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Common Fixed Point Theorems for Sequence of Mappings in Generalisation of Partial Metric Spaces

T.R.Vijayan,
Department of Mathematics, Pavai College of Technology, Namakkal - 637018, India.

Abstract: *The main purpose of this paper is to obtain fixed point theorems for sequence of mappings under partial metric spaces which generalizes theorem of four authors [5].*
Key Words: *Common fixed point, coincidence point, weakly compatible pair of mappings, partial metric space.*

I. Introduction

Partial metric spaces were introduced by Matthews [1] in 1992 as a part of the study of denotational semantics of dataflow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation.

II. Preliminaries

Before proving our results we need the following definitions and known results in this sequel [1, 2, 4].

Definition2.1. ([1]). A partial metric on a nonempty set X is a function p: $X \times X \rightarrow R_+$ such that for all x, y, z $\in$ X:

(p1) x=y <==> p(x, x) = p(x, y) = p(y, y),
(p2) p(x, x) $\leq$ p(x, y),
(p3) p(x, y) = p(y, x),
(p4) p(x,y) $\leq$ p(x,z)+ p(z,y)-p(z,z).

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Remark 2.2. It is clear that, if p(x, y) = 0, then from (p1) and (p2), x = y. But if x = y, p(x, y) may not be 0. A basic example of a partial metric space is the pair (R_+,p), where p(x,y) = max{x,y} for all x, y $\in R_+$ .Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls {Bp(x, $\in$), x $\in$ X, $\in$ >0} where Bp(x, $\in$) = {y $\in$ X: p(x, y) < p(x, x) + $\in$} for all x $\in$ X and $\in$ > 0.

If p is a partial metric on X, then the function $p^s : X \times X \rightarrow R_+$ given by$p^s(x, y)$ =2p(x, y)-p(x, x)-p(y, y) is a metric on X.

Definition 2.3. Let (X, p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to a point x $\in$ X if and only if $p(x, x) = \lim_{n \rightarrow +\infty} p(x, x_n)$

(ii) $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m \rightarrow +\infty} p(x_n, x_m)$.

Definition 2.4. A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point x $\in$ X, such that $p(x, x) = \lim_{n,m \rightarrow +\infty} p(x_n, x_m)$.

Remark 2.5. It is easy to see that every closed subset of a complete partial metric space is complete.

Lemma 2.6 ([1, 2]). Let (X, p) be a partial metric space. Then (a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X,p^s),

(b) (X, p) is complete if and only if the metric space (X,p^s) is complete. Furthermore,

$\lim_{n \rightarrow +\infty} p^s(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n \rightarrow +\infty} p(x_n, x) = \lim_{n,m \rightarrow +\infty} p(x_n, x_m)$

Matthews [1] obtained the following Banach fixed point theorem on complete partial metric spaces.

Theorem 2.7[1]. Let f be a mapping of a complete partial metric space (X, p) into itself such that there is a real number c with $0 \leq c < 1$, satisfying for all x, y $\in$ X:$p(f_x,f_y) \leq c\, p(x,y)$.Then f has a unique fixed point.

III. Main Results

Before stating the main results, we recall the following definitions.

Definition 3.1. Let X be a non-empty set and T_1, T_2: $X \rightarrow X$ are given self-maps on X. If $w = T_1x = T_2x$ for some x $\in$ X, then x is called a coincidence point of T_1 and T_2, and w is called a point of coincidence of T_1 and T_2.

Definition 3.2 [3]. Let X be a non-empty set and T_1, T_2: $X \rightarrow X$ are given self-maps on X. The pair $\{T_1, T_2\}$ is said to be weakly compatible if $T_1T_2t = T_2T_1t$, whenever $T_1t = T_2t$ for some t in X.

Our main result is the following.

Theorem 3.3. Suppose that S_i, T_i $\forall$ i are self-maps of a complete partial metric space (X, p) such that $T_iX \subseteq S_iX$ $\forall$ i, and $p(T_ix,T_iy) \le \phi$ (M(x, y)) ---- (3.1) $\forall_i$ and for all x, y $\in$ X, where $\phi \in \Phi$ and $M(x,y)=\max\{p(S_ix, S_iy),[p(T_ix,S_ix)+p(T_iy,S_iy)]/2, [p(T_iy, S_ix) +p (T_ix,S_iy)]/2\}$. If one of the ranges T_iX and S_iX $\forall$ i is a closed subset of (X, p), then S_i and T_i $\forall$ i have a coincidence point, (ii)Moreover if the pairs $\{S_i,T_i\}$ $\forall$ i is weakly compatible, then T_i and S_i $\forall$ i have a unique common fixed point.

Proof.

Let x_0 be an arbitrary point in X. Since $T_iX \subseteq S_iX$, $\forall$ i there exists $x_1 \in X$ such that $S_ix_1 = T_ix_0$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by $y_{2n} = T_ix_{2n+1}= S_ix_{2n}$ or $y_{2n+1} = T_ix_{2n+2}= S_ix_{2n+1}$ ----- (3.2) for every n $\in$ N $\forall$ i.

We claim that $\{y_n\}$ is a Cauchy sequence in the partial metric space (X, p).

We have, $M(x_{2p}, x_{2p+1}) \le \max\{p(S_ix_{2p}, S_ix_{2p+1}), [p(T_ix_{2p}, S_ix_{2p})+ p(T_ix_{2p+1},S_ix_{2p+1})]/2,$
$[p(T_ix_{2p+1}, S_ix_{2p}) + p(T_ix_{2p}, S_ix_{2p+1})]/2\}$ $\forall$ i.

$= \max\{p(y_{2p-1}, y_{2p}), [p(y_{2p-1}, y_{2p-1}) + p(y_{2p},y_{2p})]/2, [p(y_{2p}, y_{2p-1}) + p(y_{2p-1}, y_{2p})]/2\}$ $\forall$ i,

$= \max\{p(y_{2p-1}, y_{2p}), [p(y_{2p-1}, y_{2p-1}) +p(y_{2p},y_{2p})]/2\}$ $\forall$ i,

$= p(y_{2p-1}, y_{2p})$.

Using that ϕ is non-decreasing function, we get:

$\phi(M(x_{2p}, x_{2p+1})) \le \phi(\max\{p(y_{2p-1}, y_{2p})\})$ ----- (3.3)

From the contraction condition (3.1) with $x = x_{2p}$ and $y = x_{2p+1}$, we get:

$p(y_{2p}, y_{2p+1}) \le \phi(p(y_{2p-1}, y_{2p}))$

Since, $\phi(t) < t$ for each t >0, the above inequality implies that $p(y_{2p-1}, y_{2p}) =0$ and $p(y_{2p}, y_{2p+1}) = 0$ and then $y_{2p-1} = y_{2p}$ and $y_{2p} = y_{2p+1}$.

Hence we have $y_{2p-1} = y_{2p}= y_{2p+1} = y_{2p+2} =$ ……………….

Then $\{y_n\}$ is a Cauchy sequence in (X, p). The same conclusion holds if we suppose that there exists p $\in$ N such that $p(y_{2p}, y_{2p+1}) = 0$.

Now, we assume that $p(y_n, y_{n+1}) > 0$, for sufficiently large $n \in N$

Then from (3.3) as $\phi(t) < t$ for all t > 0, we have

$p(y_n, y_{n+1}) < \phi\{p(y_{n-1},y_n)\}$ for all $n \ge 1$………………….(3.4)

Repeating this inequality n time we obtain $p(y_n, y_{n+1}) \le \phi^n(p(y_0,y_1))$ -------(3.5)

By the properties (p2) and (p3) we have $\max\{p(y_n,y_n), p(y_{n+1},y_{n+1})\} \le p(y_n, y_{n+1})$

Thus from (3.5), $\max\{p(y_n,y_n), p(y_{n+1},y_{n+1})\} \le \phi^n p(y_0, y_1)$------(3.6)

Therefore, $p^s((y_n,y_{n+1})=2p(y_n,y_{n+1})- p(y_n,y_n)-p(y_{n+1},y_{n+1}) \le 2p(y_n,y_{n+1})+p(y_n,y_n)+p(y_{n+1},y_{n+1})$
$\le 4\phi^n p((y_0, y_1))$.

Now by the triangle inequality for the metric p^s and (3.6), for any k, n $\in$ N* we have

$p^s(y_n,y_{n+k}) \le p^s(y_n,y_{n+1})+ p^s(y_{n+1},y_{n+2})+$ ----------------$+p^s(y_{n+k-1},y_{n+k})$,

$\le 4\phi^n p((y_0, y_1)) + 4\phi^{n+1} p((y_0, y_1))$ +-------------$+4\phi^{n+k-1} p((y_0, y_1))$

$\le 4(\sum_{i=n}^{n+k-1} \Phi^i(p(y_0,y_1)))$

$\le 4(\sum_{i=n}^{\infty} \Phi^i(p(y_0,y_1)))$

Hence and from the property (b), lemma 2.6 of ϕ we conclude that for an arbitrary $\in > 0$ there is a positive integer n_0 such that $p^s(y_n,y_{n+k}) < \in$ for every $n \ge n_0$ and all $k \in N$

Thus we proved that $\{y_n\}$ is a Cauchy sequence in the metric space (X,p^s).

Since (X, p) is complete, then from Lemma2.6, (X,p^s) is a complete metric space.

Therefore, the sequence $\{y_n\}$ converges to some $y \in X$, that is, $\lim_{n\to+\infty} p^s(y_n, y) = 0$

From the properties (b) in Lemma 2.6, we have

$P(y, y) = \lim_{n\to+\infty} p(y_n, y) = \lim_{m\ge n\to+\infty} p(y_n, y_m)$------------ (3.7)

Moreover, since $\{y_n\}$ is a Cauchy sequence in the metric space (X,p^s), then $\lim_{n,m\to+\infty} p^s(y_n, y_m) = 0$ and so from (3.6) and the property (b) in lemma 2.6 of ϕ

We have $\lim_{n\to+\infty} p(y_n, y_n) = 0$----------------- (3.8)

Thus from the definition of p^s and (3.8), we have $\lim_{m\ge n\to+\infty} p(y_n, y_m) = 0$

Therefore, from (3.7), we have

$p(y, y) = \lim_{n\to+\infty} p(y_n, y) = \lim_{m\ge n\to+\infty} p(y_n, y_m) = 0$ ----------- (3.9)

This implies that $\lim_{n\to+\infty} p(y_{2n}, y) = \lim_{n\to+\infty} p(y_{2n-1}, y) = 0$ ------------ (3.10)

Thus from (3.10) we have $\lim_{n\to+\infty} p(T_i x_{2n}, y) = \lim_{n\to+\infty} p(S_i x_{2n+1}, y) = 0\ \forall_i$ ----------- (3.11)

Now we can suppose, without loss of generality, that $S_iX\ \forall$ i is a closed subset of the partial metric space (X, p). From (3.11), there exists $u \in X$ such that $y = S_iu$. $\forall$ i

We claim that p $(T_iu, y) = 0$. $\forall$ i Suppose, to the contrary, that p $(T_iu, y) > 0$. $\forall$ i

By (p4) and (3.1) we get

$p(y,T_iu) \le p(y,T_ix_{2n+2}) + p(T_iu,T_ix_{2n+2}) - p(T_ix_{2n+2}, T_ix_{2n+1})\ \forall i$

$\le p(y,T_ix_{2n+2}) + p(T_iu,T_ix_{2n+1})$

$\le p(y,T_ix_{2n+2}) + \phi(M(u,x_{2n+2}))$

By (3.2) we have $M(u,x_{2n+1}) = \max\{p(y,y_{2n+1}),[p(T_iu,y)+p(y_{2n+1},y_{2n+1})]/2,[p(y_{2n+1},y)+p(T_iu,y_{2n+1})]\}\ \forall i$

$\le \max\{p(y,y_{2n+1})\ [p(T_iu,y)+p(y_{2n+1},y_{2n+1})]/2,\ [p(y_{2n+1},y)+p(T_iu,y)+p(y,y_{2n+1})-p(y,y)]/2\}\ \forall i$ --------------- (3.12)

Since Φ is continuous, from (3.12), (3.9), and letting $n\to\infty$ we obtain

$p(y, T_iu) \le \lim_{n\to\infty}[p(y, y_{2n+1}) + \phi(M(u,x_{2n+2}))]$

$= \lim_{n\to\infty} p(y, y_{2n+1}) + \phi(\lim_{n\to\infty} M(u, x_{2n+2}))$

$= \phi(1/2\ p(T_iu, y))$.

Hence, as we supposed that $p(T_iu, y) > 0\ \forall$ i and as $\phi(t) < t$ for $t > 0$, we have

$P(y, T_iu) < 1/2\ p(y, T_iu)\ \forall$ i, i=1, 2, 3,........ Which is a contradiction.

Thus we deduce that $p(T_iu,y)=0$ and $y=T_iu\ \forall$ i ....(3.13)

Since $y = S_iu\ \forall$ i, then $T_iu = S_iu\ \forall$ i, that is, u is a coincidence point of S_i and $T_i\ \forall$ i

Hence the proof of (i).

Since the pair $\{T_i, S_i\}\ \forall$ i is weakly compatible, from (3.13),

We have $S_iy = S_iT_u = T_iS_iu = T_iy$. $\forall_i$

We claim that $p(T_iy, y) = 0$. $\forall$ i Suppose, to the contrary, that $p(T_iy, y) > 0$. $\forall$ i

We have $p(T_iy,y) \le p(T_iy,y_{2n+1})+p(y_{2n+1},y) = p(T_iy,S_ix_{2n+2}) + p(y_{2n+1}, y)$

$\le \phi(M(y,x_{2n+2})) + p(y_{2n+1}, y)$ --------------- (3.14)

On the other hand, we have $M(y,x_{2n+2}) = \max\{p(S_iy,S_ix_{2n+1}),[p(T_iy,S_iy)+ p(T_ix_{2n+1},S_ix_{2n+2})]/2,$
$[p(T_ix_{2n+2}, S_iy)+p(T_iy,S_i x_{2n+2})]/2\}$. $\forall$ i

$= \max\{p(T_iy,y_{2n+2}),[p(T_iy, T_iy)+p(y_{2n+1},y_{2n+2})]/2,[p(y_{2n+1},T_iy)+p(T_iy,y_{2n+2})]/2\}$. $\forall_i$

Using (3.9) and (p2), we get

For all i we have,

$M(y,x_{2n+2}) = \max\{p(T_iy,y), [p(T_iy, T_iy)+0]/2, p(T_iy,y)\}\ \forall$ i

$= p(T_iy, y)$ as $n\to+\infty$ ---------------- (3.15)

Using (3.15), the continuity of ϕ, (3.9) and letting $n\to+\infty$ in (3.14), we obtain

$p(T_iy, y) \le \phi(p(T_iy, y)) < p(T_iy, y)\ \forall$ i

Which is a contradiction. Then we deduce that $p(T_iy, y) = 0$ and $T_iy=S_iy=y\ \forall$ i --------- (3.16)

That is, y is a common fixed point of S_i, and T_i. $\forall$ i

Uniqueness

Let us suppose that $z \in X$ is a common fixed point of S_i, $T_i\ \forall$ i with $p(z, y) > 0$. $\forall$ i.

Using (3.1), we get $p(y,z)=p(T_iy,T_iz)$

$\le \phi\{\max(p(S_iy,T_iz),[p(T_iy,T_iy)+p(T_iz,T_iz)]/2,[p(T_iz,S_iy)+p(T_iy,S_iz)]/2)\}\ \forall$ i

$= \phi\{\max(p(y,z),[0+0]/2,[p(z,y)+p(y,z)]/2\} = \phi\{\max(p(y,z),p(y,z))\} < \phi(p(y,z)) < p(y,z)$

$p(y,z) < p(y,z)$

Which is a contradiction. Then we deduce that z = y.

Therefore, the uniqueness of the common fixed point is proved.

That is, the proof of the theorem is complete.

Corollary 3.4. Suppose that S and T are self-maps of a complete partial metric space (X, p) such that $TX \subseteq SX$ and $p(T_X,T_y) \le \phi(M(x, y))$ for all $x, y \in X$, where $\phi \in \Phi$ and
$M(x,y) \le \max\{p(S_x,S_y),1/2[p(T_x,S_x)+p(T_y,S_y)],1/2[p(T_y,S_x)+p(T_x,S_y)]\}$.

If one of the ranges TX and SX is a closed subset of (X, p), then (i) S and T have a coincidence point, (ii) Moreover, if the pairs {S, T} are weakly compatible, then T and S have a unique common fixed point.

Proof. The proof follows from above theorem 3.3.

Example3.5 ([Theorem 3.3]). Let X= {1, 2, 3} with the partial metric p is given by p(x,y)=max{x,y} for all $x,y \in X$. It is clear that (X,p) is a complete partial metric space.
Define the mappings $T_i,S_i:X \rightarrow X$ by $T_i1= T_i3=1$, $T_i2=2$, $\forall$ i
$S_i1=1$, $S_i2=3$, $S_i3=2$ $\forall$ i.
We have $T_iX \subseteq S_iX=X$ $\forall$ i,.
Consider the following ϕ (t)=(5/6)t, for all $t \in R$.
We have $p(T_i1, T_i2)=2 \leq [5/6](3)=(5/6)p(S_i1,S_i2)= \phi(p(S_i1,S_i2))$.
$p(T_i13, T_i2)=2 \leq [5/6](3)=(5/6)p(S_i3,S_i2)= \phi(p(S_i3,S_i2))$.
Then the contractive condition (1) is satisfied for every $x,y \in X$.
Moreover, the pair of mappings $\{T_i,S_i\}$ $\forall$ i is weakly compatible.
Now all the required hypotheses of theorem 3.3 are satisfied.
Then, we deduce that the existence an uniqueness of a common fixed point theorem of T_i and S_i. $\forall$ i .Here 1 is the unique common fixed point.
Now if X with metric d given by d(x,y)=Ix-yI for all $x,y \in X$.
We have $d (T_i1, T_i2) = I\ 1\text{-}2\ I=1 \leq \max \{d (S_i1, S_i2), [d (T_i1, S_i1) +d (T_i2, S_i2)]/2, [d (T_i2, S_i1) +d(T_i1, S_i2)]/2\}$.
$= \max\{d(1,3)[d(1,1)+d(2,3)]/2,[d(2,1)+d(1,3)]/2\}$ $\forall$ i
$= \max \{2, 1/2(0+1), 1/2(1+2)\}$
$= \max \{2, 1/2, 3/2\} =2$.
Therefore the contractive condition (1) is satisfied for any function $\phi \in \Phi$.

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