IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 6, Issue 4 (May. - Jun. 2013), PP 59-65
www.iosrjournals.org

Strongly Unique Best Simultaneous Coapproximation in Linear 2-Normed Spaces

R.Vijayaragavan
School of Advanced Sciences V I T University Vellore-632014, Tamilnadu, India.

Abstract: *This paper deals with some fundamental properties of the set of strongly unique best simultaneous coapproximation in a linear 2-normed space.*
AMS Subject classification: *41A50,41A52,41A99, 41A28.*
Keywords: *Linear 2-normed space, strongly unique best coapproximation, best Simultaneous coapproximation and strongly unique best simultaneous coapproximation.*

I. Introduction

The problem of simultaneous approximation was studied by several authors. Diaz and Mclaughlin [2,3], Dunham [4] and Ling, et al. [12] have discussed the simultaneous approximation of two real-valued functions defined on a closed interval. Many results on best simultaneous approximation in the context of normed linear space under different norms were obtained by Goel, et al. [9,10], Phillips, et al. [16], Dunham [4], Ling, et al. [12] and Geetha S. Rao, et al. [5,6,7]. Strongly unique best simultaneous approximation are investigated by Laurent, et al. [11]. D.V.Pai, et al. [13,14] studied the characterization and unicity of strongly unique best simultaneous approximation in normed linear spaces. The problem of best simultaneous coapproximation in a normed linear space was introduced by Geetha S.Rao, et al. [8]. The notion of strongly unique best simultaneous coapproximation in the context of linear 2-normed space is introduced in this paper. Section 2 provides some important definitions and results that are used in the sequel. Some fundamental properties of the set of strongly unique best simultaneous coapproximation with respect to 2-norm are established in Section 3.

II. Preliminaries

Definition 2.1. [1] Let X be a linear space over real numbers with dimension greater than one and let $\| .,. \|$ be a real-valued function on $X \times X$ satisfying the following properties for every x, y, z in X.

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent, (ii) $\| x, y\| = \| y, x \|$,
(iii) $\| \alpha x, y \| = |\alpha| \| x, y \|$, where α is a real number,
(iv) $\| x, y + z \| \le \| x, y \| + \| x, z \|$.

Then $\| .,. \|$ is called a 2-norm and the linear space X equipped with the 2-norm is called a linear 2-normed space. It is clear that 2-norm is non negative.
The following important property of 2-norm was established by Cho [1].

Theorem 2.2. [1] For any points $a, b \in X$ and any $\alpha \in R$, $\| a, b \| = \| a, b + \alpha a \|$.

Definition 2.3. Let G be a non-empty subset of a linear 2-normed space X. An element $g_0 \in G$ is called a strongly unique best coapproximation to $x \in X$ from G, if there exists a constant $t > 0$ such that for every $g \in G$,
$\| g - g_0, k \| \le \| x - g, k \| - t \| x - g_0, k \|$, for every $k \in X \setminus [G, x]$.

Definition 2.4. Let G be a non-empty subset of a linear 2-normed space X. An element $g_0 \in G$ is called a best simultaneous coapproximation to $x_1, \cdots, x_n \in X$ from G, if for every $g \in G$,
$\| g - g_0, k \| \le \max \{ \| x_1 - g, k \|, \cdots, \| x_n - g, k \| \}$, for every $k \in X \setminus [G, x_1, \cdots, x_n]$.

The definition of strongly unique best simultaneous coapproximation in the context of linear

2-normed space is introduced for the first time as follows:

Definition 2.5. Let G be a non-empty subset of a linear 2-normed space X. An element $g_0 \in G$ is called a strongly unique best simultaneous coapproximation to $x_1, \cdots, x_n \in X$ from G, if there exists a constant $t > 0$ such that for every $g \in G$,

$$\| g - g_0, k \| \le \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\} - t\max\{\| x_1 - g_0, k \|,$$

$$\| x_n - g_0, k \|\}, \text{ for every } k \in X \setminus [G, x_1, \cdots, x_n],$$

where $[G, x_1, \cdots, x_n]$ represents a linear space spanned by elements of G and $x_1, \cdots, x_n$. The set of all elements of strongly unique best simultaneous coapproximations to $x_1, \cdots, x_n \in X$ from G is denoted by $W_G(x_1, \cdots, x_n)$.

The subset G is called an existence set if $W_G(x_1, \cdots, x_n)$ contains at least one element for every $x \in X$. G is called a uniqueness set if $W_G(x_1, \cdots, x_n)$ contains at most one element for every $x \in X$. G is called an existence and uniqueness set if $W_G(x_1, \cdots, x_n)$ contains exactly one element for every $x \in X$.

For the sake of brevity, the terminology subspace is used instead of a linear 2-normed subspace. Unless otherwise stated all linear 2-normed spaces considered in this paper are real linear 2-normed spaces and all subsets and subspaces considered in this paper are existence subsets and existence subspaces with respect to strongly unique best simultaneous coapproximation.

III. Some Fundamental Properties Of $W_g(X_1, \cdots, X_n)$

Some basic properties of strongly unique best simultaneous coapproximation are obtained in the following Theorems.

Theorem 3.1. Let G be a subset of a linear 2-normed space X and $x_1, \cdots, x_n \in X$. Then the following statements hold.

(i) $W_G(x_1, \cdots, x_n)$ is closed if G is closed. (ii) $W_G(x_1, \cdots, x_n)$ is convex if G is convex. (iii) $W_G(x_1, \cdots, x_n)$ is bounded.

Proof. (i). Let G be closed.

Let $\{g_m\}$ be a sequence in $W_G(x_1, \cdots, x_n)$ such that $g_m \to \tilde{g}$.

To show that $W_G(x_1, \cdots, x_n)$ is closed, it is sufficient to show that $W_G(x_1, \cdots, x_n)$.

$\tilde{g}$ $\in$Since G is closed, $\{g_m\} \in G$ and $g_m \to$

$\tilde{g}$, we have$\tilde{g} \in G$. Since $\{g_m\} \in$

$W_G(x_1, \cdots, x_n)$, we have for all $k \in X \setminus [G, x_1, \cdots, x_n]$, $g \in G$ and for some $t > 0$ that

$$\| g - g_m, k \| \le \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$$
$$-t\max\{\| x_1 - g_m, k \|, \cdots, \| x_n - g_m, k \|\}$$

$$\Rightarrow \| g - \tilde{g}, k \| - \| g_m - \tilde{g}, k \| \le \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$$
$$-t\max\{\| x_1 - \tilde{g}, k \| - \| g_m - \tilde{g}, k \|, \cdots, \| x_n - \tilde{g}, k \| - \| g_m - \tilde{g}, k \|\}. \quad (3.1)$$

Since $g_m \to \tilde{g}$, $g_m - \tilde{g} \to 0$. So $\| g_m - \tilde{g}, k \| \to 0$, as 0 and k are linearly dependent.
Therefore, it follows from (3.1) that

$\| g - \tilde{g}, k \| \leq \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$

$-t \max\{\| x_1 - \tilde{g}, k \|, \cdots, \| x_n - \tilde{g}, k \|\},$

for all $g \in G$, $k \in X \setminus [G, x_1, \cdots, x_n]$ and for some $t > 0$, when $m \to \infty$. Thus $\tilde{g} \in W_G(x_1, \cdots, x_n)$. Hence $W_G(x_1, \cdots, x_n)$ is closed.

(ii). Let G be a convex set, $g_1, g_2 \in W_G(x_1, \cdots, x_n)$ and $\alpha \in (0, 1)$.

To show that $\alpha g_1 + (1 - \alpha) g_2 \in W_G(x_1, \cdots, x_n)$, let $k \in X \setminus [G, x_1, \cdots, x_n]$. Then

$\| g - (\alpha g_1 + (1 - \alpha) g_2), k \|$

$\leq \alpha \| g - g_1, k \| + (1 - \alpha) \| g - g_2, k \|$
$\leq \alpha (\max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$
$-t \max\{\| x_1 - g_1, k \|, \cdots, \| x_n - g_1, k \|\})$
$+(1 - \alpha) \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$
$-t \max\{\| x_1 - g_2, k \|, \cdots, \| x_n - g_2, k \|\})$
$= \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$
$-t \max\{\| \alpha x_1 - \alpha g_1, k \| \cdots \| \alpha x_n - \alpha g_1, k \|\}$
$+ \max\{\| (1 - \alpha) x_1 - (1 - \alpha) g_2, k \|, \cdots, \| (1 - \alpha) x_n - (1 - \alpha) g_2, k \|\})$
$= \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$
$-t \max\{\| x_1 - (\alpha g_1 + (1 - \alpha) g_2), k \|, \cdots, \| x_n - (\alpha g_1 + (1 - \alpha) g_2, k \|\}$.

Thus $\alpha g_1 + (1 - \alpha) g_2 \in W_G(x_1, \cdots, x_n)$. Hence $W_G(x_1, \cdots, x_n)$ is convex.

(iii). To show that $W_G(x_1, \cdots, x_n)$ is bounded, it is sufficient to show for arbitrary $g_0, \tilde{g}_0 \in W_G(x_1, \cdots, x_n)$ that $\| g_0 - \tilde{g}_0, k \| < c$ for some $c > 0$, since $\| g_0 - \tilde{g}_0, k \| < c$ implies that $\sup_{g_0, \tilde{g}_0 \in W_G(x_1, \cdots, x_n)} \| g_0 - \tilde{g}_0, k \|$ is finite and hence the diameter of $W_G(x_1, \cdots, x_n)$ is finite.

Let $g_0, \tilde{g}_0 \in W_G(x_1, \cdots, x_n)$. Then there exists a constant $t > 0$ such that for every $g \in G$ and $k \in X \setminus [G, x_1, \cdots, x_n]$,

$\| g - g_0, k \| \leq \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$

$-t \max\{\| x_1 - g_0, k \|, \cdots, \| x_n - g_0, k \|\}$

and

$\| g - \tilde{g}_0, k \| \leq \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$

$-t \max\{\| x_1 - \tilde{g}_0, k \|, \cdots, \| x_n - \tilde{g}_0, k \|\}.$

Now,

$\| x_1 - g_0, k \| \leq \| x_1 - g, k \| + \| g - g_0, k \|$
$\leq 2 \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$
$-t \max\{\| x_1 - g_0, k \|, \cdots, \| x_n - g_0, k \|\}.$

2

1 + t

$\Rightarrow \| x_1 - g_0, k \| \leq$

$\max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$, for all $g \in G$.

Hence $\| x_1 - g_0, k \| \leq \frac{d,}{1+t}$

where $d = \inf\limits_{g \in G} \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$.

2

Similarly, $\| x_1 - \tilde{g}_0, k \| \leq \frac{}{1+t} d$.

Therefore, it follows that

$$\| g_0 - \tilde{g}_0, k \| \leq \| g_0 - x_1, k \| + \| x_1 - \tilde{g}_0, k \| \leq \frac{4}{1+t} d = C.$$

Whence $W_G(x_1, \cdots, x_n)$ is bounded.

Let X be a linear2-normed space, $x \in X$ and $[x]$ denote the set of all scalar multiplications of x.

i.e., $[x] = \{\alpha x : \alpha \in R\}$.

Theorem 3.2. Let G be a subset of a linear 2-normed space $X, x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. Then the following statements are equivalent for every $y \in [k]$.

(i) $g_0 \in W_G(x_1, \cdots, x_n)$.

(ii) $g_0 \in W_G(x_1 + y, \cdots, x_n + y)$. (iii) $g_0 \in W_G(x_1 - y, \cdots, x_n - y)$.
(iv) $g_0 + y \in W_G(x_1 + y, \cdots, x_n + y)$. (v) $g_0 + y \in W_G(x_1 - y, \cdots, x_n - y)$. (vi) $g_0 - y \in W_G(x_1 + y, \cdots, x_n + y)$. (vii) $g_0 - y \in W_G(x_1 - y, \cdots, x_n - y)$. (viii) $g_0 + y \in W_G(x_1, \cdots, x_n)$.
(ix) $g_0 - y \in W_G(x_1, \cdots, x_n)$.

Proof. The proof follows immediately by using Theorem 2.2.

Theorem 3.3. Let G be a subspace of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. Then

$$g_0 \in W_G(x_1, \cdots, x_n) \Leftrightarrow g_0 \in W_G(\alpha^m x_1 + (1-\alpha^m) g_0, \cdots, \alpha^m x_n + (1-\alpha^m) g_0),$$
for all $\alpha \in R$ and $m = 0, 1, 2, \cdots$.

Proof. Claim:
$g_0 \in W_G(x_1, \cdots, x_n) \Leftrightarrow g_0 \in W_G(\alpha x_1 + (1-\alpha) g_0, \cdots, \alpha x_n + (1-\alpha) g_0)$, for all $\alpha \in R$.

Let $g_0 \in W_G(x_1, \cdots, x_n)$. Then

$\| g - g_0, k \| \leq \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\} - t \max\{\| x_1 - g_0, k \|, \cdots, k \| x_n - g_0, k \|\}$,
for all $g \in G$ and for some $t > 0$.

$\Rightarrow \quad \| \alpha g - \alpha g_0 , k \| \leq \max\{\| \alpha x_1 - \alpha g, k \|, \cdots , \| \alpha x_n - \alpha g, k \|\}$
$- t\max\{\| \alpha x_1 - \alpha g_0 , k \|, \cdots , \| \alpha x_n - \alpha g_0 , k \|\}$, for every $g \in G$.

$$\Rightarrow \left\| \alpha \left(\frac{(\alpha-1)g_0 + g}{\alpha} \right) - \alpha g_0 , k \right\|$$

$$\leq \max\left\{ \left\| \alpha x_1 - \alpha \left(\frac{(\alpha-1)g_0 + g}{\alpha} \right), k \right\|, \ldots, \left\| \alpha x_n - \alpha \left(\frac{(\alpha-1)g_0 + g}{\alpha} \right), k \right\| \right\}$$
$- t\max\{\| \alpha x_1 - \alpha g_0 , k \|, \cdots , \| \alpha x_n - \alpha g_0 , k \|\}$,

for all $g \in G$ and $\alpha = 0$, since $\frac{(\alpha - 1)g_0 + g}{\alpha} \in G$.

$\Rightarrow \quad \| g - g_0 , k \| \leq \max\{\| \alpha x_1 + (1 - \alpha)g_0 - g, k \|, \cdots , \| \alpha x_n + (1 - \alpha)g_0 - g, k \|\}$
$- t\max\{\| \alpha x_1 + (1 - \alpha)g_0 - g_0 , k \|, \cdots , \| \alpha x_n + (1 - \alpha)g_0 - g_0 , k \|\}$

$$\Rightarrow \quad g_0 \in W_G (\alpha x_1 + (1 - \alpha)g_0 , \cdots , \alpha x_n + (1 - \alpha)g_0), \text{ when } \alpha \neq 0.$$

If $\alpha = 0$, then it is clear that $g_0 \in W_G (\alpha x_1 + (1 - \alpha)g_0 , \cdots , \alpha x_n + (1 - \alpha)g_0)$.

The converse is obvious by taking $\alpha = 1$. Hence the claim is true. By repeated application of the claim the result follows.

Corollary 3.4. Let G be a subspace of a linear 2-normed space X, $x_1 , \cdots , x_n \in X$ and $k \in X \setminus [G, x_1 , \cdots , x_n]$. Then the following statements are equivalent for every $y \in [k]$, $\alpha \in R$ and $m = 0, 1, 2, \cdots$

(i) $g_0 \in W_G (x_1 , \cdots , x_n)$.

(ii) $g_0 \in W_G (\alpha^m x_1 + (1 - \alpha^m)g_0 + y, \cdots , \alpha^m x_n + (1 - \alpha^m)g_0 + y)$. (iii) $g_0 \in W_G (\alpha^m x_1 + (1 - \alpha^m)g_0 - y, \cdots , \alpha^m x_n + (1 - \alpha^m)g_0 - y)$.
(iv) $g_0 + y \in W_G (\alpha^m x_1 + (1 - \alpha^m)g_0 + y, \cdots , \alpha^m x_n + (1 - \alpha^m)g_0 + y)$. (v) $g_0 + y \in W_G (\alpha^m x_1 + (1 - \alpha^m)g_0 - y, \cdots , \alpha^m x_n + (1 - \alpha^m)g_0 - y)$. (vi) $g_0 - y \in W_G (\alpha^m x_1 + (1 - \alpha^m)g_0 + y, \cdots , \alpha^m x_n + (1 - \alpha^m)g_0 + y)$. (vii) $g_0 - y \in W_G (\alpha^m x_1 + (1 - \alpha^m)g_0 - y, \cdots , \alpha^m x_n + (1 - \alpha^m)g_0 - y)$. (viii) $g_0 + y \in W_G (\alpha^m x_1 + (1 - \alpha^m)g_0 , \cdots , \alpha^m x_n + (1 - \alpha^m)g_0)$.
(ix) $g_0 - y \in W_G (\alpha^m x_1 + (1 - \alpha^m)g_0 , \cdots , \alpha^m x_n + (1 - \alpha^m)g_0)$.

Proof. The proof follows from simple application of Theorem 2.2 and the Theorem 3.3.

Theorem 3.5. Let G be a subset of a linear 2-normed space X, $x_1 , \cdots , x_n \in X$ and $k \in X \setminus [G, x_1 , \cdots , x_n]$. Then

$$g_0 \in W_G (x_1 , \cdots , x_n) \Leftrightarrow g_0 \in W_{G+[k]}(x_1 , \cdots , x_n).$$

Proof. The proof follows from simple application of Theorem 3.2.

A corollary similar to that of Corollary 3.4 is established next as follows:

Corollary 3.6. Let G be a subspace of a linear 2-normed space X, $x_1 , \cdots , x_n \in X$ and $k \in$

$X \setminus [G, x_1, \cdots, x_n]$. Then the following statements are equivalent for every $y \in [k]$, $\alpha \in R$ and $m = 0, 1, 2, \cdots$

(i) $g_0 \in W_{G+[k]}(x_1, \cdots, x_n)$.

(ii) $g_0 \in W_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$. (iii) $g_0 \in W_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.
(iv) $g_0 + y \in W_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$. (v) $g_0 + y \in W_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$. (vi) $g_0 - y \in W_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$. (vii) $g_0 - y \in W_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$. (viii) $g_0 + y \in W_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0)$.
(ix) $g_0 - y \in W_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0)$.

Proof. The proof easily follows from Theorem 3.5 and Corollary 3.4.
Proposition 3.7. Let G be a subset of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$ and $0 \in G$. If $g_0 \in W_G(x_1, \cdots, x_n)$, then there exists a constant $t > 0$ such that

$$\| g_0, k \| \le \max\{\| x_1, k \|, \cdots, \| x_n, k \|\} - t\max\{\| x_1 - g_0, k \|, \cdots, \| x_n - g_0, k \|\}.$$

Proof. The proof is obvious.
Proposition 3.8. Let G be a subset of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. If $g_0 \in W_G(x_1, \cdots, x_n)$, then there exists a constant $t > 0$ such that for every $g \in G$,

$$||x_i - g_0, k|| \le 2\max\{||x_1 - g, k||, \cdots, ||x_n - g, k||\} - t\max\{\| x_1 - g_0, k \|, \cdots, \| x_n - g_0, k \|\}, \text{ for } i = 1, 2, \cdots, n.$$

Proof. The proof is obvious.
Theorem 3.9. Let G be a subspace of a linear 2-normed space X and $x_1, \cdots, x_n \in X$. Then the following statements hold.

(i) $W_G(x_1 + g, \cdots, x_n + g) = W_G(x_1, \cdots, x_n) + g$, for every $g \in G$. (ii) $W_G(\alpha x_1, \cdots, \alpha x_n) = \alpha W_G(x_1, \cdots, x_n)$, for every $\alpha \in R$.
Proof. (i). Let
$\tilde{g}$ be an arbitrary but fixed element of G.
Let $g_0 \in W_G(x_1, \cdots, x_n)$. It is clear that $g_0 + \tilde{g} \in W_G(x_1, \cdots, x_n) + \tilde{g}$.

To show that $W_G(x_1, \cdots, x_n) + \tilde{g} \subseteq W_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g})$, it is sufficient to show that $g_0 + \tilde{g} \in W_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g})$.

Now,
$\| g - g_0, k \| \le \max\{\| x_1 - g, k \|, \cdots, \| x_n - g, k \|\}$
$-t\max\{\| x_1 - g_0, k \|, \cdots, \| x_n - g_0, k \|\}$
for every $g \in G$ and for some $t > 0$.

$$\Rightarrow \quad \| g - (g_0 + \tilde{g}), k \| \le \max\{\| x_1 + \tilde{g} - g, k \|, \cdots, \| x_n + \tilde{g} - g, k \|\}$$

$-t\max\{\| x_1 + \tilde{g} - (g_0 + \tilde{g}), k \|, \cdots, \| x_n + \tilde{g} - (g_0 + \tilde{g}), k \|\},$

for every $g \in G$ and for some $t > 0$, since $g - \tilde{g} \in G$.

Thus $g_0 + \tilde{g} \in W_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g})$.

Conversely, let $g_0 + \tilde{g} \in W_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g})$. To show that

$$W_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g}) \subseteq W_G(x_1, \cdots, x_n) + \tilde{g},$$

it is sufficient to show that $g_0 \in W_G(x_1, \cdots, x_n)$. Let $k \in X \setminus [G, x_1, \cdots, x_n]$. Then

$\| g - g_0, k \| = \| g + \tilde{g} - (g_0 + \tilde{g}), k \|$

$\leq \max\{\| x_1 + \tilde{g} - (g + \tilde{g}), k \|, \cdots, \| x_n + \tilde{g} - (g + \tilde{g}), k \|\}$

$-t\max\{\| x_1 + \tilde{g} - (g_0 + \tilde{g}), k \|, \cdots, \| x_n + \tilde{g} - (g_0 + \tilde{g}), k \|\},$

for all $g \in G$ and for some $t > 0$, since $g + \tilde{g} \in G$.

$\Rightarrow$ $g_0 \in W_G(x_1, \cdots, x_n)$. Thus the result follows. (ii). The proof is similar to that of (i).

Remark 3.10. Theorem 3.9 can be restated as

$$W_G(\alpha x_1 + g, \cdots, \alpha x_n + g) = \alpha W_G(x_1, \cdots, x_n) + g, \text{ for all } g \in G.$$

References

[1] **Y.J.Cho**, "Theory of 2-inner product spaces", Nova Science Publications, New York, 1994.

[2] **J.B.Diaz** and H.W.McLaughlin, On simultaneous Chebyshev approximation of a set of bounded complex-valued functions, J.Approx Theory, **2** (1969), 419-432.

[3] **J.B.Diaz** and H.W.McLaughlin, On simultaneous Chebyshev approximation and Chebyshev approximation with an additive weight function, J. Approx. Theory, **6** (1972), 68-72.

[4] **C.B.Dunham**, Simultaneous Chebyshev approximation of functions on interval, Proc.Amer.Math.Soc**., 18** (1967), 472-477.

[5] **Geetha S.Rao**, Best coapproximation in normed linear spaces in Approximation Theory, V.C.K.Chui, L.L.Schumaker and J.D.Ward (eds.), Academic Press, New York, 1986,535-538.

[6] **Geetha S.Rao** and **K.R.Chandrasekaran,** Best coapproximation in normed linear spaces with property (Λ) , Math. Today, **2** (1984), 33-40.

[7] **Geetha S.Rao** and **K.R.Chandrasekaran**, Characterizations of elements of best coapproximation in normed linear spaces, Pure Appl. Math. Sci., **26**(1987), 139-147.

[8] **Geetha S.Rao** and R.Saravanan**,** Best simultaneous coapproximation, Indian J. Math., **40** No.3, (1998), 353-362.

[9] **D.S.Goel, A.S.B.Holland, C.Nasim** and **B.N.Sahney**, On best simultaneous approximation in normed linear spaces. Canad. Math. Bull**., 17** (No. 4) (1974), 523-527.

[10] **D.S.Goel, A.S.B.Holland, C.Nasim,** and **B.N.Sahney**, Characterization of an element of best L_p -simultaneous approximation, S.R.Ramanujan memorial volume, Madras, 1974, 10-14.

[11] **P.J.Laurent** and D.V.Pai, Simultaneous approximation of vector valued functions, Research Report, MRR03-97, Department of Mathematics, IIT, Bombay, 1997.

[12] **W.H.Ling, H.W.McLaughlin** and **M.L.Smith,** Approximation of random functions, AMS Notices, **22** (1975), A-161.

[13] **D.V.Pai**, Characterization of strong uniqueness of best simultaneous approximation, in "Proceedings of the Fifth Annual Conference of Indian Society of Industrial and Applied Mathematics (ISIAM)", Bhopal, 1998.

[14] **D.V.Pai** and **K.Indira**, Strong unicity in simultaneous approximation, in v"Proceedings of the Conference on Analysis, Wavelets and Applications", Feb. 12-14, 2000, being edited by H.P.Dikshit.

[15] **P.L.Papini** and **I. Singer**, Best coapproximation in normed linear spaces, Mh. Math.**88** (1979), 27-44.

[16] **G.H.Phillips** and **B.N.Sahney,** Best simultaneous approximation in the L_1 and L_2 norms, in "Proceeding of conference on theory of Approximation at University of Calgary (1975)," Academic Press, New York.