IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661,p-ISSN: 2278-8727, Volume 20, Issue 1, Ser. 3 (Jan. – Feb. 2024), PP 65-70
www.iosrjournals.org

Review Of The Conjecture On Elementary Integrals Using Computer Software Mathematica

Dharmendra Kumar Yadav[1*] & Ram Kishun Yadav[2]
[1*]*University Department Of Mathematics, Lalit Narayan Mithila University, Darbhanga, Bihar*
[2]*university Department Of Mathematics, Lalit Narayan Mithila University, Darbhanga, Bihar*

Abstract
The present paper is a review work of the Laplace's theorem based conjecture on elementary integrals using advanced computer software Mathematica. The introduction of many computer software and its applications in Mathematics and allied sciences has replaced the research methodology from traditional theoretic processes to computational techniques. Previously to find antiderivative, a lengthy process were followed but now the computer software programming languages and technique find that in a second by using their predefined codes. So it now saves both time and energy. The software Mathematica is such a tool providing technique which is used in many parts of arts, science, commerce, business, research, teaching, programming etc. In this paper we have used it to review the characteristics of the conjecture on elementary integrals and in finding the indefinite integrals. In general traditional method takes too much time and so the presentation of some error is observed. But in this review paper no such errors have been found in the previous study. The paper ends with a conclusion and a note on future scope of research in elementary and nonelementary integrals, Mathematica and other computer based mathematical software.

Key-Words: *Conjecture, Elementary Function, Elementary Integral, Nonelementary Function, Nonelementary Integral, Mathematica Software.*

Date of Submission: 09-02-2024 Date of Acceptance: 19-02-2024

I. Introduction

The integrands which have indefinite integrals in terms of elementary functions is classically known as elementary integrals and which haven't antiderivative in terms of elementary functions have been called nonelementary integrals. Based on some exceptional cases of the six conjecture propounded by Yadav & Sen (2012) and one conjecture by Chaudhary & Yadav (2024), a new Conjecture was presumed and proved using Laplace's theorem by Yadav & Yadav (2024). Recently Yadav & Chaudhary (2024) have reviewed their conjecture on nonelementary integrals using Mathematica.

Mathematica provides many computational technique systems for computing a wide range of functionalities for symbolic computations, numerical computations, data visualization, and programming. It was developed by Wolfram Research. It is a platform used in various fields like in mathematics, physics, engineering, computer science, finance, computer science, and many other areas of applications (Abbott, 1997; Abell et al., 2021; Hayes, 1990; Mathematica-Wikipedia contributors, 2023; Wolfram, 1999).

It has both numerical and symbolic computational package with extensive associated graphical capabilities and a programming language with an interactive document and a notebook interface. It is one of the well known and well used popular technical computing package in the market of computer software (Hilbe, 2006). It is widely used in academia, research, and industry for a variety of purposes, including mathematical calculations for research, scientific simulations, data analysis, and many other educational purposes. It has a broader user base across different disciplines of arts, science, commerce, engineering, management, etc. due to its versatility and powerful computational capabilities (Fitelson, 1998).

Using it we can solve and manipulate mathematical expressions and equations both symbolically and graphically (Hayes, 1990). It has extensive numerical capabilities for tasks such as solving equations, and numerical integration (Hilbe, 2006). It enables the users to manipulate and visualize data in different formats. It supports a wide range of plotting and visualization functions through graphs. It has its own programming language known as Wolfram Language (Paul, 2018; Rose et a., 2002; Stroyan, 2014). Users can write scripts, functions, and programs to perform complex computations. It contains a vast collection of built in mathematical and scientific functions, making it a comprehensive tool for various technical applications (Trott, 2007).

It uses a notebook interface, allowing users to create interactive documents that combine code, text, graphics, and other elements. It supports interactive manipulation of variables and parameters, making it useful for exploring mathematical concepts and models (Maeder, 1991; Wagon et al., 1999). It can be integrated with many other technologies and programming languages. It also supports the creation of interactive interfaces and applications (Hayes, 1990; Hilbe, 2006; Paul, 2018; Rose et a., 2002; Stroyan, 2014; Trott, 2007). In the present paper we shall use the computer software Mathematica to justify the said conjecture on elementary integrals based on Laplace's theorem propounded by Yadav & Yadav (2024).

II. Preliminary Ideas

We know that Software Mathematica excels in handling symbolic integration both in indefinite and definite forms, providing a range of functions to compute definite and indefinite integrals and work with elementary, complex and special functions. We can compute indefinite integrals using the `Integrate` function by using the code: Integrate [f(x), x], which integrates f(x) with respect to x. But it doesn't provide the result in terms of elementary and nonelementary functions (Elementary function – Wikipedia; Hardy, 2018; Victor, 2017; Nonelementary function – Wikipedia; Yadav, 2023) but in terms of traditional and special functions, which needs some more mathematical information to classify the outcomes. But since the characteristics of the conjecture are all elementary integrals and so no concept of nonelementary integrals or functions is needed in the present review work.

Conjecture: Yadav & Yadav (2024) have concluded that the indefinite integral

$$\int \frac{\mathrm{e}^{\mathrm{g}\{\mathrm{f}(\mathrm{x})\}}}{\mathrm{g}'\{\mathrm{f}(\mathrm{x})\}}\mathrm{dx} = \mathrm{I}\ (\mathrm{Let}) \qquad (1)$$

where g(x) is an inverse hyperbolic function, f(x) a polynomial of degree one and two, and $\mathrm{g}'\{\mathrm{f}(\mathrm{x})\}$ a derivative of g with respect to x, is always elementary.

III. Methodology

As far as the research methodology is concerned, we shall call those indefinite integrals or antiderivatives as elementary integrals, whose antiderivative is expressible in terms of elementary functions and those whose integrals are not expressible in terms of elementary functions or in a finite term of elementary functions, would be called as nonelementary integrals (Closed-form expression-Wikipedia; Marchisotto, et. al., 1994; Nijimbere, 2020; Risch, 2022; Ritt, 2022; Sao, 2021; Sharma, et al., 2020). Generally Mathematica software expresses the integrals in terms of either elementary or special functions or hypergeometric functions. If the result comes out in terms of hypergeometric functions, it means it is nonelementary integrals. We shall use these properties in the study.

IV. Discussion

The characteristics of the conjecture, which is to be reviewed, was proffered by Yadav & Yadav (2024), which was an extension of the six conjectures on indefinite nonintegrable functions propounded by Yadav & Sen (2012), in which they didn't consider the inverse trigonometric functions and the inverse hyperbolic functions as a component of the integrands. Recently Chaudhary & Yadav (2024) have used inverse trigonometric functions as a component and propounded a new conjecture. Thereafter Yadav & Chaudhary (2024) reviewed the same for justification using Mathematica. The conjecture, which will be reviewed, contains inverse hyperbolic functions as a component in the integrands and had been proved by Laplace's theorem and some traditional standard formulae.

To proceed further let us suppose that the indefinite integral discussed in the preliminary ideas section is denoted by I showed by (1). It has been proved that it is always elementary under certain conditions on the polynomials of degree one and two in six cases and twelve sub-cases. In the present study we shall only review and justify the results using Mathematica in six different cases for six different inverse hyperbolic functions one by one as follows:

Case-I: When $\mathrm{g}(\mathrm{x}) = \sinh^{-1}\mathrm{f}(\mathrm{x})$, $\sinh^{-1}\{\mathrm{f}(\mathrm{x})\} = \mathrm{z}$ and $\mathrm{f}(\mathrm{x}) = \mathrm{x} + \mathrm{b}$, from (1) we get

$$\mathrm{I} = \frac{1}{2}\int \mathrm{e}^{\mathrm{z}}(1 + \cosh 2\mathrm{z})\mathrm{dz}$$

Using Mathematica code of integration, we get

$$\mathrm{In}[1]\text{: Integrate}[\frac{1}{2} * \mathrm{Exp}[z] * (1 + \mathrm{Cosh}[z]), z]$$

$$\mathrm{Out}[1]\text{: }\frac{1}{4}(2e^{z} + \frac{e^{2z}}{2} + z)$$

which is elementary i.e., the integral of the input function [1] is elementary.

If we take $\mathrm{f}(\mathrm{x}) = x^2 + \mathrm{bx} + \mathrm{c}$, where b and c are arbitrary numbers and $\mathrm{K} = \frac{\mathrm{b}^2-4\mathrm{c}}{4}$, we get from (1)

$$\mathrm{I} = \frac{1}{8}\int \frac{\mathrm{e}^{\mathrm{z}}(1 + \cosh 2\mathrm{z})}{(\sinh \mathrm{z} + \mathrm{K})}\mathrm{dz} \qquad (2)$$

The simple case arise for K = 0 and for this we get from (2)

$$\mathrm{I} = \frac{1}{8}\int \frac{\mathrm{e}^{\mathrm{z}}(1+\cosh 2\mathrm{z})}{\sinh \mathrm{z}}\,\mathrm{dz}$$

Using Mathematica code of antiderivative, we get

$$\mathrm{In}[2]\!:\mathrm{Integrate}\left[\frac{1}{8}*\frac{\mathrm{Exp}[z]*(1+\mathrm{Cosh}[z])}{\mathrm{Sinh}[z]},z\right]$$

$$\mathrm{Out}[2]\!:\frac{1}{16}(1+\mathrm{Cosh}[z])(e^{z}+2\mathrm{Log}[1-e^{z}])\mathrm{Sech}\left[\frac{z}{2}\right]^{2}$$

which is elementary i.e., the integral (2) is elementary for K = 0 i.e., for $\mathrm{f(x)} = x^2 + 2\sqrt{\mathrm{c}}\,\mathrm{x} + \mathrm{c}$.
Let us consider that K ≠ 0, then we have from (2)

$$\mathrm{I} = \frac{1}{8}\int \frac{\mathrm{e}^{\mathrm{z}}(1+\cosh 2\mathrm{z})}{(\sinh \mathrm{z}+\mathrm{K})}\,\mathrm{dz}$$

Finding its integral for arbitrary K using Mathematica, we get

$$\mathrm{In}[3]\!:\mathrm{Integrate}[\frac{1}{8}*\frac{\mathrm{Exp}[z]*(1+\mathrm{Cosh}[2*z])}{(\mathrm{Sinh}[z]+K)},z]$$

$$\mathrm{Out}[3]\!:\frac{1}{16}(e^{2z}-4e^{z}K-2z+8K\sqrt{-1-K^2}\mathrm{ArcTan}[\frac{e^{z}+K}{\sqrt{-1-K^2}}]+4(1+K^2)\mathrm{Log}[-1+e^{2z}+2e^{z}K])$$

which is elementary for all K. Thus the integral (1) is elementary for both linear and quadratic f(x), when $\mathrm{g(x)} = \sinh^{-1}\mathrm{f(x)}$.

Case-II: When $\mathrm{g(x)} = \cosh^{-1}\mathrm{f(x)}$, $\cosh^{-1}\{\mathrm{f(x)}\} = \mathrm{z}$ and $\mathrm{f(x)} = \mathrm{x} + \mathrm{b}$, we get

$$\mathrm{I} = \frac{1}{2}\int \mathrm{e}^{\mathrm{z}}(\cosh 2\mathrm{z}-1)\mathrm{dz}$$

Using Mathematica code, we get its integral as

$$\mathrm{In}[4]\!:\mathrm{Integrate}[\frac{1}{2}*\mathrm{Exp}[z]*(\mathrm{Cosh}[2*z]-1),z]$$

$$\mathrm{Out}[4]\!:\frac{1}{12}e^{-z}(-3-6e^{2z}+e^{4z})$$

which is elementary. Therefore the given integral (1) is elementary for a polynomial f(x) of degree one, when $\mathrm{g(x)} = \cosh^{-1}\{\mathrm{f(x)}\}$.

For $\mathrm{g(x)} = \cosh^{-1}\mathrm{f(x)}$, $\cosh^{-1}\{\mathrm{f(x)}\} = \mathrm{z}$, $\mathrm{f(x)} = x^2 + \mathrm{bx} + \mathrm{c}$, where b and c are arbitrary and $\mathrm{K} = \frac{\mathrm{b}^2-4\mathrm{c}}{4}$, we get

$$\mathrm{I} = \frac{1}{8}\int \frac{\mathrm{e}^{\mathrm{z}}(\cosh 2\mathrm{z}-1)}{(\cosh \mathrm{z}+\mathrm{K})}\,\mathrm{dz}$$

Using Mathematica code of integration, we get

$$\mathrm{In}[5]\!:\mathrm{Integrate}[\frac{1}{8}*\frac{\mathrm{Exp}[z]*(\mathrm{Cosh}[2*z]-1)}{(\mathrm{Cosh}[z]+K)},z]$$

$$\mathrm{Out}[5]\!:\frac{1}{16}(e^{2z}-4e^{z}K+2z+8K\sqrt{1-K^2}\mathrm{ArcTan}[\frac{e^{z}+K}{\sqrt{1-K^2}}]+4(-1+K^2)\mathrm{Log}[1+e^{2z}+2e^{z}K])$$

which is elementary for all values of K. Thus the integral (1) is elementary for both linear f(x) and quadratic f(x), when $\mathrm{g(x)} = \cosh^{-1}\{\mathrm{f(x)}\}$.

Case-III: When $\mathrm{g(x)} = \tanh^{-1}\mathrm{f(x)}$, $\tanh^{-1}\{\mathrm{f(x)}\} = \mathrm{z}$ and $\mathrm{f(x)} = \mathrm{x} + \mathrm{b}$, from (1) we get

$$\mathrm{I} = \int \mathrm{e}^{\mathrm{z}}(1-\tanh^2\mathrm{z})\,\mathrm{sech}^2\mathrm{z}\,\mathrm{dz}$$

Using Mathematica technique of integration, we get

$$\mathrm{In}[6]\!:\mathrm{Integrate}[\mathrm{Exp}[z]*(1-\mathrm{Tanh}[z]*\mathrm{Tanh}[z])*\mathrm{Sech}[z]*\mathrm{Sech}[z],z]$$

$$\mathrm{Out}[6]\!:\frac{8e^{z}}{3(1+e^{2z})^3}-\frac{14e^{z}}{3(1+e^{2z})^2}+\frac{e^{z}}{1+e^{2z}}+\mathrm{ArcTan}[e^{z}]$$

which is elementary i.e., the integral of the input [6] is elementary.

For $\mathrm{f(x)} = x^2 + \mathrm{bx} + \mathrm{c}$, where b and c are arbitrary constants and $\mathrm{K} = \frac{\mathrm{b}^2-4\mathrm{c}}{4}$, we get

$$\mathrm{I} = \frac{1}{4}\int \frac{\mathrm{e}^{\mathrm{z}}(1-\tanh^2\mathrm{z})\,\mathrm{sech}^2\mathrm{z}}{(\tanh \mathrm{z}+\mathrm{K})}\,\mathrm{dz} \qquad (3)$$

The simple case arise for K = 0. For this we get from (3)

$$\mathrm{I} = \frac{1}{4}\int \frac{\mathrm{e}^{\mathrm{z}}(1-\tanh^2\mathrm{z})\,\mathrm{sech}^2\mathrm{z}}{\tanh \mathrm{z}}\,\mathrm{dz}$$

Using Mathematica technique of integration, we get

$$\mathrm{In}[7]\!:\mathrm{Integrate}[\frac{1}{4}*\frac{\mathrm{Exp}[z]*(1-\mathrm{Tanh}[z]*\mathrm{Tanh}[z])*\mathrm{Sech}[z]*\mathrm{Sech}[z]}{\mathrm{Tanh}[z]},z]$$

$$\mathrm{Out}[7]: \frac{1}{4}(-\frac{2e^z}{(1+e^{2z})^2}+\frac{3e^z}{1+e^{2z}}+\mathrm{ArcTan}[e^z]+\mathrm{Log}[-1+e^z]-\mathrm{Log}[1+e^z])$$

which is elementary i.e., the integral (3) is elementary for K = 0.
Let us consider that K ≠ 0. Then we have from (3)

$$\mathrm{I}=\frac{1}{4}\int \frac{\mathrm{e^z(1-tanh^2\,z)\,sech^2 z}}{(\tanh z+\mathrm{K})}\mathrm{dz}$$

Using Mathematica technique of integration, we get

$$\mathrm{In}[8]: \mathrm{Integrate}[\frac{1}{4}*\frac{\mathrm{Exp}[z]*(1-\mathrm{Tanh}[z]*\mathrm{Tanh}[z])*\mathrm{Sech}[z]*\mathrm{Sech}[z]}{\mathrm{Tanh}[z]+K},z]$$

$$\mathrm{Out}[8]: \frac{1}{4(K+\mathrm{Tanh}[z])}(\frac{e^z(1+e^{2z}(3-2K)-2K)}{(1+e^{2z})^2}+(1+2K-2K^2)\mathrm{ArcTan}[e^z] + 2(-1+K)^{3/2}\sqrt{1+K}\,\mathrm{ArcTan}[\frac{e^z}{\sqrt{\frac{-1+K}{1+K}}}])\mathrm{Sech}[z](K\mathrm{Cosh}[z]+\mathrm{Sinh}[z])$$

which is elementary for all K i.e., the integral (3) is elementary for non-zero K also. Thus the integral (1) is elementary for both linear and quadratic f(x), when $\mathrm{g(x)} = \tanh^{-1}\{\mathrm{f(x)}\}$.

Case-IV: When $\mathrm{g(x)} = \coth^{-1}\mathrm{f(x)}$ and $\coth^{-1}\{\mathrm{f(x)}\} = \mathrm{z}$, we get

$$\mathrm{I}=-\int \frac{\mathrm{e^z(1-coth^2 z)}}{\{\mathrm{f'(x)}\}^2}\mathrm{cosech^2\,z\,dz} \qquad (4)$$

For $\mathrm{f(x)} = \mathrm{x} + \mathrm{b}$, from (4) we get

$$\mathrm{I}=\int \mathrm{e^z(coth^2 z-1)\;cosech^2 z\;dz}$$

Using Mathematica technique of integration, we get

$$\mathrm{In}[9]: \mathrm{Integrate}[\mathrm{Exp}[z]*(\mathrm{Coth}[z]*\mathrm{Coth}[z]-1)*\mathrm{Csch}[z]*\mathrm{Csch}[z],z]$$

$$Out[9]: -\frac{8e^z}{3(-1+e^{2z})^3}-\frac{14e^z}{3(-1+e^{2z})^2}-\frac{e^z}{-1+e^{2z}}-\frac{1}{2}\mathrm{Log}[-1+e^z]+\frac{1}{2}\mathrm{Log}[1+e^z]$$

which is elementary i.e., the integral (4) is elementary for linear f(x).
For $\mathrm{f(x)} = x^2 + \mathrm{bx} + \mathrm{c}$, where b and c are arbitrary constants and $\mathrm{K} = \frac{\mathrm{b^2-4c}}{4}$, from (4) we get

$$\mathrm{I}=\frac{1}{4}\int \frac{\mathrm{e^z(coth^2\,z-1)\;cosech^2 z}}{(\coth z+\mathrm{K})}\mathrm{dz} \qquad (5)$$

The simple case arise for K = 0. For this we get from (5)

$$\mathrm{I}=\frac{1}{4}\int \frac{\mathrm{e^z(coth^2\,z-1)cosech^2 z}}{\coth z}\mathrm{dz}$$

Using Mathematica code of integration, we get

$$\mathrm{In}[10]: \mathrm{Integrate}[\frac{1}{4}*\frac{\mathrm{Exp}[z]*(\mathrm{Coth}[z]*\mathrm{Coth}[z]-1)*\mathrm{Csch}[z]*\mathrm{Csch}[z]}{\mathrm{Coth}[z]},z]$$

$$\mathrm{Out}[10]: \frac{1}{4}(-\frac{2e^z}{(-1+e^{2z})^2}-\frac{3e^z}{-1+e^{2z}}-2\mathrm{ArcTan}[e^z]-\frac{1}{2}\mathrm{Log}[-1+e^z]+\frac{1}{2}\mathrm{Log}[1+e^z])$$

which is elementary i.e., the integral (5) is elementary for K = 0.
Let us consider K ≠ 0. Then we have from (5)

$$\mathrm{I}=\frac{1}{4}\int \frac{\mathrm{e^z(coth^2\,z-1)\;cosech^2 z}}{(\coth z+\mathrm{K})}\mathrm{dz}$$

Using Mathematica technique of integration, we get

$$\mathrm{In}[11]: \mathrm{Integrate}[\frac{1}{4}*\frac{\mathrm{Exp}[z]*(\mathrm{Coth}[z]*\mathrm{Coth}[z]-1)*\mathrm{Csch}[z]*\mathrm{Csch}[z]}{\mathrm{Coth}[z]+K},z]$$

$$\mathrm{Out}[11]: \frac{1}{8(K+\mathrm{Coth}[z])}\mathrm{Csch}[z](-\frac{4e^z}{(-1+e^{2z})^2}+\frac{2e^z(-3+2K)}{-1+e^{2z}}-4(1-K)^{3/2}\sqrt{1+K}\,\mathrm{ArcTan}[\frac{e^z\sqrt{1+K}}{\sqrt{1-K}}] + (-1-2K+2K^2)\mathrm{Log}[-1+e^z]+(1+2K-2K^2)\mathrm{Log}[1+e^z])(\mathrm{Cosh}[z]+K\mathrm{Sinh}[z])$$

which is elementary ie., the integral (5) is elementary for non-zero K also. Thus the integral (1) is elementary for both linear f(x) and quadratic f(x), when $\mathrm{g(x)} = \coth^{-1}\{\mathrm{f(x)}\}$.

Case-V: When $\mathrm{g(x)} = \mathrm{sech}^{-1}\mathrm{f(x)}$, $\mathrm{sech}^{-1}\{\mathrm{f(x)}\} = \mathrm{z}$ and $\mathrm{f(x)} = \mathrm{x} + \mathrm{b}$, from (1) we get

$$\mathrm{I}=\int \mathrm{e^z\,sech^2\,z\,tanh^2\,z\,dz}$$

Using Mathematica technique of integration, we get

$$\mathrm{In}[12]: \mathrm{Integrate}[\mathrm{Exp}[z]*\mathrm{Tanh}[z]*\mathrm{Tanh}[z]*\mathrm{Sech}[z]*\mathrm{Sech}[z],z]$$

$$Out[12]: -\frac{8e^z}{3(1+e^{2z})^3}+\frac{14e^z}{3(1+e^{2z})^2}-\frac{3e^z}{1+e^{2z}}+\text{ArcTan}[e^z]$$

which is elementary i.e., the integral of input [12] is elementary.

For $f(x) = x^2 + bx + c$, where b and c are arbitrary constants and $K = \frac{b^2-4c}{4}$, we get

$$I = \frac{1}{4}\int \frac{e^z\left[\operatorname{sech} z\sqrt{1-\{\operatorname{sech} z\}^2}\right]}{(\operatorname{sech} z + K)} \operatorname{sech} z \tanh z \, dz \qquad (6)$$

The simple case arise for K = 0. For this we get from (6)

$$I = \frac{1}{4}\int e^z \tanh^2 z \operatorname{sech} z \, dz$$

Using Mathematica code of integration, we get

$$\text{In}[13]: \text{Integrate}[\frac{1}{4} * \text{Exp}[z] * \text{Tanh}[z] * \text{Tanh}[z] * \text{Sech}[z], z]$$

$$\text{Out}[13]: \frac{1}{4}(\frac{2+4e^{2z}}{(1+e^{2z})^2}+\text{Log}[-2(1+e^{2z})])$$

which is elementary.

Let us consider that K ≠ 0. Then we have from (6)

$$I = \frac{1}{4}\int \frac{e^z \operatorname{sech}^2 z \tanh^2 z}{(\operatorname{sech} z + K)} dz$$

Using Mathematica technique of integration, we get

$$\text{In}[14]: \text{Integrate}[\frac{1}{4} * \frac{\text{Exp}[z] * \text{Tanh}[z] * \text{Tanh}[z] * \text{Sech}[z] * \text{Sech}[z]}{\text{Sech}[z] + K}, z]$$

$$\text{Out}[14]: \frac{1}{4(K+\text{Sech}[z])}(1+K\text{Cosh}[z])(-\frac{2}{(1+e^{2z})^2}+\frac{4}{1+e^{2z}}-\frac{2e^zK}{1+e^{2z}}+2K\text{ArcTan}[e^z]$$
$$-2\sqrt{-1+K^2}\text{ArcTan}[\frac{1+e^zK}{\sqrt{-1+K^2}}]+\text{Log}[1+e^{2z}]-K^2\text{Log}[1+e^{2z}]-\text{Log}[2e^z+K$$
$$+e^{2z}K]+K^2\text{Log}[2e^z+K+e^{2z}K])\text{Sech}[z]$$

which is elementary for non-zero K also. Thus the integral (1) is elementary for both linear f(x) and quadratic f(x), when $g(x) = \operatorname{sech}^{-1}\{f(x)\}$.

Case-VI: When $g(x) = \operatorname{cosech}^{-1} f(x)$, $\operatorname{cosech}^{-1}\{f(x)\} = z$ and $f(x) = x + b$, from (1) we get

$$I = -\int e^z \operatorname{cosech}^2 z \coth^2 z \, dz$$

Using Mathematica and ignoring negative sign, we get

$$\text{In}[15]: \text{Integrate}[\text{Exp}[z] * \text{Coth}[z] * \text{Coth}[z] * \text{Csch}[z] * \text{Csch}[z], z]$$

$$Out[15]: -\frac{8e^z}{3(-1+e^{2z})^3}-\frac{14e^z}{3(-1+e^{2z})^2}-\frac{3e^z}{-1+e^{2z}}+\frac{1}{2}\text{Log}[-1+e^z]-\frac{1}{2}\text{Log}[1+e^z]$$

which is elementary.

For $f(x) = x^2 + bx + c$, where b and c are arbitrary constants and $K = \frac{b^2-4c}{4}$, we get

$$I = -\frac{1}{4}\int \frac{e^z\left[\operatorname{cosech} z\sqrt{\{\operatorname{cosech} z\}^2+1}\right]}{(\operatorname{cosech} z + K)} \operatorname{cosech} z \coth z \, dz \qquad (7)$$

The simple case arise for K = 0. For this we get from (7)

$$I = -\frac{1}{4}\int e^z \operatorname{cosech} z \coth^2 z \, dz$$

Using Mathematica technique and ignoring negative sign and coefficient factor, we get

$$\text{In}[16]: \text{Integrate}[\text{Exp}[z] * \text{Coth}[z] * \text{Coth}[z] * \text{Csch}[z], z]$$

$$\text{Out}[16]: \frac{2-4e^{2z}}{(-1+e^{2z})^2}+\text{Log}[2-2e^{2z}]$$

which is elementary.

Let us consider that K ≠ 0. Then we have from (7) that

$$I = -\frac{1}{4}\int \frac{e^z \operatorname{cosech}^2 z \coth^2 z}{(\operatorname{cosech} z + K)} dz$$

Using Mathematica technique and ignoring negative sign and coefficient factor, we get

$$\text{In}[17]: \text{Integrate}[\frac{\text{Exp}[z] * \text{Coth}[z] * \text{Coth}[z] * \text{Csch}[z] * \text{Csch}[z]}{\text{Csch}[z] + K}, z]$$

$$\text{Out}[17]\text{:}\frac{1}{K+\text{Csch}[z]}\text{Csch}[z](-\frac{2}{(-1+e^{2z})^2}+\frac{2(-2+e^zK)}{-1+e^{2z}}-2\sqrt{-1-K^2}\text{ArcTan}[\frac{1+e^zK}{\sqrt{-1-K^2}}]+(1-K+K^2)\text{Log}[-1+e^z]+(1+K+K^2)\text{Log}[1+e^z]-(1+K^2)\text{Log}[2e^z-K+e^{2z}K])(1+K\text{Sinh}[z])$$

which is elementary for non-zero K also.

Thus the integral (1) is elementary for both linear and quadratic f(x), when $g(x)=\text{cosech}^{-1}\{f(x)\}$.

V. Conclusion

From above discussion and review of the characteristics of the conjecture, we conclude that the integral (1) is always elementary. The difference between the previous and the present study is that in previous one, K was considered for particular values 1 and 0 only in quadratic polynomial but in this study K has been considered as an arbitrary constant. So the present study is more general and wider than the previous study.

VI. Future Scope of Research

The conjecture has been reviewed and discussed for only two cases of the polynomial of linear and quadratic nature. A big scope is available for research for higher degree polynomials and its special cases. Mathematica doesn't contain the concepts of elementary and nonelementary functions as their names in its library but contains their components like polynomial, trigonometric, etc. functions, so there is a scope of research for Computer Software programmers also to induce such functions in different Computer software.

References

[1]. Abbott, P. (1997). Teaching Mathematics Using Mathematica. In Proceedings Of The 2nd Asian Technology Conference In Mathematics (Pp. 24-40).

[2]. Abell, M. L., & Braselton, J. P. (2021). Mathematica By Example. Academic Press.

[3]. Chaudhary, M. K. & Yadav, D. K. (2024). Strong Liouville's Theorem Based Conjecture On Nonelementary Integrals, IOSR Journal Of Mathematics (IOSR-JM), 20 (1), Ser. 1 (Jan.-Feb.), 35-44.

[4]. Fitelson, B. (1998). Using Mathematica To Understand The Computer Proof Of The Robbins Conjecture. Mathematica In Education And Research, 7, 17-26.

[5]. Hardy, G. H. (2018). The Integration Of Functions Of A Single Variable, Hawk Press, 1-12.

[6]. Hayes, B. (1990). Thoughts On Mathematica. Pixel, 1, 28-34.

[7]. Hilbe, J. M. (2006). Mathematica 5.2: A Review. The American Statistician, 60(2), 176-186.

[8]. Marchisotto, E. A., & Zakeri, G. A. (1994). An Invitation To Integration In Finite Terms. The College Mathematics Journal, 25(4), 295-308.

[9]. Maeder, R. (1991). Programming In Mathematica. Addison-Wesley Longman Publishing Co., Inc..

[10]. Nijimbere, V. (2020). Evaluation Of Some Non-Elementary Integrals Involving The Generalized Hypergeometric Function With Some Applications. Arxiv Preprintarxiv:2003.07403.

[11]. Paul, C. A. (2018). Mathematica. In Revival: The Handbook Of Software For Engineers And Scientists (1995) (Pp. 926-962). CRC Press.

[12]. Risch, R. H. (2022). On The Integration Of Elementary Functions Which Are Built Up Using Algebraic Operations. In Integration In Finite Terms: Fundamental Sources (Pp. 200-216). Cham: Springer International Publishing.

[13]. Ritt, J. F. (2022). Integration In Finite Terms Liouville's Theory Of Elementary Methods. In Integration In Finite Terms: Fundamental Sources (Pp. 31-134). Cham: Springer International Publishing.

[14]. Rose, C., & Smith, M. D. (2002). Mathstatica: Mathematical Statistics With Mathematica. In Compstat: Proceedings In Computational Statistics (Pp. 437-442). Physica-Verlag HD.

[15]. Sao, G. S. (2021). Special Functions, 3rd Revised Edition, Shree Shiksha Sahitya Prakashan, Meerut, Pp. 1-3, 40-45.

[16]. Sharma, J. N. & Gupta, R. K. (2020). Special Functions, Krishna Prakashan Media (P) Ltd., Meerut, 34th Edition, Pp. 70-72.

[17]. Stroyan, K. D. (2014). Calculus Using Mathematica. Academic Press.

[18]. Trott, M. (2007). The Mathematica Guidebook For Symbolics. Springer Science & Business Media.

[19]. Victor, N. (2017). EVALUATION OF THE NON-ELEMENTARY INTEGRAL∫ Eλx^ Adx, A≥ 2, AND OTHER RELATED INTEGRALS. Ural Mathematical Journal, 3(2 (5)), 130-142.

[20]. Wagon, S., & Wagon, S. (1999). Mathematica In Action. Springer Science & Business Media.

[21]. Wikipedia Contributors. (2023, January 1). Elementary Function. In Wikipedia, The Free Encyclopedia. Retrieved 12:05, April 10, 2023, From Https://En.Wikipedia.Org/W/Index.Php?Title=Elementary_Function&Oldid=1130910903

[22]. Wikipedia Contributors. (2023, March 29). Closed-Form Expression. In Wikipedia, The Free Encyclopedia. Retrieved 12:48, April 10, 2023, From Https://En.Wikipedia.Org/W/Index.Php?Title=Closed-Form_Expression&Oldid=1147215178

[23]. Wikipedia Contributors. (2022, August 13). Nonelementary Integral. In Wikipedia, The Free Encyclopedia. Retrieved 13:20, April 10, 2023, From Https://En.Wikipedia.Org/W/Index.Php?Title=Nonelementary_Integral&Oldid=1104142356

[24]. Wikipedia Contributors. (2023, December 22). Wolfram Mathematica. In Wikipedia, The Free Encyclopedia. Retrieved 11:08, January 24, 2024, From Https://En.Wikipedia.Org/W/Index.Php?Title=Wolfram_Mathematica&Oldid=1191274595

[25]. Wolfram, S. (1999). The MATHEMATICA® Book, Version 4. Cambridge University Press.

[26]. Yadav, D. K. & Sen, D. K. (2012). Ph.D. Thesis: A Study Of Indefinite Nonintegrable Functions, (Vinoba Bhave University, Hazaribag, Jharkhand, India). GRIN Verlag, Open Publishing Gmbh, Germany.

[27]. Yadav, D. K. (2023). Nonelementary Integrals: Indefinite Nonintegrable Functions, Notion Press, India, 36-54.

[28]. Yadav, R. K. & Yadav, D. K. (2024). Laplace's Theorem Based Conjecture On Elementary Integrals, IOSR Journal Of Mathematics (IOSR-JM), 20 (1), Ser.1 (Jan. – Feb.), 45-55.

[29]. Yadav, D. K. & Chaudhary, M. K. (2024). Review Of The Conjecture On Elementary Integrals Using Computer Software Mathematica, IOSR Journal Of Mathematics (IOSR-JM), 20 (1), Ser.3 (Jan. – Feb.), 01-09.