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Total Coloring Of Snake Graphs And Other Standard Graphs

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Abstract:
In this paper, I have discussed the total chromatic number of Snake graphs viz., like Triangular Snake and its sub classes of graphs, Quadrilateral Snake graphs and its sub classes of graphs, Ladder Graph and Polygonal Chain. The total chromatic number of a graph G is defined as the assignment of minimum number of colors, to color the vertices, V(G) and the edges, E(G) of a graph in such a way that no two adjacent vertices nor no two adjacent edges receive same color, and no incident vertex and edge receive the same color.

Keywords: *Snake graph, total chromatic number, triangular snake, double star, Triangular Snake, Double Triangular Snake, Alternating Triangular Snake, Alternating Double Triangular Snake, Quadrilateral Snake, Double Quadrilateral Snake, Alternating Double Quadrilateral Snake, Ladder Graph, Polygonal Chain.*

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I. Introduction

Unless mentioned otherwise, for terminology and notation the reader may refer D B West [4] and G Chartrand and P Zhang [5], new ones will be introduced as and when found necessary.

In this paper, by a graph G = (V(G), E(G)), we mean a simple, undirected, connected graph without self-loops. The order, V(G) and size, E(G) are respectively the number of vertices denoted by *n* and the number of edges denoted by *m*.

A coloring of a graph is to assign colors to the vertices , edges or both vertices and edges. A coloring is said to be a proper one if no two vertices (edges) receive the same color. Many variants of proper colorings are available in the literature such as a - coloring, b - coloring, list coloring, total coloring etc. In this paper, the concept of total coloring for few standard classes of graphs is been worked.

The concept of total coloring was introduced independently by Behzad [8] and Vizing [13]. A function $\pi : V(G) \cup E(G) \rightarrow N$ is called a total coloring if no two adjacent or incident graph elements are assigned the same color. The total chromatic number of G, denoted by $\chi_{TC}(G)$, is the smallest positive integer k for which there exists a total coloring $\pi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$. The total chromatic number is denoted by $\chi''(G)$ or $\chi_{TC}(G)$ or $\chi_T(G)$. Behzad and Vizing a have also posed the following conjecture termed as Total Coloring Conjecture (TCC) , which is stated below.

Conjecture 1.1.[8],[13]: $\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2$ where $\Delta(G)$ denotes the maximum degree of G.

This conjecture was confirmed for $\Delta(G) = 3$ by Rosenfeld [10] and Vijayaditya [11] and for $\Delta(G) \leq 3$ by Kostochka [2]. The total chromatic number for complete graph K_n is determined by Behzad et al [9] while Yap [6] have determined the total chromatic number for cycle C_n. According to the survey by J Geetha [7] et al, has listed the research that is been covered from 1996 till date. The user can refer to the list of articles by J Geetha et.al.[7]. Yap [6] gave a nice survey on total colorings that covers the results till 1995.

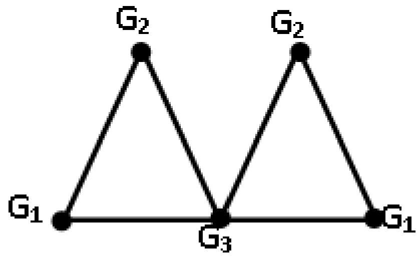
If $\chi''(G) = \Delta(G) + 1$, then G is known as Type -I graph and $\chi''(G) = \Delta(G) + 2$, then G is known as Type - II graph.

In this paper I find the total chromatic number of Ladder graph, Triangular snake, Quadrilateral snake, Double triangular snake, Double quadrilateral snake, Alternate triangular snake, Alternate quadrilateral snake, polygonal snake, and which are proved to be a class of edge rotation distance graphs by Chitra Ramaprakash et al.,[3], full binary tree and a double star. Here Triangular snake, Alternate triangular snake, Double triangular snake all belong to Type - I category, where as the rest fall into Type-II category.

The notations for the above mentioned graphs are as follows. A triangular snake is denoted by T_n and a quadrilateral snake is denoted by Q_n, A Ladder graph is denoted by L_n, an alternating triangular snake is denoted by AT_n, a double triangular snake is denoted by DT_n, an alternate quadrilateral snake is denoted by AQ_n and an alternating double quadrilateral snake is denoted by ADQ_n.

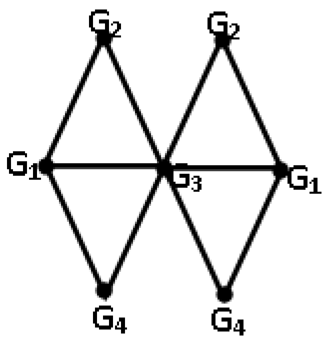
DEFINITION 1:

A triangular snake is a connected graph in which all blocks are triangles and the block cut point graph is a path. These were defined by Rosa [1].



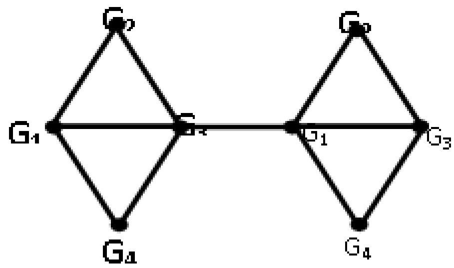
DEFINITION 2 [1],[3],[12] : A Triangular Snake T_n is obtained from a path $u_1, u_2, u_3, \ldots, u_n$ by joining u_i and u_{i+1} to a new vertex v_i for $1 \leq i \leq n - 1$.

DEFINITION 3 [1],[3],[12]: A Double Triangular Snake $D(T_n)$ consists of two triangular snakes that have a common path.

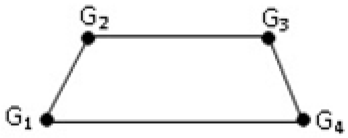


DEFINITION 4 [1],[3],[12]: An Alternate Triangular Snake AT_n is obtained from a path $u_1, u_2, u_3, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to a new vertex v_i.

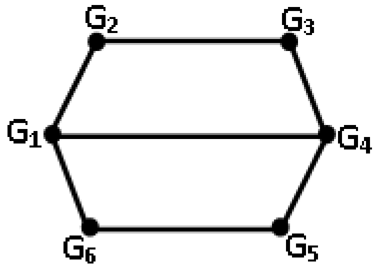
DEFINITION 5 [1],[3],[12] : An Alternate Double Triangular Snake ADT_n consists of two alternate triangular snakes that have a common path.



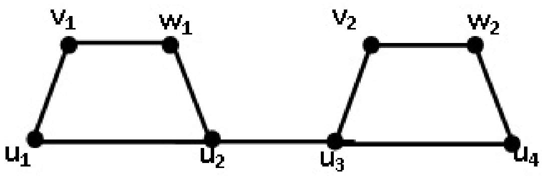
DEFINITION 6 [1],[3],[12] A Quadrilateral Snake Q_n is obtained from a path $u_1, u_2, u_3, \ldots, u_n$ by joining u_i and u_{i+1} to new vertices v_i and $_{wi}$ respectively and then joining v_i and w_i.



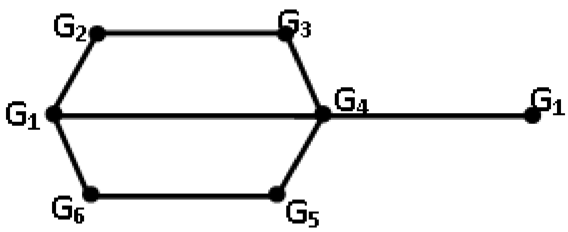
DEFINITION: 7 [1],[3],[12]: A Double Quadrilateral Snake consists of two quadrilateral snakes that have a common path and is denoted by DQ_n.



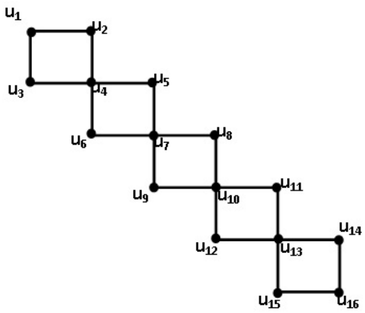
DEFINITION 7. [1],[3],[12] An Alternate Quadrilateral Snake AQ_n is obtained from a path $u_1, u_2, u_3, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i and w_i respectively and then joining v_i and w_i.



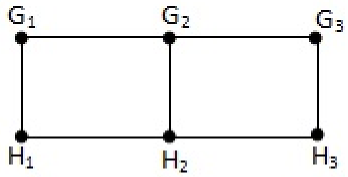
DEFINITION 8. [1],[3],[12] An Alternate Double Quadrilateral Snake, ADQ_n consists of two alternate quadrilateral snakes that have a common path.



DEFINITION 9. [1],[3],[12] A Polygonal Chain $G_{m,n}$ is a connected graph all of whose m blocks are polygons on n sides.



DEFINITION 10. A Ladder, L_n is defined as the Cartesian product of a path with a K_2, that is, $L_n = P_n \times K_2$.



II. Results:

Theorem 1: For a Triangular Snake T_n, $\chi_{TC}(T_n) = 5$

Proof: A triangular snake consists of triangles, which requires a minimum of three colors to color the vertices and the edges properly. Using the concept of permutation, the vertices and the edges are colored following the definition of total coloring. Since the $\Delta(T_n) = 4$, a proper coloring cannot be obtained by using only three colors. If the vertex with maximum degree is assigned with color λ_1, then its vertices adjacent to it can be colored with λ_2 and λ_3. Thus, in a T_3, all the vertices are uniquely colored. Now for the edges, that are incident with a vertex of maximum degree, the colors λ_2, λ_3, λ_4 and λ_5 can be used, resulting in minimum number of colors resulting in a proper coloring. Also, to note that, the maximum degree vertex is a cut vertex, resulting in two or more components, along with the graph having a perfect matching or a one factor and thus, $\chi_{TC}(T_n) = 5$, and for any T_n . Also, this graph is of Type-I and hence $\chi''(G) = \Delta(G) + 1 = 4+1 = 5$.

Theorem 2: For a Double Triangular Snake DT_n, $\chi_{TC}(DT_n) = 7$.
Proof: A double triangular snake consists of two triangular snakes that has a common path. The $\Delta(DT_n) = 6$. Since this vertex is incident with 6 edges, we require atmost 6 colors(λ_1 , ...,λ_6) to color them properly and the vertex incident is colored with a new color say λ_7.

Also, Kostochka,A.V in [8], mentioned that the total chromatic number of any multigraph with maximum degree five is atmost 7. Thus, $\chi_{TC}(DT_n) = 7$. Also to note that the double triangular snake does not have a perfect matching and neither is k-factorable and this graph is of Type-I and hence $\chi''(G) = \Delta(G) + 1 = 6+1 = 7$.

Corollary : For an Alternating Triangular Snake AT_n, $\chi_{TC}(AT_n) = 4$.
Proof: In this graph, as the triangles are placed at every alternate edge, it suffices to properly color the edges and the vertices with only three colors, but as the edge needs to be uniquely colored, we require one more color, hence the total coloring number of $AT_n = 4$. Also to note that this graph a perfect matching/one factor and this graph is of Type-I and hence $\chi''(G) = \Delta(G) + 1 = 3+1 = 4$.

Theorem 3: For an Alternating Double Triangular Snake, $\chi_{TC}(ADT_n) = 5$.
Proof: There are two triangles lying on either sides of the path and alternatively, the maximum degree is 4. To color one such double triangle lying on a path we require at least 5 colors and the adjoining path can be colored with any of the used colors itself. Since the upper triangle uses atmost 3 colors to properly color the edges and the vertices , the remaining two edges incident on either of the vertices on the path, requires two different colors as they are adjacent to each other, thus requiring in total 5 colors to obtain a total coloring of the graph. Thus, $\chi_{TC}(ADT_n) = 5$ and this graph is of Type-I and hence $\chi''(G) = \Delta(G) + 1 = 4+1 = 5$.

Theorem 4: For a Quadrilateral Snake, $\chi_{TC}(Q_n) = 6$.
Proof: The Quadrilateral Snake, Q_2 has two pairs of 1-factor or perfect matching and these can be colored using the same set of colors, which includes coloring the vertices with λ_1 and λ_2 and the edge receiving λ_3. Thus the remaining edge will receive a unique color , λ_4, accounting to a total of 4 colors required to properly color a Quadrilateral Snake Q_2.

But for a generalized Q_n, the maximum degree is 4 and does not have a perfect matching. Here the vertex with maximum degree uses, λ_1 as its vertex color and the 4 edges incident requires 4 (λ_2,...,λ_5) colors to color these edges. The other vertex with which these edges are incident will receive a different color and hence gets λ_6 as its color, thus resulting in a total of 6 colors being used to properly color the snake and this graph is of Type-II and hence $\chi''(G) = \Delta(G) + 2 = 4+2 = 6$.

Theorem 5: For a Double Quadrilateral Snake, $\chi_{TC}(DQ_n) = 8$.
Proof: In this graph, the $\Delta(DQ_n) = 6$, this being the cut vertex, after being removed disconnects the graphs into more than two components. To color the edges incident on the maximum degree vertex we require atmost six colors and the vertex also receives a unique color, accounting to 7 colors being used and the remaining vertices incident on this vertex(which are not adjacent) receives a different color and thus a total of 8 colors is required to properly color the entire double quadrilateral snake. This graph is of Type-II and hence $\chi''(G) = \Delta(G) + 2 = 6 + 2 = 8$.

Corollary: For an Alternating Quadrilateral Snake, $\chi_{TC}(AQ_n) = 5$
Proof: As the quadrilaterals are placed alternatively on a path, to color a single quadrilateral, we require atmost 4 colors, but to the edge that joins the other quadrilateral, needs to be assigned a different color as the maximum degree is three and hence we require a fifth color, λ_5. Thus, this results in using atmost 5 colors to have a proper coloring. This graph is of Type-II and hence $\chi''(G) = \Delta(G) + 2 = 3 + 2 = 5$.

Corollary: For an Alternating Double Quadrilateral $\chi_{TC}(ADQ_n) = 6$.
Proof: The maximum degree of a vertex is 4 and as there are four edges incident on this vertex, we require atmost four colors to color them. The end vertices of these vertices are not adjacent to each other and hence can be colored with a single color. In the next alternate quadrilateral, which lies on the path requires a new color to color one of its edge. This can be continued for any ADQn. Thus, a total of 6 colors are required to properly color the edges and the vertices. Also, This graph is of Type-II and hence $\chi''(G) = \Delta(G) + 2 = 4 + 2 = 6$.

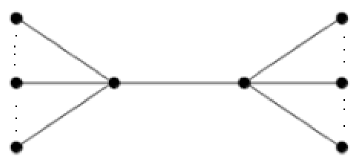
Theorem 6: The Ladder graph , $\chi_{TC}(L_n) = 5$.
Proof: For, the ladder graph L_2 consists of two paths of length P_2 and the vertices here can be colored alternatively with two colors, λ_1 and λ_2 and for the edges that lie can be colored with two new colors λ_3 and λ_4. Thus, resulting in usage of 4 colors for a proper coloring.

But for a generalized L_n, we require an extra color to color the incident(adjoining) edge. Also to note that this has a 1 - factor and belongs to type-II category as the maximum degree here is $\Delta(G) = 3$. Thus, $\chi''(G) = \Delta(G) + 2 = 3 + 2 = 5$.

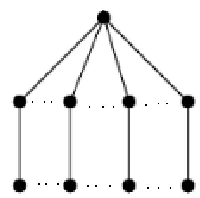
Theorem 7: For a polygonal Chain, $\chi_{TC}(G_{m,n}) = 6$.
Proof: The $\Delta(G_{m,n})$= 4. In, $\Delta(G_{1,4})$= 2. We require two colors to color the vertices alternatively and two more colors to color the edges. But in Gm,n for m polygons with n sides , the $\Delta(Gm,n) = 4$. As the vertex of maximum degree is incident with 4 edges, we require 4 new colors to properly color them, along with a new color for the vertex. The vertices adjacent to this maximum degree vertex can be assigned with a new color, thus resulting in a total of 6 colors being used to properly color the vertices and edges of this graph. This belongs to type-II category as the maximum degree here is $\Delta(G) = 4$. Thus, $\chi''(G) = \Delta(G) + 2 = 4 + 2 = 6$. Also to note that the polygonal chain has 2-factor.

Definition: Double Star
There are two ways a double star can be defined.

A double star graph ST(m, n) is a graph that is formed by two stars ST(m) and ST(n) via joining their centers by an edge.



Double star, $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge for the existing n pendant vertices. It has 2n + 1 vertices and 2n edges.



Theorem 8: For a double star, $\chi_{TC}(\mathrm{ST(m,n)}) = \begin{cases} 3+m, if\ m>n \\ 3+n, if\ n>m \end{cases}$.

Proof: We shall discuss this proof in two cases. First for m > n and second n > m and third for m = n.
All the vertices of a double star requires just two colors to color them.

Case 1: m > n

Now for the edges, on either sides, if there are m edges on the left side, since all of them are adjacent , they require "m" colors and for the n (n<m) edges on the right, requires those m colors that are used for coloring the edges on the left. For the edge that acts a cut edge requires one more new color to have a proper coloring. Thus in total, two colors used from the vertices, one from the central edge, results in (3 + m) colors suffice to obtain a total coloring of the graph.
Case 2: n >m. Following the above pattern, here also we require 3 + n colors to properly color them.
Case 3: If n = m , then we require 3 + m = 3 + n colors to color them properly.

Corollary: For a double star, $\chi_{TC}(K_{1,n,n}) = n+2$.
Proof: In $K_{1,n,n}$, the root vertex has maximum degree "n". The vertices requires just two colors λ_1 and λ_2, which can alternate as it consists of paths. The first level and the second level has n vertices each and n edges. All of these n vertices in the first level are adjacent to the root vertex (central vertex of maximum degree) and the edges are incident on the root vertex. Since these edges are adjacent to each other, they require atmost "n" colors , i.e., λ_3 , λ_4, λ_5, ..., λ_n to obtain a proper coloring. The 2nd level of vertices are one on one adjacent to the first level vertices. These can be colored using any of the above λ_3 , λ_4, λ_5, ..., λ_n colors excluding λ_1 and λ_2, where we just requires a minimum of only colors from the above mentioned list. Thus in total, we require n colors plus the two colors used for the vertices for having a total coloring of a double star.

NOTE: Here, I shall be denoting binary tree as BT.

Corollary: For a full binary tree, $\chi_{TC}(BT) = 5$.

Proof: The nodes of the full binary tree can be colored alternatively using λ_1 and λ_2 colors. For a full binary tree on n levels, each node has two children. To color the edges of the tree, we require at-least a minimum of three colors as the maximum degree for any node at any level is three. Hence, these three colors along with the colors used for node coloring, results to usage of atmost 5 colors of a full binary tree. This belongs to type-II category as the maximum degree here is $\Delta(G) = 3$. Thus, $\chi''(G) = \Delta(G) + 2 = 3 + 2 = 5$.

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