IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661,p-ISSN: 2278-8727, Volume 20, Issue 2, Ser. 1 (Mar. – Apr. 2024), PP 59-62
www.iosrjournals.org

Proof Of Continuum Hypothesis

Michael Oser Rabin And Duggirala Ravi

Abstract:
Background*: The continuum hypothesis was originally proposed by Georg Cantor. The continuum hypothesis has remained a prominent conjecture, mainly because the mathematical tools of truth and provability have been developed much later. The continuum hypothesis has been shown to be independent of the axiom of choice.*
Materials and Methods*: The continuum hypothesis asserts that there is no set of cardinality strictly between the cardinalities of the set of natural numbers* $\mathbb{N}$ *and its power set* $2^{\mathbb{N}}$*. In this paper, a set* X *of cardinality at most t* $2^{\mathbb{N}}$ *is assumed to consist of binary sequences with index set* $\mathbb{N}$*. The set can be described by means of an infinite binary tree.*
Results*: It is shown that if a set* $X \subseteq 2^{\mathbb{N}}$ *is uncountable, then the binary tree encoding it includes a subtree isomorphic to the complete binary tree encoding* $2^{\mathbb{N}}$*.*
Conclusion: *Well-founded recursion and inductive constructability are prerequisites for a mathematical theory to become free from paradoxes.*
Key Word*: Set Theory; Mathematical Induction; Well-founded Recursion*

Date of Submission: 08-04-2024 Date of Acceptance: 18-04-2024

I. Introduction

The continuum hypothesis was originally formulated by Georg Cantor. It asserts that there is no set of cardinality strictly between the cardinalities of the set of natural numbers $\mathbb{N}$ and its power set $2^{\mathbb{N}}$.

In this paper, a given set $X \subseteq 2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ consists of binary sequences, with indexes in $\mathbb{N}$. Any subset X of $2^{\mathbb{N}}$ can be described by means of a binary tree, T_X, such that for every binary sequence $(x_1, x_2, x_3, \dots) \in X$, and positive integer $k \in \mathbb{N}$, there is a path from the root of the binary tree to a node in the binary tree with path label $(x_1, \dots, x_k)$, and conversely, if there is a node in the binary tree T_X, with its path label from the root to the given node is $(x_1, \dots, x_k)$, for some $k \in \mathbb{N}$, then there are binary digits $y_{k+i} \in \{0, 1\}$, $i \in \mathbb{N}$ such that the infinite sequence $(x_1, \dots, x_k, y_{k+1}, y_{k+2}, y_{k+3}, \dots)$ belongs to X. Thus, the binary tree T_X encodes the finite length prefixes of sequences in X.

As the set of finite length prefixes of $2^{\mathbb{N}}$ is countable, the binary tree is T_X countable. Nevertheless, it encodes every element of X, as an infinite sequence. For the purpose of reconciliation, it may be useful to refer to arithmetic coding.

The arcs in the binary tree T_X are doubly labelled : one label describes the binary digit in the finite length prefix at the current location of an infinite sequence in X, and the second label indicates whether the subtree rooted at the node to which the arc points encodes a countable or an uncountable subset of X. If a node corresponds to the path prefix $(x_1, \dots, x_k)$, for some $k \in \mathbb{N}$, then the inward arc to the given node has label countable or uncountable depending on whether the set $\{ (x_1, \dots, x_k, y_{k+1}, y_{k+2}, y_{k+3}, \dots) \in X : for\ some\ y_{k+i} \in \{0, 1\}, i \in \mathbb{N} \}$ is a countable or an uncountable subset of X. It is shown that if X is uncountable, then the binary tree T_X encoding X includes a subtree isomorphic to the complete binary tree W_X encoding the full set $2^{\mathbb{N}}$. Thus, if X is uncountable, then there is a one-to-one correspondence between X and $\{0, 1\}^{\mathbb{N}}$.

II. Material And Methods

Let $\mathbb{N}$ be the set of positive integers, and let $\mathbb{R}$ the set of real numbers. Any reference to recursion must be understood to be well-founded. For constructible sets, the following set operations are defined:

(1) Union $\cup$: for an inductively or recursively constructible index set I, and inductively or recursively constructible sets, A_i, $i \in I$, the set union is $\bigcup_{i \in I} A_i$,

(2) Intersection $\cap$: for an inductively or recursively constructible index set I, and inductively or recursively constructible sets, A_i, $i \in I$, the set intersection is $\bigcap_{i \in I} A_i$,

(3) Cartesian Product $\times$ or Π : for an inductively or recursively constructible index set I, and inductively or recursively constructible sets, A_i, $i \in I$, the set intersection is $\prod_{i \in I} A_i$,

(4) Power Set $\mathcal{P}$: for an inductively or recursively constructible set A , the set of all its subsets, denoted by $\mathcal{P}(A)$, is the power set of A, and
(5) Set Difference $A - B$ and Symmetric Difference $A \Delta B$: for two inductively or recursively defined sets A and B, $A - B$ is the collection of elements in A that are not in B, and $A \Delta B$ is the union of the two sets $A - B$ and $B - A$, i.e., $A \Delta B = A - B \cup B - A$.

The set complementation by itself is undefined. In item (4), the subsets of a set are not attempted to be defined further. If a single subset is required, it may have to be constructible appropriately. But, since the power set is taken as a whole, when referring to the subsets of a set, no further constructability conditions are implied in (4). The power set $\mathcal{P}(A)$ is also written as 2^A , as a subset can be identified with its binary valued indicator function defined on the set A.

The complementation of a set is undefined, unless a constructible superset is explicitly or implicitly specified, owing to the discovery of paradoxes originating from the notion of the set of all sets. Instead, for two inductively or recursively defined sets A and B , the set differences, $A - B$ and $B - A$, as well as the symmetric difference, $A \Delta B = B \Delta A$, are defined.

In view of the fact that unbounded recursion is mainly responsible for the unreasonableness of the paradoxes discovered, recursive or inductive constructability is specifically included as a prerequisite. In this paper, whenever a reference to a set is made, it is implicitly assumed that the set is inductively or recursively constructible. From the standpoint of formal logic, except for some few sets that are taken to be basic and subject to no further definability conditions, each set may need to be definable recursively or inductively, by means of functions, that are recursively or inductively constructible.

Two sets A and B may be compared with respect to a partial order $\preccurlyeq$, defined as follows: if there exists a one-to-one mapping from A into B or there exists a surjective mapping from B onto A , then $A \preccurlyeq B$. If $A \preccurlyeq B$, then the cardinality of A is no larger than the cardinality of B .

Proposition 1 *There is no surjective mapping from a finite set A onto the set of natural numbers $\mathbb{N}$, and there is no one-to-one mapping from the set of natural numbers $\mathbb{N}$ into a finite set A* , i.e., *the partial relation $\mathbb{N} \preccurlyeq A$ never holds, for a finite set A.*
Proof Both statements can be proved by induction the number of elements of A.
▢

For an infinite set A , for some strictly proper subset $B \subseteq A$, it may hold that $A \preccurlyeq B$.

Proposition 2 (Cantor) *For any set A , there is no surjective mapping from A onto its power set $\mathcal{P}(A)$, and there is no one-to-one mapping from the power set $\mathcal{P}(A)$ into A* , i.e., *the partial relation $\mathcal{P}(A) \preccurlyeq A$ never holds, for any set A.*
Proof Cantor introduced the diagonal argument in the proof this proposition. ▢

Assumption 1 *There is no set B of cardinality strictly larger than that of every finite set, but strictly smaller than that of the set of natural numbers $\mathbb{N}$* , i.e., *if the partial relation $A \preccurlyeq B$ holds, for every finite set A, then the partial relation $\mathbb{N} \preccurlyeq B$ holds, too.*

Proposition 2 $\mathbb{R} \preccurlyeq \mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}} \preccurlyeq \mathbb{R}$ *and* $\mathbb{N}^k \preccurlyeq \mathbb{N}$, *for every* $k \in \mathbb{N}$.
Proof Known to be standard results. ▢

III. Result

The binary sequences in X can be encoded using an infinite binary tree T_X, as follows: The descending arcs of T_X are labeled with a binary entry in $\{0, 1\}$, as, for example, if the arc points to a left child of the node from where the arc begins, then the label is 0, and 1 otherwise (*i.e.*, if the arc points to the right child of its predecessor). For each node of the tree, the arc to the left child and the arc to the right child are labeled with distinct binary digits. A finite length path starting from the root of T_X to an interior node (not the root itself) in the tree corresponds to a finite length sequence, $(x_1, x_2, x_3 \dots, x_k)$, for some $k \in \mathbb{N}$, such that, by traversing the path from the root to the interior node and concatenating the labels on the arcs, the sequence $(x_1, x_2, x_3 \dots, x_k)$ of length k is formed. The label of the node is the finite length path from the root to the given node. The finite length path from the root to itself is the null sequence ( ), of length zero, with $k = 0$, in the path $(x_1, x_2, x_3 \dots, x_k)$.

The subset $A_{(x_1, x_2, x_3 \ldots, x_k)}$ of X encoded by the subtree rooted at a node, corresponding to the path with label $(x_1, x_2, x_3 \ldots, x_k)$ is the set of all infinite binary sequences $(y_1, y_2, y_3 \ldots)$, with $y_i \in \{0, 1\}$, for $i \in \mathbb{N}$, such that $y_j = x_j$, for $1 \le j \le k$, where $k \in \{0\} \cup \mathbb{N}$.

The arcs are further labeled as *countable* or *uncountable*, depending on whether the subset encoded by the subtree rooted at the node to which the arc points is of the cardinality same as the label on the arc. The arcs now have two labels: one for the binary digit for the element in the sequence, and another for the cardinality of the subset encoded by the subtree which it points to. The label for the binary digit can be detected by checking whether a node is the left or right child of its parent. It is still convenient to label the arcs using the two components, as just described.

As an uncountable set must include at least two elements (infinite sequences), if a node corresponding to the finite length path $(x_1, x_2, x_3 \ldots, x_k)$, with $k \in \{0\} \cup \mathbb{N}$, is the root of a subtree that encodes an uncountable subset of X, there is an index $l \in \mathbb{N}$, such that both the finite length paths $(x_1, x_2, x_3 \ldots, x_k, x_{k+1}, \ldots, x_{k+l-1}, 0)$ and $(x_1, x_2, x_3 \ldots, x_k, x_{k+1}, \ldots, x_{k+l-1}, 1)$ are the prefixes of length $(k + l)$ of some elements (infinite sequences) in X. To begin with, there are two sequences in X with their prefixes of length k equal to $(x_1, x_2, x_3 \ldots, x_k)$, and for definiteness, let $l \ge 1$, be the least positive integer such that the $(k + l)$-th bits in the sequences differ. Let $A_{(x_1, x_2, x_3 \ldots, x_k)}$ be uncountable. For every $l \in \mathbb{N}$, let $A_{(x_1, x_2, x_3 \ldots, x_k, 1-y_{k+1}, \ldots, 1-, y_{k+l-1}, 1-y_{k+l})}$ be uncountable. If the subset encoded by the other child $A_{(x_1, x_2, x_3 \ldots, x_k, 1-y_{k+1}, \ldots, 1-, y_{k+l-1}, y_{k+l})}$ is countable, then $A_{(x_1, x_2, x_3 \ldots, x_k)}$ can be expressed as the countable union of countable sets $\bigcup_{l \in \mathbb{N}} A_{(x_1, x_2, x_3 \ldots, x_k, 1-y_{k+1}, \ldots, 1-, y_{k+l-1}, y_{k+l})} \cup \{ (x_1, x_2, x_3 \ldots, x_k, 1 - y_{k+1}, \ldots, 1-, y_{k+l-1}, 1 - y_{k+l}, \ldots.) \}$, and becomes countable. Hence, there is an index $l \in \mathbb{N}$, such that both the finite length paths $(x_1, x_2, x_3 \ldots, x_k, x_{k+1}, \ldots, x_{k+l-1}, 0)$ and $(x_1, x_2, x_3 \ldots, x_k, x_{k+1}, \ldots, x_{k+l-1}, 1)$ are the prefixes of length $(k + l)$ of some elements (infinite sequences) in X, such that the subsets encoded by the subtrees rooted at these nodes are both uncountable.

Let W_X be the subtree of T_X, consisting of the arcs with label pair $(path,\ cadinality\ uncountable)$, where the $path$ component corresponds to the path from a node in T_X , which is the root of a subtree encoding an uncountable subtree, to the child nodes of the nearest descendent of the node, with both left and right arcs labeled uncountable in T_X. If the node in T_X itself has two children, the arcs to both child nodes bearing cardinality label uncountable, then $path$ component coincides with the binary digit label of the corresponding arc in T_X. It is easy to show that if T_X is uncountable, then the binary tree W_X is the <u>complete binary tree</u>, encoding $2^{\mathbb{N}}$ exactly. The proof depends on the inductive construction of W_X, and must be more carefully described.

1. The root of W_X corresponds to the least descendent $(l.d.)$ of the root r_T of T_X, such that the least descendent (which may also be the root of T_X) has both left and right child nodes in T_X, with both arcs to the child nodes bearing the cardinality label uncountable. The path label of the root of W_X corresponds to that of the least descendent of the root of T_X, that has both arcs to the child nodes bearing cardinality label uncountable.
2. For a node n_W of the binary tree W_X, corresponding to a node n_T of the binary treeT_X , the left and right child nodes of n_W are defined as follows: by the construction, in the previous step, as the induction hypothesis, <u>the node n_T of the binary tree T_X has both child nodes, and the arcs pointing to the child nodes are both labeled cardinality uncountable</u>. The left child of n_W in W_X corresponds to the <u>closest descendent of the left child of n_T</u> in T_X (also called closest left descendent of n_T) <u>that has arcs to both of its child nodes labeled with cardinality uncountable</u>. The right child of n_W in W_X corresponds to the <u>closest descendent of the right child of n_T</u> in T_X (also called closest right descendent of n_T) <u>that has arcs to both of its child nodes labeled with cardinality uncountable</u>.
3. **(continued from the previous step)** Now, the node n_W in W_X has two child nodes in W_X, as found in the previous step. The binary subsequence labels on the arcs to the child nodes of n_W are the paths from n_T to the corresponding descendent nodes of n_T in W_X. The path labels on the child nodes of n_W are the path labels of the descendants of n_T in T_X corresponding to the child nodes of n_W in W_X.

Let $T_{X,uncountable}$ be the binary tree obtained by collecting the arcs in T_X with cardinality label uncountable. Then the paths obtained by descending the arcs in $T_{X,uncountable}$, traversing the nodes that have only one child node each, are compressed into single arcs to get W_X, to make W_X a complete binary tree. In summary, if a subset $X \subseteq 2^{\mathbb{N}}$ is uncountable, then there is a one-to-one correspondence between X and $2^{\mathbb{N}}$. For an infinite set, the full set can be in one-to-one a subset of itself.

IV. Discussion

Well-foundedness of recursion is an indispensable prerequisite for constructability. Well-founded recursion can be shown to be some form of induction. For inductive constructability, the number of basic operations starting from basic sets is required to be finite. a given set $X \subseteq 2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ consists of binary sequences, with indexes in $\mathbb{N}$. Any subset X of $2^{\mathbb{N}}$ can be described by means of a binary tree, T_X,

V. Conclusion

It is shown that if a subset $X \subseteq 2^{\mathbb{N}}$ is uncountable, then there is a one-to-one correspondence between X and $2^{\mathbb{N}}$, establishing the continuum hypothesis.

References

[1]. Willem Conradie And Valentin Goranko. Logic And Discrete Mathematics : A Concise Introduction. John Wiley & Sons Ltd. 2015
[2]. Daniel W. Cunningham. Set Theory : A First Course. Cambridge University Press. New York, 2016
[3]. Keith Devlin. Sets, Functions, And Logic : An Introduction To Abstract Mathematics. Chapman & Hall / Crc Press Company. 2005
[4]. Robert R. Stoll. Set Theory And Logic. Dover Publications Inc. New York, 1979, Corrected Republication Of The Work Originally Published By W. H. Freeman And Company In 1963
[5]. George Tourlakis. Logic And Set Theory. Volume 1 : Mathematical Logic. Cambridge University Press. New York, 2003
[6]. George Tourlakis. Logic And Set Theory. Volume 2 : Set Theory. Cambridge University Press. New York, 2003