the sub-criteria, the weights of $SC_{j,k}$ are unchanged for $2 \le j \le N$, $1 \le k \le n_j$. On the other hand, deleting the wash criterion $SC_{1,0}$, the weights of $SC_{1,k}$ for $1 \le k \le n_1$ is changed from $b_{1,k}$ into $\frac{b_{1,k}}{1-b_{1,0}}$, for $1 \le k \le n_1$.

Hence, without the wash criterion $SC_{1,0}$, the weight of A_i is computed as

$$w\left(\cancel{SC}_{1,0}, A_i\right) = \sum_{k=1}^{n_1} \frac{a_1\left(1-b_{1,0}\right)}{1-a_1 b_{1,0}} \frac{b_{1,k}}{1-b_{1,0}} \alpha_{1,k,i} + \sum_{j=2}^{N}\sum_{k=1}^{n_j} \frac{a_j}{1-a_1 b_{1,0}} b_{j,k}\alpha_{j,k,i}. \qquad (3.11)$$

In the following, we assume that $w\left(SC_{1,0}, A_i\right) \le w\left(SC_{1,0}, A_s\right)$, then we will prove that $w\left(\cancel{SC}_{1,0}, A_i\right) \le w\left(\cancel{SC}_{1,0}, A_s\right)$.

From $w\left(SC_{1,0}, A_i\right) \le w\left(SC_{1,0}, A_s\right)$, owing to $\alpha_{1,0,i} = \frac{1}{M} = \alpha_{1,0,s}$ it yields that

$$\sum_{k=1}^{n_1} a_1 b_{1,k}\alpha_{1,k,i} + \sum_{j=2}^{N}\sum_{k=1}^{n_j} a_j b_{j,k}\alpha_{j,k,i} \le \sum_{k=1}^{n_1} a_1 b_{1,k}\alpha_{1,k,s} + \sum_{j=2}^{N}\sum_{k=1}^{n_j} a_j b_{j,k}\alpha_{j,k,s}. \qquad (3.12)$$

According to Equation (3.12), we rewrite $w\left(\cancel{SC}_{1,0}, A_i\right)$ in the following,

$$w\left(\cancel{SC}_{1,0}, A_i\right) = \frac{1}{1-a_1 b_{1,0}}\left[\sum_{k=1}^{n_1} a_1 b_{1,k}\alpha_{1,k,i} + \sum_{j=2}^{N}\sum_{k=1}^{n_j} a_j b_{j,k}\alpha_{j,k,i}\right]. \qquad (3.13)$$

Similarly, we rewrite $w\left(\cancel{SC}_{1,0}, A_s\right)$ as follows

$$w\left(\cancel{SC}_{1,0}, A_s\right) = \frac{1}{1-a_1 b_{1,0}}\left[\sum_{k=1}^{n_1} a_1 b_{1,k}\alpha_{1,k,s} + \sum_{j=2}^{N}\sum_{k=1}^{n_j} a_j b_{j,k}\alpha_{j,k,s}\right]. \qquad (3.14)$$

If we combine Equations (3.12), (3.13) and (3.14), then it derives that

$$w\left(\cancel{SC}_{1,0}, A_i\right) \le w\left(\cancel{SC}_{1,0}, A_s\right). \qquad (3.15)$$

We summarize our findings in the next theorem.

Theorem 2. If a wash criterion is deleted, then the rank of alternatives will keep the same order.

IV. Direction for Further Research

We invite researchers to consider an example in Yao and Yao [5] for the medical diagnosis to illustrate the possible application in the practical environment. The decision problem has a four level hierarchy, with (1) the top level: goal, one patient, (2) the second level: three symptoms, C_1 : headache, C_2 : fever, C_3 : phlegm, (3) the third level: characters. For headache, three are three sub-criteria, $SC_{1,0}$: minor headache, $SC_{1,1}$: median headache, and $SC_{1,2}$: strong headache. For fever, there are three sub-criteria, $SC_{2,1}$: low fever, $SC_{2,2}$: median fever, and $SC_{2,3}$: high fever. For phlegm, there are two sub-criteria, $SC_{3,1}$: light phlegm, $SC_{3,2}$: thick phlegm, and (4) the fourth (bottom) level: three diseases, A_1 : cold, A_2 : pulmonary tuberculosis, and A_3 : pertussis.

If we assume that cold, pulmonary tuberculosis and pertussis have the same weight for the sub-criterion, minor headache with weight $1/3$, respectively. Hence, the minor headache is a wash criterion. Consequently, if we consider the comparison matrix for sub-criteria respect to headache, then with wash criterion, we have to execute three comparisons among minor, median and strong headache. When the wash criterion, minor headache, is deleted, then we only need to compare median and strong headache. It may save some money and time in medical diagnosis. From our theorem 2, deleting a wash criterion, the rank of alternatives will keep the same order as with the wash criterion. Hence, our findings provide a theoretical support to simplify the AHP procedure.

Up to now, our results are limited to the bottom level with two alternatives. Therefore, how to generalize our findings from two alternatives to arbitrary finite alternatives will be an interesting problem for the further investigation. We may provide a partial solution for the future study.

Theorem 3. If there are n alternatives in the bottom level, and J_0 is a wash criterion for $A_1, \ldots, A_n$, under the condition $w\left(J_0, A_1\right) < \left(1/n\right)$, then $w(\cancel{J}_0, A_1) < w\left(J_0, A_1\right)$. Moreover, if $w\left(J_0, A_2\right) > \left(1/n\right)$, then $w(\cancel{J}_0, A_2) > w\left(J_0, A_2\right)$.

V. Conclusion

In this article, we provide a theoretical proof for the four level hierarchy decision problem of Finan and Hurley [1] to show that when deleting wash criterion, then the rank reversal phenomenon will not occur. Hence, the challenge proposed by Finan and Hurley [1] to challenge the validity of analytic hierarchy process is based on their questionable approach. Our findings will serve as a patch work for Liberatore and Nydick [2] and Saaty and Vargas [4] to defend analytic hierarchy process from arbitrary accusation.

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