IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 8, Issue 2 (Sep. - Oct. 2013), PP 35-40
www.iosrjournals.org

Uniformly Stability of a Class of Quasi-linear Parabolic Distributed Parameter Systems

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Abstract: *The paper is devoted to the stability of a class of quasi-linear parabolic distributed parameter system (DPS). By using the linear matrix inequality (LMI) and Lyapunov functional methods, a sufficient condition is derived to ensure the uniformly stability of quasi-linear parabolic DPS. The numerical simulations are presented.*
Keywords: *Quasi-linear distributed parameter system, Lyapunov functional, linear matrix inequality*

I. INTRODUCTION

DPS is called the infinite-dimensional systems, whose dynamics are described by partial differential equations, delay equations, or functional differential equations depended on infinite dimensional spaces[1-3]. It should be pointed out that, in past decades, the stability analysis problem for parabolic DPS has received very more research attention, and many authors have discussed the stability problem [4-8]. It should be pointed out that, up to now, the stability analysis problem for quasi-linear DPS has received very little research attention, despite its practical importance.

In this paper, we consider via the Lyapunov functional and LMI approach a case of DPS described by the quasi-linear parabolic partial differential equation (PDE). The nonlinear term in system dynamics is assumed to be norm bounded, which shows that the assumptions are not so restrictive.

The outline of this paper is as follows. The system model is described in Section 2. The main results are derived in Section 3. We give a numerical example to illustrate the usage of the theoretical results in Section 4. Conclusions are given in the last section.

Notations: Ω denotes a compact set with smooth boundary $\partial\Omega$ and measure $\mu(\Omega)$ in R^l. $L^2\left(\Omega\times R,\ R^l\right)$ means the space of real Lebesgue measurable functions on $\Omega\times R$. $\|\cdot\|$ denotes the Euclidean norm of a vector or its induced matrix norm. $\|f\|_{L^2}$ denotes the norm on $L^2(\Omega)$, i.e., $\|f\|_{L^2}^2=\int_\Omega\|f\|^2dx$, where $\Omega=\{x\mid \|x\|<\varpi<+\infty\}\in R^l$ is a bounded region with the smooth boundary $\partial\Omega$, ϖ is a known constant. The superscript T denotes the transpose. ∇ indicates the gradient operator. For the vector function $p(x,t)=(p_1(x,t)\ \ \dots\ \ p_n(x,t))^T\ (x\in\Omega,t\geq 0)$, define

$$\nabla p_i(x,t)=\begin{pmatrix}\frac{\partial p_i(x,t)}{\partial x_1} & \frac{\partial p_i(x,t)}{\partial x_2} & \dots & \frac{\partial p_i(x,t)}{\partial x_l}\end{pmatrix}^{\mathrm{T}},\ i=1,2,\cdots,l.$$

Denote $\nabla p(x,t)=\left(\nabla p_1(x,t)\ \ \nabla p_2(x,t)\ \ \dots\ \ \nabla p_n(x,t)\right)^{\mathrm{T}}$ and $\|\nabla p(x,t)\|_{L^2}=(\int_\Omega\sum_{i=1}^n\sum_{j=1}^l(\frac{\partial}{\partial x_j}p_i(x,t))^2\mathrm{d}x)^{\frac{1}{2}}$. For $Y_i=(y_{i1}\ y_{i2}\ \ \dots\ \ y_{il})^T, i=1,2,\cdots,n$ and matrix $Y=(Y_1\ \ Y_2\ \ \dots\ \ Y_n)^T$, we denote $\nabla\bullet Y_i=\frac{\partial y_{i1}}{\partial x_1}+\frac{\partial y_{i2}}{\partial x_2}+\dots+\frac{\partial y_{il}}{\partial x_l}, \nabla\bullet Y=\left(\nabla\bullet Y_1\ \ \nabla\bullet Y_2\ \ \dots\ \ \nabla\bullet Y_n\right)^T$. $A=\left(a_{ij}\right)_{n\times n}>0(<0)$ denotes A is a positive (negative) definite matrix, i.e., $x^TAx>0(<0)$ for any $x\in R^n$, $x\neq 0$. $A=\left(a_{ij}\right)_{n\times n}\geq 0$ denotes A is a semi-positive definite matrix, i.e., $x^TAx\geq 0$ for any $x\in R^n, x\neq 0$. $A\geq B$ (respectively, $A>B$) means the matrix $A-B$ is a semi-positive definite matrix (respectively, positive definite). I means a identity matrix with compatible dimension. For matrix $A\in R^{n\times n}$, $\lambda_{\max}(A)$ indicates its biggest eigenvalue. The symbol $\mathrm{rank}(\cdot)$ indicates the rank number of matrix.

II. PROBLEM FORMULATION

Consider the quasi-linear parabolic PDE systems of the form

$$\frac{\partial p(x,t)}{\partial t} = \nabla \bullet (D \circ \nabla p(x,t)) + \left(F \circ \nabla p(x,t)\right)E + Ap(x,t) + f(p(x,t)) \tag{1}$$

where $p(x,t) = (p_1(x,t) \quad ... \quad p_n(x,t))^T \in R^n$ is the state vector. $D = \left(D_{ij}\right)_{n\times l}$, $A = \left(a_{ij}\right)_{n\times n}$ and $F = \left(F_{ij}\right)_{n\times l}$ are known real constant matrices, $E = \left(1,\ 1,\ \cdots\ ,\ 1\right)^T$ is a l-dominations column vector. $\nabla p_i(x,t) = (\frac{\partial p_i(x,t)}{\partial x_1} \quad ... \quad \frac{\partial p_i(x,t)}{\partial x_l})^T$, $i = 1,...,n$. $D \circ \nabla p = (D_{ij}\frac{\partial p_i}{\partial x_j})_{n\times l}$ is Hadamard product of matrix D and $\nabla p(x,t)$. $F \circ \nabla p(x,t) = (F_{ij}\frac{\partial p_i}{\partial x_j})_{n\times l}$ is Hadamard product of matrix F and $\nabla p(x,t)$.

The vector function $f(p(x,t)) = (f^1(p_1(x,t)) \quad ... \quad f^n(p_1(x,t)))^T \in R^n$ is smooth. We suppose that the vector function $f(p(x,t))$ satisfies the following condition

$$\left\|f(p(x,t))\right\| \le \sqrt{\mu}\left\|p(x,t)\right\|, \tag{2}$$

where μ is a known positive constant.

The initial and boundary conditions of Eq. (1) are, respectively, as follows

$$p(x,0) = \phi(x), \quad x \in \Omega, \tag{3}$$

$$\frac{\partial p(x,t)}{\partial \vec{n}} = 0, \quad (x,t) \in \partial\Omega \times [0 \quad +\infty) \tag{4}$$

where $\vec{n}$ indicates outward unit normal on $\partial\Omega$, $\phi(x)$ is an appropriate smooth vector function on Ω.

Obviously, the origin O is an equilibrium point. System (1) with (3) and (4) has a unique global solution [7]. Denote that $\Psi = \max_{i,j}\left\{F_{ij}^2 - D_{ij}\right\}, \tilde{D} = \min_{i,j}\left\{D_{ij}\right\}$.

Assumption 1: $\Psi < 0$. (5)

Assumption 2: $\tilde{D} > 1$.
(6)

First give the following lemmas, which will be frequently used in the proofs of our main results in this paper.

Lemma 1 [9]. The LMI $\begin{pmatrix} Q(u) & S(u) \\ S^T(u) & R(u) \end{pmatrix} > 0$,where $Q(u) = Q^T(u), R(u) = R^T(u)$, and $S(u)$ depend on u, is equivalent to any one of the following conditions:

(L1) $R(u) > 0, Q(u) - S(u)R^{-1}(u)S^T(u) > 0;$;

(L2) $Q(u) > 0, R(u) - S(u)Q^{-1}(u)S^T(u) > 0.$

Lemma 2 [10]. Given real matrices with compatible dimension $Q_1 = \left(q_{ij}^1\right)_{m\times n}, Q_2 = \left(q_{ij}^2\right)_{m\times n}, Q_3 = \left(q_{ij}^3\right)_{m\times m}$ and $Q_3 = Q_3^T > 0$. Then, for any constant $\beta > 0$, $Q_2^TQ_1 + Q_1^TQ_2 \le \beta^{-1}Q_1^TQ_3^{-1}Q_1 + \beta Q_2^TQ_3Q_2$.

Lemma 3[9] If $p(x,t)$ is a solution of (1), then

$$\int_\Omega p^T(x,t)\nabla \bullet \left(D \circ \nabla p\right)dx = -\int_\Omega \left(D \bullet \left(\nabla p \circ \nabla p\right)\right)Edx \tag{7}$$

where $\nabla p \circ \nabla p = \left(\nabla p_1 \circ \nabla p_1 \quad \cdots \quad \nabla p_n \circ \nabla p_n\right)^T$, $\nabla p_i \circ \nabla p_i = \left(\left(\frac{\partial p_i}{\partial x_1}\right)^2 \quad \cdots \quad \left(\frac{\partial p_i}{\partial x_l}\right)^2\right)^T$,

$D \bullet \left(\nabla p \circ \nabla p\right) = \left(D_1 \bullet \left(\nabla p_1 \circ \nabla p_1\right) \quad \cdots \quad D_n \bullet \left(\nabla p_n \circ \nabla p_n\right)\right)$, $D = \left(D_1 \quad \cdots \quad D_n\right)^T$, $D_i = \left(D_{i1} \quad \cdots \quad D_{il}\right)^T$, $i = 1,\cdots,n$, $E = \left(1,\cdots,1\right)^T$.

III. MAIN RESULTS

The main results of this paper are given as follows.

Theorem 1 *Given matrices* A *.Under condition (5)and (6) the null solution of the sliding motion equation (1) with (3) and (4) is uniformly convergent to zero, i.e.* $\lim_{t\to\infty} p(x,t)=0, x\in\Omega$ *, if there exist scalars* $\lambda>0$ *, and a matrix K such that the following linear matrix inequality*

$$\Lambda=(A^T+A)+(2+\mu+\lambda)I<0 \tag{8}$$

holds.

Proof Define a Lyapunov-Krasovskii functional candidate by $N(t)=\frac{1}{2}h\int_\Omega (p(x,t))^T p(x,t)\mathrm{d}x$. Calculating the time derivative of functional $N(t)$ along the trajectory of Eq. (1) yields

$$\begin{aligned}
\frac{\mathbf{d}N(t)}{\mathbf{d}t} &= \frac{1}{2}h\int_\Omega \frac{\partial}{\partial t}p^T(x,t)p(x,t)\mathbf{d}x+\frac{1}{2}h\int_\Omega p^T(x,t)\frac{\partial}{\partial t}p(x,t)\mathbf{d}x\\
&=\frac{1}{2}h\int_\Omega\Big(\big(\nabla\bullet(D\circ\nabla p)\big)p(x,t)+p^T(x,t)\nabla\bullet(D\circ\nabla p)\Big)\mathbf{d}x\\
&+\frac{1}{2}h\int_\Omega\Big(E^T\left(F\circ\nabla p(x,t)\right)^T p(x,t)+p^T(x,t)\left(F\circ\nabla p(x,t)\right)E\Big)\mathbf{d}x\\
&+\frac{1}{2}h\int_\Omega p^T(x,t)\left(A^T+A\right)p(x,t)\mathbf{d}x+h\int_\Omega p^T(x,t)f(p(x,t))\mathbf{d}x\\
&=h\int_\Omega p^T(x,t)\nabla\bullet\left(D\circ\nabla p\right)\mathbf{d}x+h\int_\Omega p^T(x,t)\left(F\circ\nabla p(x,t)\right)E\mathbf{d}x\\
&+\frac{1}{2}h\int_\Omega p^T(x,t)((A^T+A)+h\int_\Omega p^T(x,t)f(p(x,t))\mathbf{d}x
\end{aligned} \tag{9}$$

From Lemma 3, we have

$$h\int_\Omega p^T(x,t)\nabla\bullet\left(D\circ\nabla p\right)\mathbf{d}x=-h\int_\Omega\left(D\bullet\left(\nabla p\circ\nabla p\right)\right)E\mathbf{d}x \tag{10}$$

In view of Lemma 2 and (4), we have

$$\begin{aligned}
h\int_\Omega p^{\mathrm{T}}\ (x,t)f(p(x,t))\mathbf{d}x&\le\frac{1}{2}h\int_\Omega\left(p^T(x,t)p(x,t)+(f(p(x,t))^T f(p(x,t))\right)\mathbf{d}x\\
&\le\frac{1}{2}h\int_\Omega p^T(x,t)(1+\mu)Ip(x,t)\mathbf{d}x
\end{aligned} \tag{11}$$

In addition, we have

$$\begin{aligned}
h\int_\Omega p^T(x,t)\left(F\circ\nabla p(x,t)\right)E\mathbf{d}x&=h\sum_{i=1}^{n}\left(\int_\Omega p_i(x,t)\sum_{j=1}^{l}F_{ij}\frac{\partial p_i(x,t)}{\partial x_j}\mathbf{d}x\right)\\
&\le\frac{1}{2}h\sum_{i=1}^{n}\ (\int_\Omega(p_i(x,t))^2\mathbf{d}x+\int_\Omega(\sum_{j=1}^{l}F_{ij}\frac{\partial p_i(x,t)}{\partial x_j})^2\mathbf{d}x)\\
&\le\frac{1}{2}h\sum_{i=1}^{n}\int_\Omega(p_i(x,t))^2\mathbf{d}x+h\sum_{i=1}^{n}\sum_{j=1}^{l}\int_\Omega F_{ij}^2(\frac{\partial p_i(x,t)}{\partial x_j})^2\mathbf{d}x\\
&=\frac{1}{2}h\int_\Omega p^T(x,t)p(x,t)\mathbf{d}x+h\int_\Omega\overline{F}\bullet\left(\nabla p\circ\nabla p\right)E\mathbf{d}x
\end{aligned} \tag{12}$$

where $\overline{F}=\left(F_{ij}^2\right)_{n\times l}$. Substituting (10), (11) and (12) into (9) gets

$$\frac{\mathbf{d}N(t)}{\mathbf{d}t} \le -h\int_{\Omega}\left(D\bullet\left(\nabla p\circ\nabla p\right)\right)E\mathbf{d}x + \frac{1}{2}h\int_{\Omega}p^T(x,t)p(x,t)\mathbf{d}x + h\int_{\Omega}\overline{F}\bullet\left(\nabla p\circ\nabla p\right)E\mathbf{d}x$$
$$+\frac{1}{2}h\int_{\Omega}p^T(x,t)\left((A^T+A)+(1+\mu)I\right)p(x,t)\,\mathbf{d}x = -h\int_{\Omega}\left((D-\overline{F})\bullet\left(\nabla p\circ\nabla p\right)\right)E\mathbf{d}x$$
$$+\frac{1}{2}h\int_{\Omega}p^T(x,t)\left((A^T+A)+(2+\mu+\lambda)I\right)p(x,t)\,\mathbf{d}x - \frac{1}{2}h\lambda\int_{\Omega}p^T(x,t)p(x,t)\,\mathbf{d}x$$

In view of Lemma 1 and (8), we have $\frac{\mathbf{d}N(t)}{\mathbf{d}t} < h\Psi\left\|\nabla p(x,t)\right\|_{L^2}^2 - \frac{1}{2}h\lambda\left\|p(x,t)\right\|_{L^2}^2$ .Integrating the above form from T to t gets $N(t) - h\Psi\int_T^t \left\|\nabla p(x,s)\right\|_{L^2}^2 ds + \frac{1}{2}h\lambda\int_T^t \left\|p(x,s)\right\|_{L^2}^2 ds \le N(T)$ .Obviously, $N(T)$ is bounded. From the definition of functional $N(t)$, one obtains $N(t)\ge 0,\ 0\le t<\infty$. So from (5) one gets that $N(t)$ is bounded, $\int_T^\infty \left\|p(x,s)\right\|_{L^2}^2 ds < \infty$ and $\int_T^\infty \left\|\nabla p(x,s)\right\|_{L^2}^2 ds < \infty$. These imply that $\left\|p(x,s)\right\|_{L^2}^2$ and $\left\|\nabla p(x,s)\right\|_{L^2}^2$ are bounded, $\left\|p(x,s)\right\|_{L^2}^2 \in L_1(0,\infty)$ and $\left\|\nabla p(x,s)\right\|_{L^2}^2 \in L_1(0,\infty)$. Furthermore, we can prove that (see [11]) $\frac{d}{dt}\left\|p(x,t)\right\|_{L^2}^2 \in L_1(0,\ \infty)$ and $\frac{d}{dt}\left\|\nabla p(x,t)\right\|_{L^2}^2 \in L_1(0,\ \infty)$.

Based on that Barbalat theorem (see[12]), $\left\|p(x,t)\right\|_{L^2}^2 \in L_1(0,\ \infty)$, $\left\|\nabla p(x,t)\right\|_{L^2}^2 \in L_1(0,\ \infty)$, $\frac{d}{dt}\left\|p(x,t)\right\|_{L^2}^2 \in L_1(0,\ \infty)$ and $\frac{d}{dt}\left\|\nabla p(x,t)\right\|_{L^2}^2 \in L_1(0,\ \infty)$, one obtains $\lim_{t\to\infty}\left\|p(x,t)\right\|_{L^2}^2 = 0$, $\lim_{t\to\infty}\left\|\nabla p(x,t)\right\|_{L^2}^2 = 0$.

Based on that $\left\|p(x,t)\right\|_{L^2}^2$ and $\left\|\nabla p(x,t)\right\|_{L^2}^2$ are bounded, $p(x,t)$ is smooth, Ω is bounded and its boundary $\partial\Omega$ is smooth, we have $p(x,t)$ and $\nabla p(x,t)$ are bounded on $\Omega\times R^+$. So $\left\|p(x,t)\right\|_{L^\infty}$ and $\left\|\nabla p(x,t)\right\|_{L^\infty}$ are bounded. Thus using Hōlder inequality (see[13]) we have that for $\kappa > 2$, $\left\|p(x,t)\right\|_{L^\kappa} \le \left\|p(x,t)\right\|_{L^\infty}^{(\kappa-2)/\kappa}\left\|p(x,t)\right\|_{L^2}^{2/\kappa}$ and $\left\|\nabla p(x,t)\right\|_{L^\kappa} \le \left\|\nabla p(x,t)\right\|_{L^\infty}^{(\kappa-2)/\kappa}\left\|\nabla p(x,t)\right\|_{L^2}^{2/\kappa}$.

Due to $\lim_{t\to\infty}\left\|p(x,t)\right\|_{L^2}^2 = 0, \lim_{t\to\infty}\left\|\nabla p(x,t)\right\|_{L^2}^2 = 0$, we have

$$\lim_{t\to\infty}\left\|p(x,t)\right\|_{L^\kappa} = \lim_{t\to\infty}\left\|\nabla p(x,t)\right\|_{L^\kappa} = 0. \tag{13}$$

Based on Sobolev inequality (see [13]), we have that if $\kappa > l$, there exists constant ξ associated with Ω, l and κ such that

$$\left\|p(x,t)\right\|_{L^\infty} \le \xi\left\{\left\|p(x,t)\right\|_{L^\kappa} + \left\|\nabla p(x,t)\right\|_{L^\kappa}\right\}. \tag{14}$$

From (13) and (14), one obtains $\lim_{t\to\infty}\left\|p(x,t)\right\|_{L^\infty} = 0$. This implies $\lim_{t\to\infty} p(x,t) = 0$. The proof completes.

IV. SIMULATION RESULTS

Consider the parabolic PDE systems (1) with $n=2,\ m=2,\ l=1,\ \varpi=10,\ D=\begin{pmatrix}1.5\\1.2\end{pmatrix}$, $F=\begin{pmatrix}-1.2\\-1\end{pmatrix}$, $A=\begin{pmatrix}0.6310 & -1.9\\0.6085 & 0.4575\end{pmatrix}$, $f(p(x,t))=\begin{pmatrix}p_1(x,t)e^{\sin(p_2(x,t))}\\ \frac{\sqrt{30}}{10}\sin(p_1(x,t)+p_2(x,t))\end{pmatrix}$. Select $\mu = 8$.

Obviously, $\left\|f(p(x,t))\right\| \le \sqrt{\mu}\left\|p(x,t)\right\|$ holds.

The initial and boundary value conditions are given as follows:

$$\phi(x)=\begin{pmatrix} 0.0116(x^{1/2}\sin(\frac{\pi}{10}(x-10)))^2 \\ 0.0092x^{1/3}(sin(\frac{\pi}{10}(x-10))) \end{pmatrix},\ x\in(-10\quad 10),$$

and

$$\frac{\partial p(x,t)}{\partial x}=0,\quad x=\pm 10,\quad t\in[0\quad +\infty).$$

By simply computing, we can obtain that $\Psi=-0.06<0$ and $\tilde{D}=1.2>1$.Thus, Assumption 1 and 2 are satisfied. $\Lambda=\begin{pmatrix} -13.2413 & -2.9039 \\ -2.9039 & -15.3006 \end{pmatrix}$, then $\lambda_{\max}(\Lambda)=-11.1899<0$. Fig.1 and Fig.2 present the state variables of the system. We can see that the system is stable.

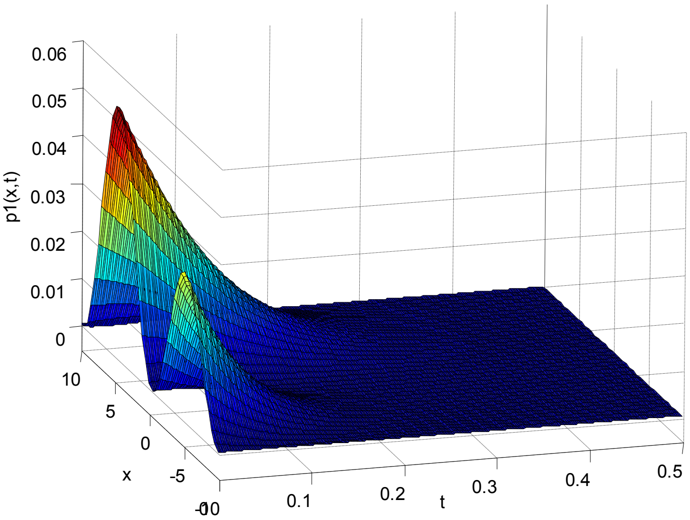

Figure 1 Trajectory of state variable $p_1(x,t)$

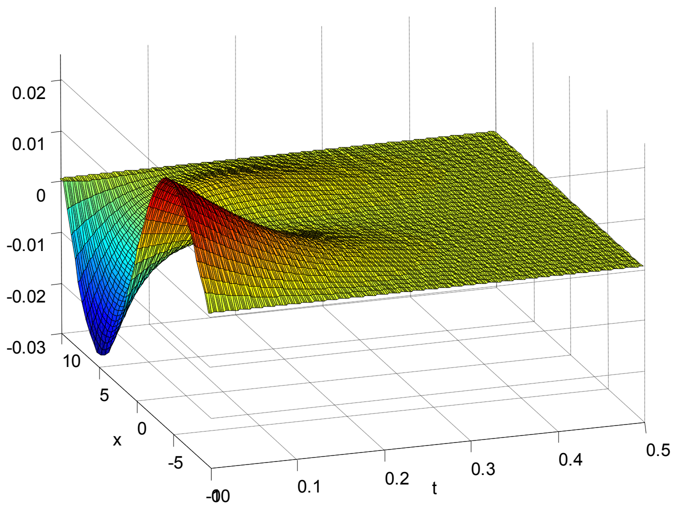

Figure 2 Trajectory of state variable $p_2(x,t)$

V. CONCLUSIONS

The stability problem of quasi-linear parabolic DPS is discussed via the Lyapunov-Krasovskii functional and the LMI approach. The nonlinear term in the system is only required to be norm-bounded. A new simple sufficient condition of uniformly stability is presented, which is easily verifiable. These works will have some theoretical significance in stability analysis of quasi-linear DPS.

ACKNOWLEDGEMENTS

This work was supported by the High-level Talent Foundation of Qingdao Agriculture University under Grant 631118.

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