*IOSR Journal of Mathematics (IOSR-JM)*
*e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 7, Issue 2 (Jul. - Aug. 2013), PP 30-34*
*www.iosrjournals.org*

# Some properties of two-fuzzy Nor med spaces

Noori F.AL-Mayahi, Layth S.Ibrahaim
[1](Department of Mathematics ,College of Computer Science and Mathematics ,University of AL –Qadissiya)
[2](Department of Mathematics ,College of Computer Science and Mathematics ,University of AL –Qadissiya)

***Abstract:*** *The study sheds light on the two-fuzzy normed space concentrating on some of their properties like convergence, continuity and the in order to study the relationship between these spaces*
***Keywords:*** *fuzzy set, Two-fuzzy normed space, α-norm,*
**2010 MSC:** 46S40

## I. Introduction

The concept of fuzzy set was introduced by Zadeh [2] in 1965 as an extension of the classical notion of set. A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gahler in [4]. The concept of fuzzy norm and $\alpha - norm$ were introduced by Bag and Samanta and the notions of convergent and Cauchy sequences were also discussed in [6]. zhang [1] has defined fuzzy linear space in a different way. RM. Somasundaram and ThangarajBeaula defined the notion of fuzzy 2-normed linear space ($F(X)$;$N$) or 2- fuzzy 2-normed linear space. Some standard results in fuzzy 2- normed linear spaces were extended .The famous closed graph theorem and Riesz Theorem were also established in 2-fuzzy 2-normed linear space. In [5] , we have introduced the new concept of 2-fuzzy inner product space on $F(X)$, the set of all fuzzy sets of $X$. This paper is about the concepts related to two-fuzzy normed spaces (fuzzy convergence and fuzzy continuity).

## II. PRELIMINARIES

**Definition2.1.[3]**Let X be a real vector space of dimension greater than one and let $\|.,.\|$be a real valued function on $X \times X$ satisfying the following conditions:

(1) $\|x,y\| = 0$ if and only if x and y are linearly dependent,

(2) $\|x,y\| = \|y,x\|$

(3) $\|\alpha x,y\| = |a|\|x,y\|$ $where\ \alpha\ is\ real,$

(4) $\|x,y+z\| < \|x,y\| + \|x,z\|$

$\|.,.\|$ is called a two-norm on X and the pair $(X,\|.,.\|)$ is called a two normed liner space.

**Definition2.2.[3]**Let X be a vector space over K (the field of real or complex numbers). A fuzzy subset N of $X \times \mathbb{R}$ ($\mathbb{R}$, the set of real numbers) is called a fuzzy norm on X if and only if for all

$$x,y \in X\ and\ c \in K.$$

(N1) For all $t \in \mathbb{R}\ with\ t \le 0, N(x,t) = 0$

(N2) For all $t \in \mathbb{R}\ with\ t > 0, N(x,t) = 1\ if\ and\ only\ if\ x = 0$

(N3) For all $t \in \mathbb{R}\ with\ t > 0, N(cx,t) = N(x,C) = N\left(x,\frac{x}{|c|}\right), if\ c = 0$

(N4) For all $s,t \in \mathbb{R}, x,y \in X, N(f + g, s + t) > min\{N(x,s),N(y,t)\}$

(N5) $N(x,.)$is a non decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$

The pair $(X,N)$ will be referred to be as a fuzzy normed linear space.

**Theorem2.3.[3]**Let $(X,N)$ be a fuzzy normed linear space . Assume $further\ that\ (N6)\ N(x,t) > 0\ for\ all\ t > 0\ implies\ x = 0.$

$Define\ \|x\|_\alpha = inf\{t : N(x,t) > \alpha\}\ where\ \alpha \in (0,1).$

$Then$ $\{\|x\|_\alpha : \alpha \in (0,1)\}$ $is\ an\ ascending\ family\ of\ norms\ on\ X\ (or)\alpha -$ norms on $X$ corresponding to the fuzzy norm on $X$.

**Definition2.4.[3]**A fuzzy vector space $\tilde{X} = X \times (0,1]$over the number field K, where the addition and scalar multiplication operation on $X$ are defined by

$$(x,\lambda) + (y,\mu) = (x + y, \lambda \wedge \mu)\ and\ k(x,\lambda) = (kx,\lambda)$$

is a fuzzy normed space if to every $(x,\lambda) \in \tilde{X}$there is associated a non-negative real number, $\|(x,\lambda)\|$, called the fuzzy norm of $(x,\lambda)$, in such a way that

(1) $\|(x,\lambda)\| = 0\ iff\ x = 0\ the\ zero\ element\ of\ X, \lambda \in (0,1]$

(2) $\|(kx,\lambda)\| = |k|\|(x,\lambda)\|,\ for\ all\ (x,\lambda) \in \widetilde{X}\ and\ all\ k\ G\ K$

(3) $\|(x,\lambda) + (y,\mu)\| < \|(x,\lambda \wedge \mu)\| + \|(y,\lambda \wedge \mu)\|\ for\ all\ (x,\lambda) and$
$(y,\mu) \in \widetilde{X}$

(4) $\|U(x,\vee\lambda_t)\| = \vee\|(x,\lambda_t)\|\ for\ \lambda_t \in (0,1]$

Let $X$ be a nonempty and $F(X)$be the set of all fuzzy sets in X. If $f \in F(X)\ then\ f = \{(x,\mu): x \in X\ and\ \mu \in (0,1]\}$. Clearly f is a bounded function for$|f(x)| \leq 1$ . Let K be the space of real numbers, then F(X) is a linear space over the field K where the addition and scalar multiplication are defined by

$$f + g = \{(x,\mu) + (y,\eta)\} = \{(x + v, \mu\wedge\eta): (x,\mu) \in f, and\ (y,n) \in g\}$$

$kf = \{(kf,\mu): (x,\mu) \in f\}\ where\ k \in K$. The linear space $F(X)$ is said to be normed space if to every$f \in F(X)$, there is associated a non-negative real number $\|f\|$ called the norm of f in such a way that

(1) $\|f\| = 0\ if\ and\ only\ if\ f = 0$ *For,*

$$\|f\| = 0 \Leftrightarrow \{\|x,\mu\|: (x,\mu) \in f\} = 0$$
$$\Leftrightarrow x = 0, \quad \mu \in (0,1]$$
$$\Leftrightarrow f = 0.$$

(2) $\|kf\| = |k|\|f\|, k \in K$ *For,*

$\|kf\| = \{\|kx,\mu\|: (x,\mu) \in f, k \in K\}$
$= \{|k|\|x,\mu\|: (x,\mu) \in f\} = |k|\ \|f\|$

(3) $\|f + g\| \leq |f| + |g|\ for\ every\ f,g \in F(X)$

*For,* $\|f + g\| = \{\|(x,\mu) + (y,\eta)\|/x,y \in X, \mu,\eta \in (0,1]\}$
$$= \{\|(x + y), \mu\wedge\eta\|: x,y \in X, \mu,\eta \in (0,1]\}$$
$$\leq \{\|x,\mu\| + \|y,\mu\wedge\eta\|: (x,\mu) \in f and (y,\eta) \in g\}$$
$$= \|f\| + \|g\|$$

*And so* $(F(X), \|.\|)$ *is a normed linear space.*

**Definition2.6.[3]**Let $X$ be any non-empty set and $F(X)$ be the set of all fuzzy sets on $X$. Then for$U,V \in F(X) and\ k \in K$ the field of real numbers, define

$$U + V = \{(x + y, \lambda \wedge \mu): (x,\lambda) \in U, (y,\mu) \in V\} and$$

$kU = \{(kx,\lambda) : (x,\lambda) \in U\}$.

**Definition2.7.[3]**A two-fuzzy set on $X$ is a fuzzy set on $F(X)$.

**Definition2.8.[3]**Let F(X) be a vector space over the real field K. A fuzzy subset N of $F(X) \times \mathbb{R}$,($\mathbb{R}$, the set of real numbers) is called a 2-fuzzy norm on $F(X)$ if and only if,

(N1) $For\ all\ t \in \mathbb{R}\ with\ t \leq 0, N(f,t) = 0$

(N2) $For\ all\ t \in \mathbb{R}\ with\ t > 0, N(f,t) = 1\ if\ and\ only\ if\ f = 0$

(N3) $For\ all\ t \in \mathbb{R}, with\ t \geq 0, N(cf,t) = N(f,\frac{y}{|c|})\ if\ c \neq 0, c \in K\ (field)$

(N4) $For\ all\ s,t \in \mathbb{R}, N(f_1 + f_2, s,t) \geq min\{N(f_1,s), N(f_2,t)\}$

(N5) $N(f,\bullet) : (0,\infty) \rightarrow [0,1]$ is continuous

(N6) $\lim_{t\to\infty} N(f,t) = 1$

Then the pair $(F(X), N)$ is a fuzzy two-normed vector space.

## III. Mainresult

**Definition 3.1.**Let $(F(X), N, *)$ be a two-fuzzy normed space, then:

(a) A sequence$\{f_n\}$ in $F(X)$ is said to fuzzy converges to $f$ in $F(X)$ if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in Z^+$ such that $N(f_n - f, t) > 1 - \varepsilon$ for all $n \geq n_0$ (Or equivalently $\lim_{n\to\infty} N(f_n - f, t) = 1$ ).

(b) A sequence $\{f_n\}$ in $F(X)$ is said to be fuzzy Cauchy if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in Z^+$ such that $N(f_n - f_m, t) > 1 - \varepsilon$ for all $n,m \geq n_0$ (Or equivalently
$$\lim_{n,m\to\infty} N(f_n - f_m, t) = 1).$$

(c) A two-fuzzy normed space in which every fuzzy Cauchy sequence is fuzzy convergent is said to be complete.

**Theorem 3.2.** Let $(F(X), N, *)$ be a two-fuzzy normed space and let $\{f_n\}$, $\{g_n\}$ be two sequences in two-fuzzy normed space $F(X)$, and for all $\alpha_1 \in (0,1)$ there exist $\alpha \in (0,1)$ such that $\alpha * \alpha \geq \alpha_1$.

(1) Every fuzzy convergent sequence is fuzzy Cauchy sequence.

(2) Every sequence in $F(X)$ has a unique fuzzy limit.
(3) If $f_n \to f$ then $cf_n \to cf$, $c \in \mathbb{F}/\{0\}$.
(4) If $f_n \to f$, $g_n \to g$, then $f_n + g_n \to f + g$.

**Proof:** (1) Let $\{f_n\}$ be a sequence in $F(X)$ such that $f_n \to f$ then for all $t > 0$, $\lim_{n\to\infty} N\left(f_n - f, \frac{t}{2}\right) = 1$,
$N(f_n - f_m, t) = N((f_n - f) - (f_m - f), t) \geq N\left(f_n - f, \frac{t}{2}\right) * N\left(f_m - f, \frac{t}{2}\right)$, by taking limit: $\lim_{n,m\to\infty} N(f_n - f_m, t) \geq \lim_{n\to\infty} N(f_n - f, \frac{t}{2}) * \lim_{m\to\infty} N(f_m - f, \frac{t}{2}) = 1 * 1 = 1$ but $\lim_{n,m\to\infty} N(f_n - f_m, t) \leq 1$ then $\lim_{n,m\to\infty} N(f_n - f_m, t) = 1$ therefore $\{f_n\}$ is a Cauchy sequence in $F(X)$.

(2) Let $\{f_n\}$ be a sequence in $F(X)$ such that $f_n \to f$ and $f_n \to g$ as $n \to \infty$ and $f \neq g$ then for all $t > s > 0$, $\lim_{n\to\infty} N(f_n - f, s) = 1$, $\lim_{n\to\infty} N(f_n - g, t - s) = 1$ $N(f - g, t) \geq N(f_n - f, s) * N(f_n - g, t - s)$
Taking limit as $n \to \infty$:
$N(f - g, t) \geq 1 * 1 = 1. \Longrightarrow but N(f - g, t) \leq 1 \Longrightarrow N(f - g, t) = 1$.
Then by axiom (ii) $f - g = 0 \Longrightarrow f = g$.
(3) Since $f_n \to f$ then if for all $\varepsilon \in (0,1)$ and for all $t > 0$, there exists $n_0 \in Z^+$ such that $N(f_n - f, t) > 1 - \varepsilon$ for all $n \geq n_0$ put $t = \frac{t1}{|c|}$.
$N(c f_n - cf, t_1) = N(f_n - f, \frac{t1}{|c|}) = N(f_n - f, t) > 1 - \varepsilon$
Then $c f_n \to c f$.
(4) For each $\varepsilon_1 \in (0,1)$ there exists $\varepsilon \in (0,1)$ such that $(1-\varepsilon) * (1-\varepsilon) \geq (1-\varepsilon_1)$. Since $x_n \to x$ then for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_1 \in Z^+$ such that $N(f_n - f, \frac{t}{2}) > 1 - \varepsilon$ for all $n \geq n_1$, since $g_n \to g$ then if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_2 \in Z^+$ such that $N(g_n - g, \frac{t}{2}) > 1 - \varepsilon$ for all $n \geq n_2$. Take $n_0 = min\{n_1, n_2\}$, and for each $t > 0$, there exists $n_0 \in Z^+$ such that
$N((f_n + g_n) - (f + g), t) = N((f_n - f) + (g_n - g), t) \geq$
$N(f_n - f, \frac{t}{2}) * N(g_n - g, \frac{t}{2}) > (1-\varepsilon) * (1-\varepsilon) \geq (1-\varepsilon_1)$ for all $n \geq n_0$. Then $f_n + g_n \to f + g$.

**Theorem 3.3.** Let $(F(X), N, *)$, $(F(Y), N, *)$ be a two-fuzzy normed spaces and let $f_n \to f$, $g_n \to g$, such that $\{f_n\}$ and $\{g_n\}$ are two sequences in $F(X)$ and $\alpha, \beta \in \mathbb{F}/\{0\}$ then $\alpha\Psi(f_n) + \beta\omega(g_n) \to \alpha\Psi(f) + \beta\omega(g)$ whenever $\Psi$ and $\omega$ are two identity fuzzy functions.
**Proof:** For all $\varepsilon \in (0,1)$ there exist $\varepsilon_1 \in (0,1)$ such that $(1-\varepsilon_1) * (1-\varepsilon_1) \geq (1-\varepsilon)$, since $f_n \to f$, then for all $\varepsilon_1 \in (0,1)$ and $t > 0$ there exists $n_1 \in Z^+$ such that
$N(f_n - f, \frac{t}{2|\alpha|}) > (1-\varepsilon_1)$ for all $n \geq n_1$, and since $g_n \to g$ then for all $\varepsilon_1 \in (0,1)$ and t $>0$ there exists $n_2 \in Z^+$ such that $N(g_n - g, \frac{t}{2|\beta|}) > (1-\varepsilon_1)$ for all $n \geq n_2$. Take $n_0 = min\{n_1, n_2\}$, $n \geq n_0$
$N((\alpha\Psi(f_n) + \beta\omega(g_n)) - (\alpha\Psi(f) + \beta\omega(g)), t) = N\left(\alpha\alpha\left(\Psi(f_n) - \Psi(f)\right) + \beta\left(\omega(g_n) - \omega(g)\right), t\right) \geq$
$N(\Psi(f_n) - \Psi(f), \frac{t}{2|\alpha|}) * N(\omega(g_n) - \omega(g), \frac{t}{2|\beta|})$
$= N(f_n - f, \frac{t}{2|\alpha|}) * N(g_n - g, \frac{t}{2|\beta|}) > (1-\varepsilon_1) * (1-\varepsilon_1) \geq (1-\varepsilon)$
$\alpha\Psi(f_n) + \beta\omega(g_n) \to \alpha\Psi(f) + \beta\omega(g)$.

**Theorem 3.4.** A two-fuzzy normed space $(F(X), N, *)$ is complete two-fuzzynormed space if every fuzzy Cauchy sequence $\{f_n\}$ in $F(X)$ has a fuzzy convergent subsequence.
**Proof:** Let $\{f_n\}$ be a fuzzy Cauchy sequence in $F(X)$ and $\{f_{nm}\}$ be a subsequence of $\{f_n\}$ such that $f_{nm} \to f$, $f \in F(X)$.
Now to prove $f_n \to f$. For all $\varepsilon \in (0,1)$ there exist $\varepsilon_1 \in (0,1)$ such that $(1-\varepsilon_1) * (1-\varepsilon_1) \geq (1-\varepsilon)$. Since $\{f_n\}$ is a fuzzy Cauchy sequence then for all $t > 0$ and $\varepsilon_1 \in (0,1)$ there exists $n_0 \in Z^+$ such that: $N(f_n - f_m, \frac{t}{2}) > 1 - \varepsilon_1$, for all $n, m \geq n_0$.
Since $\{f_{nm}\}$ is fuzzy convergent to $f$, there exists $im \geq n_0$ such that $N(f_{im} - f, \frac{t}{2}) > 1 - \varepsilon_1$
$N(f_n - f, \mathrm{t}) = N((f_n - f_{im}) + (f_{im} - x), \frac{t}{2} + \frac{t}{2}) \geq$
$N(x_n - x_{im}, \frac{t}{2}) * N(f_{im} - f, \frac{t}{2}) > (1-\varepsilon_1) * (1-\varepsilon_1) \geq (1-\varepsilon)$.
Therefore $f_n \to f$, $\{f_n\}$ is fuzzy convergent to $f$ Hence $(F(X), N, *)$ is complete two-fuzzy normed space.

**Definition 3.5.** Let $(F(X), N,*)$ and $(F(Y), N,*)$ be two-fuzzy normed spaces. The function $\Psi: F(X) \to F(Y)$ is said to be fuzzy continuous at $f_0 \in F(X)$if for all $\varepsilon \in (0,1)$ and all $t > 0$ there exist $\delta \in (0,1)$ and $s > 0$ such that for all $f \in F(X)$
$N(f - f_0, s) > 1 - \delta$ implies $N(\Psi(f) - \Psi(f_0), t) > 1 - \varepsilon$.
The function $f$ is called a fuzzy continuous function if it is fuzzy continuous at every point of $F(X)$.

**Theorem 3.6.** Every identity fuzzy function is fuzzy continuous function in two-fuzzy normed space.
**Proof:** For all $\varepsilon \in (0,1)$ and $t > 0$ there exist $s = t$ and $< \varepsilon, \delta \in (0,1)$ , $N(f_n - f, s) > 1 - \delta$
$N(\Psi(f_n) - \Psi(f), t) = N(f_n - f, s) > 1 - \delta > 1 - \varepsilon$ therefore $\Psi$ is a fuzzy continuous at $f$, since $f$ is an arbitrary point then $\Psi$ is a fuzzy continuous function.
**Theorem 3.7**.Let $F(X)$ be a two-fuzzy normed space over $\mathbb{F}$. Then the functions $\Psi: F(X) \times F(X) \to F(X), \Psi(f, g) = f + g$ and
$\omega: \mathbb{F} \times F(X) \to F(X), \omega(\lambda, f) = \lambda f$ are fuzzy continuous functions.
**Proof:** (1) Let $\varepsilon \in (0,1)$ then there exists $\varepsilon_1 \in (0,1)$ such that $(1 - \varepsilon_1) * (1 - \varepsilon_1) \geq (1 - \varepsilon)$.
let $f, g \in F(X)$ and $\{f_n\}$, $\{g_n\}$ in $F(X)$ such that $f_n \to f$ and $g_n \to g$ as $n \to \infty$, then for each $\varepsilon_1 \in (0,1)$ and each $\frac{t}{2} > 0$ there exists $n_1 \in Z^+$such that $N(f_n - f, \frac{t}{2}) > 1 - \varepsilon_1$ for all $n \geq n_1$ , and for each $\varepsilon_1 > 0$ and $\frac{t}{2} > 0$ there exists $n_2 \in Z^+$such that $N\left(g_n - g, \frac{t}{2}\right) > 1 - \varepsilon_1$for all $n \geq n_2$ , put $n_0 = min\{n_1, n_2\} N(f(x_n, y_n) - f(x, y), t) = N((f_n + g_n) - (f + g), t) = N((f_n - f) + (g_n - g), t) \geq N\left(f_n - f, \frac{t}{2}\right) * N\left(g_n - g, \frac{t}{2}\right) > (1 - \varepsilon_1) * (1 - \varepsilon_1) \geq 1 - \varepsilon$ for all $n \geq n_0$, therefore $\Psi(f_n, g_n) \to \Psi(f, g)\ as\ n \to \infty, \Psi$ is fuzzy continuous function at $(x, y)\ and\ (x, y)$ is any point in $F(X) \times F(X)$, hence $\Psi$ is fuzzy continuous function.
(2) Let $f \in F(X)$, $\lambda \in \mathbb{F}$ and $\{f_n\}$ in $F(X)$, $\{\lambda_n\}$ in $\mathbb{F}$ such that $f_n \to f$ and $\lambda_n \to \lambda$ as $n \to \infty$, then for each $\frac{t}{2|\lambda_n|} > 0$, $\lim_{n\to\infty} N\left(f_n - f, \frac{t}{2|\lambda_n|}\right) = 1$, $|\lambda_n - \lambda| \to 0$ as $n \to \infty$,
$N(\omega(\lambda_n, f_n) - \omega(\lambda, f), t) = N(\lambda_n f_n - \lambda f, t) = N((\lambda_n f_n - \lambda_n f) + (\lambda_n f - f\lambda), t) \geq N(\lambda_n(f_n - f), \frac{t}{2}) * N(f(\lambda_n - \lambda), \frac{t}{2}) = N\left(f_n - f, \frac{t}{2|\lambda_n|}\right) * N(f, \frac{t}{2|\lambda_n - \lambda|})$, by taking limit: $\lim_{n\to\infty} N(\omega(\lambda_n, f_n) - \omega(\lambda, f), t) \geq \lim_{n\to\infty} N\left(f_n - f, \frac{t}{2|\lambda_n|}\right) * \lim_{n\to\infty} N\left(f, \frac{t}{2|\lambda_n - \lambda|}\right) = 1 * 1 = 1$ but $\lim_{n\to\infty} N(\omega(\lambda_n, f_n) - \omega(\lambda, f), t) \leq 1$ then $\lim_{n\to\infty} N(\omega(\lambda_n, f_n) - \omega(\lambda, f), t) = 1$ then
$\omega(\lambda_n, f_n) \to \omega(\lambda, f)\ as\ n \to \infty, \omega$ is fuzzy continuous at $(\lambda, \omega)$ and $(\lambda, f)$ is any point in $\mathbb{F} \times F(X)$, hence $\omega$ is fuzzy continuous.

**Theorem 3.8.** Let $(F(X), N,*)$ and $(F(Y), N,*)$be two-fuzzy normed spaces and let $\Psi : F(X) \to F(Y)$ be a linear function. Then $\Psi$ is a fuzzy continuous either at every point of $F(X)$ or at no point of $F(X)$.
Proof: Let $f_1$and $f_2$ be any two points of $F(X)$ and suppose $\Psi$ is fuzzy continuous at $f_1$.Then for each $\varepsilon \in (0,1)$, $t > 0$ there exist $\delta \in (0,1)$such that $f \in F(X), N(f - f_1, s) > 1 - \delta \Longrightarrow N(\Psi(f) - \Psi(f_1), t) > 1 - \varepsilon$Now: $N(f - f_2, s) > 1 - \delta$ , $N((f + f_1 - f_2) - f_1, s) > 1 - \delta \Longrightarrow N(\Psi(f + f_1 - f_2) - \Psi(f_1), t) > 1 - \varepsilon \Longrightarrow N(\Psi(f) + \Psi(f_1) - \Psi(f_2) - \Psi(f_1), t) > 1 - \varepsilon \Longrightarrow N(\Psi(f) - \Psi(f_2), t) > 1 - \varepsilon$ , $\Psi$ is a fuzzy continuous at $f_1$, since $f_2$is arbitrary point. Hence $\Psi$ is a fuzzy continuous.

**Corollary 3.9.**Let $(F(X), N,*)$ and $(F(Y), N,*)$ be two-fuzzy normed spaces and let $\Psi : F(X) \to F(Y)$ be a linear function. If $\Psi$ is fuzzy continuous at 0 then it is fuzzy continuous at every point.
**Proof:** Let $\{f_n\}$ be a sequence in $F(X)$such that there exist $f_0$, and$f_n \to f_0$, since $\Psi$ is fuzzy continuous at 0 then:
For all $\varepsilon \in (0,1), t > 0$ there exist $\delta \in (0,1), s > 0$: $(f_n - f_0) \in F(X)$
$, N((f_n - f_0) - 0, s) > 1 - \delta \Rightarrow N(\Psi(f_n - f_0) - \Psi(0), t) > 1 - \varepsilon,$
$N(f_n - f_0, s) > 1 - \delta \Rightarrow N(\Psi(f_n) - \Psi(f_0) - \Psi(0), t) > 1 - \varepsilon$
$N(f_n - f_0, s) > 1 - \delta \Rightarrow N(\Psi(f_n) - \Psi(f_0) - 0, t) > 1 - \varepsilon$
$N(f_n - f_0, s) > 1 - \delta \Rightarrow N(\Psi(f_n) - \Psi(f_0), t) > 1 - \varepsilon$
$f_n \to f_0 \Rightarrow \Psi(f_n) \to \Psi(f_0)$ therefore $\Psi$ is fuzzy continuous at $f_0$ since $f_0$ is arbitrary point, then $\Psi$ is fuzzy continuous function.

**Theorem 3.10.**Let $(F(X), N,*), (F(Y), N,*)$ be a two-fuzzy normed spaces, then the function $\Psi : F(X) \to F(Y)$ is fuzzy continuous at $f_0 \in F(X)$if and only if for all fuzzy sequence $\{f_n\}$ fuzzy convergent to $f_0$ in $X$ then the sequence $\{\Psi(f_n)\}$ is fuzzy convergent to $\Psi(f_0)$ in $Y$.

**Proof:** Suppose the function $\Psi$ is fuzzy continuous in $f_0$ and let $\{f_n\}$ is a sequence in $F(X)$such that $f_n \to f_0$. Let $\varepsilon \in (0,1), t > 0$, since $\Psi$ is fuzzy continuous in$f_0 \Longrightarrow$ there exist $\delta \in (0,1), s > 0$, such that for all

$f \in F(X)$: $N\,(f - f_0, s) > 1 - \delta \Longrightarrow N\,(\psi(f) - \psi(f_0), t) > 1 - \varepsilon$
Since $f_n \to f_0$, $\delta \in (0,1), s > 0$ , there exist $k \in Z^+$ such that
$N\,(f_n - f_0, s) > 1 - \delta$ for all $n \geq k$ hence $N\,(\psi(f_n) - \psi(f_0), t) > 1 - \varepsilon$ for all $n \geq k$ therefore $\psi(f_n) \to \psi(f_0)$.
Conversely suppose the condition in the theorem is true.
Suppose $\psi$ is not fuzzy continuous at $f_0$.
There exist $\varepsilon \in (0,1), t > 0$ such that for all $\delta \in (0,1), s > 0$ there exist $f \in F(X)$ and $N\,(f - f_0, s) > 1 - \delta \Longrightarrow N\,(\psi(f) - \psi(f_0)\, t) \leq 1 - \varepsilon \Longrightarrow$ for all $n \in Z^+$ there exist $f_n \in F(X)$such that
$N\,(f_n - f_0, s) > 1 - \frac{1}{n} \Longrightarrow N\,(\psi(f_n) - \psi(f_0), t) \leq 1 - \varepsilon$ that is mean $f_n \to f_0$in $F(X)$, but $\psi(f_n) \nrightarrow \psi(f_0)$ in $Y$ this contradiction , $\psi$ is fuzzy continuous at $f_0$.

**Theorem3.11.** Let $(F(X), N_1,*)$ $(F(Y), N_2,*)$ be two-fuzzy normed spaces. If the functions $\psi : F(X) \to F(Y)$, $\omega : (X) \to F(Y)$ are two fuzzy continuous functions and with for all $a$ there exist $a_1$ such that $a_1 * a_1 \geq a \text{ and } a, a_1 \in (0,1)$ then:
$(1) f + g, (2) kf \text{ where } k \in \mathbb{F}/\{0\}$, are also fuzzy continuous functions over the same filed $\mathbb{F}$.
**Proof:** (1)Let $\varepsilon \in (0,1)$ then there exists $\varepsilon_1 \in (0,1)$ such that $(1 - \varepsilon_1) *(1 - \varepsilon_1) \geq (1 - \varepsilon)$.
Let $\{f_n\}$ be a sequence in $F(X)$ such that $f_n \to f$. Since $\psi$, $\omega$ are two fuzzy continuous functions at $f$ thus for all $\varepsilon_1 \in (0,1)$ and all $t > 0$ there exist $\delta \in (0,1)$ and $s > 0$ such that for all $f \in F(X)$: $N_1(f_n - f, s) > 1 - \delta$ implies $N_2\,(\psi(f_n) - \psi(f), \frac{t}{2}) > 1 - \varepsilon_1$.
And $N_1(f_n - f, s) > 1 - \delta$ implies $N_2\,(\omega(f_n) - \omega(f), \frac{t}{2}) > 1 - \varepsilon_1$
Now: $N_2\big((\psi + \omega)(f_n) - (\psi + \omega)(f), t\big) = N_2(\psi(f_n) + \omega(f_n) - \psi(f) - \omega(f), t) \geq N_2\left(\psi(f_n) - \psi(f), \frac{t}{2}\right) * N_2\left(\omega(f_n) - \omega(f), \frac{t}{2}\right)$

$$> (1 - \varepsilon_1) * (1 - \varepsilon_1) \geq (1 - \varepsilon)$$

Then $\psi + \omega$ is fuzzy continuous function.
(2) Let $\{f_n\}$ be a sequence in $F(X)$ such that $f_n \to f$.Thus for all $\varepsilon_1 \in (0,1)$ and for all $t > 0$, there exist $\delta \in (0,1)$ and $s > 0 \ni N_1(f_n - f, s) > 1 - \delta$ implies $N_2\,(\psi(f_n) - \psi(f), t) > 1 - \varepsilon_1$ , take $t_1 = t|k|$.
Then for all $\varepsilon_1 \in (0,1)$ and for all $t_1 > 0$, there exist $\delta \in (0,1)$ and $s > 0 \ni N_1(f_n - f, s) > 1 - \delta$implies

$$\mathrm{N}_2((k\,\psi)(f_n) - (k\psi)(f), t_1) = N_2(k(\psi(f_n) - \psi(f)), t_1)$$
$$= N_2(\psi(f_n) - \psi(f), t) > 1 - \varepsilon_1$$

Then $k\psi$ is a fuzzy continuous function.

## References


[1]. J. Zhang, The continuity and boundedness of fuzzy linear operators in fuzzy normed space, J.Fuzzy Math. 13(3) (2005) 519-536.
[2]. L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.
[3]. RM. Somasundaram and ThangarajBeaula, Some Aspects of 2-fuzzy 2-normed linear spaces, Bull. Malays. Math. Sci. Soc. 32(2) (2009) 211-222.
[4]. S. Gahler, Lineare 2-normierte Raume, Math. Nachr. 28 (1964) 1- 43.
[5]. THANGARAJBEAULA, R. ANGELINE SARGUNAGIFTA. Some aspects of 2-fuzzy inner product space. Annals of Fuzzy Mathematics and Informatics Volume 4, No. 2, (October 2012), pp. 335-342
[6]. T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11(3) (2003) 687-705.