IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 7, Issue 2 (Jul. - Aug. 2013), PP 42-47
www.iosrjournals.org

Third-kind Chebyshev Polynomials $V_r(x)$ in Collocation Methods of Solving Boundary value Problems

Olagunju, A. S. and Joseph Folake L.
(Department of Mathematics and Statistical Sciences, Kwara state University, Malete, Nigeria)
(Department of Mathematics, Bingham University, Karu Nasarawa State, Nigeria)

Abstract : *This paper proposed the use of third-kind Chebyshev polynomials as trial functions in solving boundary value problems via collocation method. In applying this method, two different collocation points are considered, which are points at zeros of third-kind Chebyshev polynomials and equally-spaced points. These points yielded different results on each considered problem, thus possessing different level of accuracy. The method is computational very simple and attractive. Applications are equally demonstrated through numerical examples to illustrate the efficiency and simplicity of the approach.*
Keywords *- Collocation method, equally-spaced point, third-kind Chebyshev polynomial, trial function, zeros of Chebyshev polynomial*

I. INTRODUCTION

The use of first- and second-kind Chebyshev polynomials has over the years been thoroughly explored in literatures, this cuts across a good number of areas such as approximation, series expansion, interpolation, solution of differential and integral equations [1]. The idea that there are four kinds of these polynomials as projected by Mason and Handscomb strongly leads to an extended range of application [2]. Properties of these polynomials and available proven results further enhanced their use in improving the performance of many available numerical methods. For instance Taiwo and Olagunju applied the first-kind polynomials in enhancing results produced by collocation and comparison methods of solving 4^{th} order differential equations (see [3]). A good number of researchers have equally applied this first-kind polynomial in formulation of basis function and as perturbation tools in solving differential and integral equations. This in essence is due to minimax property of this polynomial. In this paper however, our aim is to apply this Jacobi related polynomials $V_r(x)$ which is referred to as third-kind Chebyshev polynomials [2] in solving nth order Boundary value problem of the form:

$$\alpha_n \frac{d^n y}{dx^n} + \alpha_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \alpha_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + \alpha_1 \frac{dy}{dx} + \alpha_0 y = f(x) \qquad (1)$$

with sufficient conditions attached to the physical boundaries of the problem.

This problem arises in a good number of scientific studies, for instance equation (1) when order n=1 finds relevant application in growth and decay problems, temperature problems, falling body problems and orthogonal trajectories while n=2 (second order equation) adequately models spring problems, buoyancy problems, electrical circuits problems and etc. ([1], [4],[5])

The purpose of this study is to apply third-kind Chebyshev polynomial $V_r(x)$ in the construction of basis function for solving (1) via collocation method. In applying this method, two basic type of collocation points are applied, the first is according to Lanczos [6-7], who introduced it as collocation points at the roots of orthogonal polynomials (orthogonal collocation), while the second is collocation at equi-distance points. By the use of this polynomial as trial function, equation (1) is converted into system of algebraic equations which in essence are simpler to handle. For problems existing in intervals other than natural interval [-1 1] of the polynomial, a shifting technique is adopted.

II. DEFINITIONS OF CHEBYSHEV POLYNOMIALS

In this section we briefly look at basic definition of Chebyshev polynomials; this is needed in order to understand the relationship between the applied polynomial.

2.1 Chebyshev polynomial of the first kind $T_r(x)$

The Chebyshev polynomial of the first kind $T_r(x)$ is defined as:

$$T_r(x) = \cos r\theta \qquad where\ x = \cos\theta\ \ or\ \ \theta = \cos^{-1} x$$

$$= \sum_{r=0}^{N} C_i^{(r)} x^r, \qquad r = 0, 1, \ldots N;$$

where N is a fixed positive integer.
The 3-term recurrence relation:

$$T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x) \tag{2}$$

with initial conditions;

$$T_0(x) = 1$$
$$T_1(x) = x \qquad r = 1, 2, \ldots$$

is used in generating $T_r(x)$ *for* $r \geq 1$

2.2 The second-kind polynomial $U_r(x)$

The Chebyshev polynomial $U_r(x)$ of the second kind is a polynomial of degree *r* in *x* defined by:

$$U_r(x) = \frac{\sin(r+1)\theta}{\sin\theta} \tag{3}$$

where $x = \cos\theta$

The ranges of x and θ are the same as that of $T_r(x)$ and it satisfies the 3-term recurrence relation:

$$U_{r+1}(x) = 2xU_r(x) - U_{r-1}(x)$$

with initial conditions;

$$U_0(x) = 1$$
$$U_1(x) = 2x \qquad r = 1, 2, \ldots$$

2.3 The third-kind polynomial $V_r(x)$

The Chebyshev polynomial $V_r(x)$ of the third-kind has trigonometric definitions involving the half angle $\theta/2$ $(where\ x = \cos\theta)$. This polynomial is referred to 'Air-flow polynomials' but Gautsichi named it "third-kind Chebyshev polynomials" [2]. It is the polynomials of degree *r* in *x* defined by:

$$V_r(x) = \frac{\cos(r+\frac{1}{2})\theta}{\cos(\frac{1}{2}\theta)} \tag{4}$$

where $x = \cos\theta$

$V_r(x)$ shares precisely the same recurrence relation as $T_r(x)$ *and* $U_r(x)$ and its generation differs only in the prescription of the initial condition for *r* =1, hence we have:

$$V_{r+1}(x) = 2xV_r(x) - V_{r-1}(x) \tag{5}$$

with initial conditions:

$$V_0(x) = 1$$
$$V_1(x) = 2x - 1 \qquad r = 1, 2, \ldots$$

From these we have the first four polynomials as:

$V_0(x)=1$

$V_1(x)=2x-1$

$V_2(x)=4x^2-2x-1$

$V_3(x)=8x^3-4x^2-4x+1$

2.4 The fourth-kind polynomial $W_r(x)$

The Chebyshev polynomial $W_r(x)$ of the fourth-kind has trigonometric definitions involving the half angle $\theta/2$ $(where\ x=\cos\theta)$. Gautsichi referred to this "Air-flow polynomials" as "fourth-kind Chebyshev polynomials" [2]. It is the polynomials of degree *r* in *x* defined by:

$$W_r(x)=\frac{\sin(r+\frac{1}{2})\theta}{\sin(\frac{1}{2}\theta)} \tag{6}$$

where $x=\cos\theta$

2.5 The Shifted Polynomial $V_r^*(x)$

Generally, in this sub-section, we define third-kind Chebyshev polynomials $V_r^*(x)$ appropriate to any given finite range $a \le x \le b$ of *x* by making the interval correspond to the interval $-1 \le x \le 1$ of a new variable *s* under the linear transformation:

$$s=\frac{2(x-a)}{b-a}-1 \tag{7}$$

The third-kind polynomial appropriate to $a \le x \le b$ are thus given by $V_r(s)$, where *s* is given in equation (7). Using this in conjunction with (5) yields:

$$V_0^*(x)=1$$

$$V_1^*(x)=2\left[\frac{2(x-a)}{b-a}-1\right]-1$$

$$V_2^*(x)=4\left[\frac{2(x-a)}{b-a}-1\right]^2-2\left[\frac{2(x-a)}{b-a}-1\right]-1$$

$$V_3^*(x)=8\left[\frac{2(x-a)}{b-a}-1\right]^3-4\left[\frac{2(x-a)}{b-a}-1\right]^2-4\left[\frac{2(x-a)}{b-a}-1\right]+1$$

With the recursive formula given as:

$$V_{r+1}^*(x)=2\left\{\frac{2(x-a)}{b-a}-1\right\}V_r^*(x)-V_{r-1}^*(x) \tag{8}$$

2.6 Zeros of Third-kind polynomial $V_r(x)$

The zeros of $V_r(x)$ corresponds to zeros of $\cos(r+\frac{1}{2})\theta$ and these occur at

$$x=\cos\frac{(k-\frac{1}{2})\pi}{r+\frac{1}{2}} \qquad (k=1,2,\ ...\ r) \tag{9}$$

This in the natural order is

$$x=\cos\frac{(r-k+\frac{1}{2})\pi}{r+\frac{1}{2}} \qquad (k=1,2,\ ...\ r) \tag{10}$$

III. TRIAL SOLUTION FORMULATION

Most common methods in numerical approximation of differential equations are based on approximation by orthogonal basis. In fact every continuous function can be approximated by orthogonal polynomials [5]. On this note, function *f(x)* can be written in the form:

$$f(x)=\sum_{i=0}^{N} a_i \phi_i(x) \tag{11}$$

Where N is a natural number and $\phi_i(x)$, $i = 0, 1, \ . \ . \ . \ N$ are orthogonal basis function.

The construction of a trial solution consists of constructing expressions for each of the trial functions. In choosing expressions for the trial functions, it has over the years been established that an important consideration is the use of functions that are algebraically as simple as possible and also easy to work with because derivatives and integrals of $\phi_i(x)$ will frequently be calculated [5]. Polynomials or trigonometric sines or cosines are certainly easiest for these operations [9]. Therefore a logical choice for trial functions used in this study are the Chebyshev polynomials of the third-kind, this choice is likewise being influenced by the convergent property exhibited by this polynomial. The trial solution is therefore written as:

$$\bar{y}_N(x;a)=\sum_{r=0}^{N} a_r V_r(x) \tag{12}$$

Where x represents the entire dependent variables in the problem, coefficient a_r are specialized coordinates and $V_r(x)$ are Chebyshev polynomial of the third kind in variable x.

IV. IMPLEMENTATION OF $V_r(x)$ ON BOUNDARY VALUE PROBLEM VIA COLLOCATION METHOD

The central technique in this method involves the determination of approximants a_r by first substituting trial solution (12) into equation (1) so as to yield a residual equation of the form:

$$\alpha_n \frac{d^n \bar{y}}{dx^n}+\alpha_{n-1}\frac{d^{n-1}\bar{y}}{dx^{n-1}}+\alpha_{n-2}\frac{d^{n-2}\bar{y}}{dx^{n-2}}+\ldots+\alpha_1\frac{d\bar{y}}{dx}+\alpha_0 y\neq f(x) \tag{13}$$

It should be noted that the boundary condition of the problem under consideration is equally imposed on the trial solution to yield a set of equations that are equal in number to the boundary conditions. The residual equation (13) is in the first technique collocated at the zeros of the polynomial $V_r(x)$ defined by (9) and (10), this is chosen to have r which in addition to the given boundary conditions must be equal to the number of unknown approximants a_r in the trial solution. This is to guide against over-determined and under-determined cases. This method in essence, requires that at the zeros of relevant polynomial $V_r(x)$, the residual equation is satisfied thus yielding collocation equations whose number is equal to the number of zeros of $V_r(x)$. The collocation equations in conjunction with the resulting equations from the boundary conditions are solved to determine the unknown approximants a_r , which are thereafter substituted into (12) to yield the desired approximate solution $\bar{y}(x)$.

On the other hand, a different technique of selecting the collocation points is also explored; this is the use of equally-spaced points defined by:

$$x_i = a + \frac{(b-a)i}{D+2} \qquad i=1,2,\ . \ . \ ., D+1 \tag{14}$$

Where D is the difference between the degree of the trial solution and the order n of the equation. *a* and *b* are respectively the lower and the upper bound of the interval of consideration. The collocation equations produced through these points are equally solved alongside with the ones derived from the boundary conditions. The system of N+1 equations arrived at, is then solved to obtain the numerical values for a_i, i=0(1)N. The values of a_i are thereafter substituted into the trial solution (12) to obtain the required approximate solution.

V. Illustrative examples

Example 5.1

Solve the boundary value problem:

$$(x+1)\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$$y(1)=1 \quad and \quad \left[-(x+1)\frac{dy}{dx}\right]_{x=2} = 1$$

The analytical solution is: $y(x) = 1-\ln\left[{(x+1)}/{2}\right]$

Example 5.2

Consider the differential equation:

$$x^3\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - 2 = 0$$

Subject to boundary conditions:

$$y(1)=2 \quad and \quad \left[-x\frac{dy}{dx}\right]_{x=0} = \frac{1}{2}$$

The analytical solution is : $y(x) = \dfrac{2}{x} + \dfrac{\ln x}{2}$

Example 5.3

Solve the boundary value problem:

$$12x^2\frac{d^2y}{dx^2} + 24x\frac{dy}{dx} = -30x^4 + 204x^3 - 351x^2 + 110x$$

$$y(0)=1 \quad and \quad y(1) = 2$$

The analytical solution is: $y(x) = \dfrac{1}{24}\left(-3x^4 + 34x^3 - 117x^2 + 110x + 24\right)$

Example 5.4

Solve the fourth-order differential equation:

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - y = e^x - 2x$$

Subject to boundary conditions:

$$y(0) = y''(0) = 1$$
$$y(1) = 2+e$$
$$y''(1) = e$$

The exact solution is: $y(x) = 2x + e^x$

Given below are tables of absolute maximum errors accrued from solving examples 5.1-5.4. For each N the maximum error is defined as:

$$Max\left|y(x) - \bar{y}_N(x)\right|$$

Table 5.1Table of Maximum Errors for example 5.1

N	Collocation Points at Zeros of Third kind Chebyshev Polynomials	Collocation Points at equally spaced points
4	$2.3100e^{-05}$	$3.0831e^{-04}$
6	$1.4395e^{-07}$	$7.7483e^{-06}$
8	$1.1225e^{-09}$	$1.8657e^{-07}$
10	9.0093^{e-12}	$4.3940e^{-09}$

Table 5.2 Table of Maximum Errors for example 5.2

N	Collocation Points at Zeros of Third kind Chebyshev Polynomials	Collocation Points at equally spaced points
4	$2.1345e^{-03}$	$2.4638e^{-02}$
6	$5.3737e^{-05}$	$2.6065e^{-03}$
8	$1.3477e^{-06}$	$2.3658e^{-04}$
10	4.0381^{e-08}	$1.9737e^{-05}$

Table 5.3 Table of Maximum Errors for example 5.3

N	Collocation Points at Zeros of Third kind Chebyshev Polynomials	Collocation Points at equally spaced points
4	$8.8818e^{-016}$	$8.8818e^{-016}$
6	$8.8818e^{-016}$	$8.8818e^{-016}$
8	$8.8818e^{-016}$	$8.8818e^{-016}$
10	$8.8818e^{-016}$	$8.8818e^{-016}$

Table 5.4Table of Maximum Errors for example 5.4

N	Collocation Points at Zeros of Third kind Chebyshev Polynomials	Collocation Points at equally spaced points
4	$2.7035e^{-002}$	$2.2539e^{-003}$
6	$2.4247e^{-005}$	$1.0658e^{-005}$
8	$6.7176e^{-009}$	$3.3456e^{-008}$
10	4.6443^{e-012}	$6.2863e^{-011}$

VI. CONCLUSION

This paper presents a new approach in the use of Chebyshev polynomials of the third-kind for the numerical solution of boundary value problems via collocation method. The two types of collocation points applied produced results that are close enough to the exact solution to be useful in practical applications. We equally noticed that on a good number of problems, points at zeros of third-kind polynomials produced better results but in situations where equation produced from boundary conditions are more than the collocation equations, equally-spaced points' results are closer and are much better especially as long as N is close to the order of the problem. We thus conclude that these points are not only feasible means of achieving good approximate solution via collocation method but that the approach are equally simple and useful in practical applications.

References

[1]. Richard Bronson, *Differential Equations*, Schaum's Outline series, McGRAW-HILL (2003).

[2]. Mason J.C. and Handscomb, D.C., *Chebyshev Polynomials*, Rhapman & Hall – *CRC*, Roca Raton, London, New York, Washington D.C , 2003

[3]. Taiwo O. A, Olagunju A. S, *Chebyshev methods for the numerical solution of fourth-order differential equations,* International Journal of Physical Sciences Vol. 7(13), pp. 2032 - 2037, 2012

[4]. Hermann, B. *Collocation Methods for Volterra integral and related functional Differential Equations,* Cambridge University press, New York, (2004)

[5]. David, S.B. Finite Element Analysis, from concepts to applications, *AT&T Bell Laboratory*, Whippany, New Jersey, (1987)

[6]. Lanczos, C. *Legendre Versus Chebyshev polynomials.* Miller topics in Numerial analysis, Academic press, London, 1973.

[7]. Lanczos, C. *Applied Analysis,* Prentice Hall, Endlewood Cliffs, New Jersey, 1957.

[8]. Erwin Kreyszig, *Advance engineering mathematics,* 8th ed. John Wiley & sons. Inc. New York.

[9]. Taiwo O. A, Olagunju A. S, *Perturbed segmented Domain collocation Tau- method for the numerical solution of second order BVP,* Journal of the Nigerian Assoc. of mathematical physics. Vol. 10 pp293-298, 2006

[10]. Olagunju A. S., *Chebyshev Series Representation For Product of Chebyshev Polynomials and some notable functions,* IOSR Journal of Maths., vol. 2(2), pp 09-13, 2012

[11]. Olagunju, A.S, Olaniregun D. G, *Legendre-coefficients Comparison methods for the Numerical solution of a class of Ordinary differential equation,* IOSR Journal of Maths. vol. 2(2), pp 14-19, 2012