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Integral points on the homogeneous cone $z^2 = 3x^2 + 6y^2$

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Abstract: *The homogeneous cone represented by the ternary quadratic equation $z^2 = 3x^2 + 6y^2$ is analysed for its non-zero integral solutions. Five different patterns of solutions are illustrated. In each pattern, interesting relations among the solutions and some special polygonal and pyramidal numbers are exhibited.*
Keywords: *Homogeneous cone, Integral solutions, Polygonal numbers, Pyramidal numbers, Ternary quadratic.*

I. Introduction

The ternary quadratic Diophantine equations offer an unlimited field for research by reason of variety[1-2]. For an extensive review of various problems one may search refer [3-15]. This communication concerns with yet another interesting ternary quadratic equation representing a homogeneous cone for determining its infinitely many non-zero integral solutions.Also a few interesting relations among the solutions and some special polygonal and pyramidal numbers are presented. Further three different general forms for generating sequence of integral points based on the given point on the considered cone are exhibited.

II. Notations

Polygonal Numbers	**Notations for rank 'n'**	**Definitions**
Triangular number	T_n	$\frac{1}{2}n(n+1)$
Pentagonal number	Pen_n	$\frac{1}{2}(3n^2 - n)$
Hexagonal number	Hex_n	$2n^2 - n$
Octagonal number	Oct_n	$3n^2 - 2n$
Nanogonal number	Nan_n	$\frac{1}{2}(7n^2 - 5n)$
Decagonal number	Dec_n	$4n^2 - 3n$
Hendecagonal number	HD_n	$\frac{1}{2}(9n^2 - 7n)$
Dodecagonal number	DD_n	$\frac{1}{2}(10n^2 - 8n)$
Tridecagonal number	TD_n	$\frac{1}{2}(11n^2 - 9n)$
Tetradecagonal number	TED_n	$\frac{1}{2}(12n^2 - 10n)$
Octadecagonal number	OD_n	$\frac{1}{2}(16n^2 - 14n)$
Icosagonal number	IC_n	$\frac{1}{2}(18n^2 - 16n)$
Centered Square number	CS_n	$n^2 + (n-1)^2$

Centered hexagonal number	CH_n	$3n^2 - 3n + 1$
Gnomonic number	Gno_n	$2n – 1$
Pronic number	Pro_n	$n (n + 1)$
Stella Octangula number	SO_n	$n (2n^2 - 1)$
Star number	$Star_n$	$6 n (n - 1) + 1$
Pentagonal Pyramidal number	PP_n	$\frac{1}{2} n^2 (n + 1)$
Hexagonal Pyramidal number	HXP_n	$\frac{1}{6} n (n + 1) (4n - 1)$
Tetrahedral number	$Tetra_n$	$\frac{1}{6} n (n + 1) (n + 2)$
Pentatope number	PT_n	$\frac{1}{24} n (n + 1)(n + 2)(n + 3)$

III. Method Of Analysis

The equation under consideration to be solved is $z^2 = 3x^2 + 6y^2$ ……(1)

Five different patterns to (1) are illustrated below:

1. Pattern 1:

By applying the transformations, $x = X + 6T, y = X - 3T, z = 3W$ ……(2)

equation (1) reduces to $W^2 = X^2 + 18T^2$ ……(3)

which is satisfied by $T = 2rs, X = 18r^2 - s^2, W = 18r^2 + s^2$

Thus in view of (2), the non-zero distinct integral points on the homogeneous cone (1) are given by

$$x(r,s) = 18r^2 + 12rs - s^2$$

$$y(r,s) = 18r^2 - 6rs - s^2$$

$$z(r,s) = 54r^2 + 3s^2$$

1.1 Properties:

1. Each of the following represents a Nasty number:
   (i) $x(r,1) + 2y(r,1) + 3$
   (ii) $z(r,1) - 3$
   (iii) $z(r,s) - x(r,s) - 2y(r,s)$

2. Each one of the following is a perfect square:
   (i) $3x(r,s) + 6y(r,s) + 3z(r,s)$
   (ii) $12x(1,s) + 6y(1,s) + 3star_s - 120T_s + 30Hex_s - 3$
3. $3x(r,1) - z(r,1) - 18Gno_r \equiv 0 \pmod{12}$
4. $x(1,s) + Dec_s - Oct_s - 18 \equiv 0 \pmod{11}$
5. $y(1,s) + Pro_s - 18 \equiv 0 \pmod{5}$
6. $x(r,1) - 36T_r + 3Gno_r + 4 = 0$
7. $y(r,1) - 6[TED_r - DD_r + CS_r + Gno_r] + 1 = 0$
8. $2z(1,s) - 3[TED_s - DD_s + Pro_s] - 108 = 0$

II. Pattern 2:

Consider (3) as $X^2 + 18T^2 = W^2 * 1$ ……(4)

Let $W = a^2 + 18b^2$ and write 1 as $1 = \frac{(1+i2\sqrt{2})(1-i2\sqrt{2})}{9}$

Then (4) $\Rightarrow (X + i3\sqrt{2}T)(X - i3\sqrt{2}T) = (a + i3\sqrt{2}b)^2 (a - i3\sqrt{2}b)^2 \frac{(1+i2\sqrt{2})(1-i2\sqrt{2})}{9}$

Let us define $X + i3\sqrt{2}T = (a + i3\sqrt{2}b)^2 \frac{(1+i2\sqrt{2})}{3}$

By equating the real and imaginary parts on both sides, we get

$$X = \frac{1}{3}(a^2 - 24ab - 18b^2)$$

$$T = \frac{1}{9}(2a^2 + 6ab - 36b^2)$$

Using these values of X and T the values of x, y and z are obtained as

$$x = \frac{1}{3}(5a^2 - 12ab - 90b^2)$$

$$y = \frac{1}{3}(-a^2 - 30ab + 18b^2)$$

$$z = 3a^2 + 54b^2$$

Since our aim is to get integral values, we may choose a = 3A and b = 3B
Then the solutions of (1) are given by

$$x(A,B) = 15A^2 - 36AB - 270B^2$$

$$y(A,B) = -3A^2 - 90AB + 54B^2$$

$$z(A,B) = 27A^2 + 486B^2$$

2.1 Properties:

1. Each of the following represents a perfect square:
   (i) $3[x(A,A) + z(A,A)] + 15y(A,A)$
   (ii) $-3y(A,1) - 135Gno_A + 27$
2. $y(A,1) + z(A,1) - x(A,1) - 3Oct_A + 24Gno_A - 786 = 0$
3. $z(A,1) - y(A,1) - 60\Pr o_A + 15CS_A \equiv 0 \pmod{477}$
4. $x(A,1) + 5y(A,1) = 54DD_A - 540T_A$
5. Each one of the following represents a Nasty number:
   (i) $x(A,1) + z(A,1) - 12Nan_A + 12Hex_A - 6Dec_A$
   (ii) $2z(A,1) - 972$
6. $x(A,1) - 6(OD_A) + 6TD_A + 42HD_A - 21IC_A \equiv 0 \pmod{270}$
7. $5DD_z - 4TED_z = 2T_x + Hex_x + 18Oct_y - 12Dec_y$
8. $\left(\frac{PP_z}{T_z}\right)^2 = 3\left(\frac{4PT_x}{Tetra_x} - 3\right)^2 + 6\left(\frac{6Tetra_y}{\Pr o_y} - 2\right)^2$

III. Pattern 3:

Taking the transformations $x = 2u + 1, y = u - 1, z = 3v$

equation (1) is reduced to $v^2 = 2u^2 + 1$ ……(5)

whose general solution is $v_s = \frac{f}{2}, u_s = \frac{g}{2\sqrt{2}}$

Here $f = (3 + 2\sqrt{2})^{s+1} + (3 - 2\sqrt{2})^{s+1}$

$g = (3+2\sqrt{2})^{s+1} - (3-2\sqrt{2})^{s+1}$

From these values of v_s, u_s,the solutions of (1) are given by

$$x_s = \frac{g}{\sqrt{2}} + 1, y_s = \frac{g}{2\sqrt{2}} - 1, z_s = \frac{3f}{2}$$

The recurrence relations for the solutions are

$$x_{s+2} - 6x_{s+1} + x_s = -4$$
$$y_{s+2} - 6y_{s+1} + y_s = 4$$
$$z_{s+2} - 6z_{s+1} + z_s = 0 \qquad s = 0,1,2,3,\ldots$$

Using the above recurrence relations, few integral solutions for (1) are presented as follows:

s	x_s	y_s	z_s
0	5	1	9
1	25	11	51
2	141	69	297
3	817	407	1731
4	4757	2377	10089
5	27721	13859	58803
6	161565	80781	342729

3.1 Properties:

1. $x_s - 2y_s \equiv 0 \pmod{3}$
2. $x_{s+2} - 12y_{s+1} + x_s \equiv 0 \pmod{14}$
3. $63x_{s+1} - 378y_s - 84z_s$ is a perfect square.
4. $6z_{s+2} - 216x_s - 102z_s + 216 = 0$
5. $2z_{s+1} - 6z_s - 24y_s$ is a Nasty number.
6. $2z_{s+1} - 4x_{s+1} - y_{s+1} + 3y_s \equiv 0 \pmod{6}$
7. $(x_s - 1)(y_s + 1)$ is a perfect square.

IV. Pattern 4:

Equation (1) can be rewritten as $z^2 - 6y^2 = 3x^2$ ……(6)

Let $x = a^2 - 6b^2$ and $3 = (3+\sqrt{6})(3-\sqrt{6})$

Then (6) $\Rightarrow (z+\sqrt{6}y)(z-\sqrt{6}y) = (3+\sqrt{6})(3-\sqrt{6})[(a+\sqrt{6}b)(a-\sqrt{6}b)]^2$

Define $z + \sqrt{6}y = (3+\sqrt{6})(a+\sqrt{6}b)^2$

By equating the real and imaginary parts, the values of y and z are

$$y = a^2 + 6ab + 6b^2$$
$$z = 3a^2 + 12ab + 18b^2$$

Then the solutions of (1) are represented by $x(a,b) = a^2 - 6b^2$

$$y(a,b) = a^2 + 6ab + 6b^2$$
$$z(a,b) = 3a^2 + 12ab + 18b^2$$

4.1 Properties:

1. Each of the following is a perfect square:
   (i) $x(a,1) + 6$ is a perfect square.
   (ii) $z(a,1) - 3x(a,1) - 6\Pr o_a + 3CS_a - 3$
2. $y(a,1) - 2T_a - 6 \equiv 0 \pmod{5}$

3. $2z(a,1) - 3\Pr o_a - CH_a - 12Gno_a - 47 = 0$
4. $z(a,1) - 2y(a,1) - x(a,1) \equiv 0 \pmod{12}$
5. $4x(a,1) - \dfrac{3HXP_{a^2}}{T_{a^2}} + 23 = 0$
6. $6z(a,1) - 12y(a,1) - 36$ is a Nasty number.
7. $7x(a,1) - y(a,1) - Star_a + 49 = 0$
8. $6x(a,1) - CH_a - 6T_a \equiv 0 \pmod{37}$

V. Pattern 5:

Equation (1) can also be written as $z^2 - 3x^2 = 6y^2$ …….(7)

Let $y = a^2 - 3b^2$ and $6 = (3+\sqrt{3})(3-\sqrt{3})$

From (7), $(z+\sqrt{3}x)(z-\sqrt{3}x) = (3+\sqrt{3})(3-\sqrt{3})[(a+\sqrt{3}b)(a-\sqrt{3}b)]^2$

Let us define $z+\sqrt{3}x = (3+\sqrt{3})(a+\sqrt{3}b)^2$

By Equating real and imaginary parts, we get

$$x = a^2 + 6ab + 3b^2$$
$$z = 3a^2 + 6ab + 9b^2$$

Then the solutions of (1) are
$$x(a,b) = a^2 + 6ab + 3b^2$$
$$y(a,b) = a^2 - 3b^2$$
$$z(a,b) = 3a^2 + 6ab + 9b^2$$

5.1 Properties:

1. $z(a,1) - y(a,1) - x(a,1)$ can be written as the sum of two perfect squares.
2. Each of the following is a perfect square
   (i) $2[x(a,1) + y(a,1)] - 24\Pr o_a + 48PP_a - 12SO_a$
   (ii) $z(1,b) - 18Hex_b + 12Oct_b - 3$
3. $6z(a,1) - 60T_a + 6Hex_a$ is a Nasty number.
4. $x(a,1) + y(a,1) - CS_a - 4Gno_a \equiv 0 \pmod{3}$
5. $z(a,1) - x(a,1) - 2CS_a + CH_a - \Pr o_a \equiv 0 \pmod{5}$
6. $2y(1,b) + 3Pen_b + 3T_b = 2$
7. $4x(1,b) - 3CS_b - 6\Pr o_b - 12Gno_b \equiv 0 \pmod{13}$

VI. Remarkable observations:

If (x_0, y_0, z_0) is any given solution of (1), then each of the following three triples (i to iii) of integers satisfies (1).

(i) (x_n, y_0, z_n), where $x_n = Y_{n-1}x_0 + X_{n-1}z_0$, $z_n = 3X_{n-1}x_0 + Y_{n-1}z_0$, n = 1,2,3,…

[(X_{n-1}, Y_{n-1}) is the general solution of $Y^2 = 3X^2 + 1$]

(ii) (x_0, y_n, z_n), where $y_n = Y_{n-1}y_0 + X_{n-1}z_0$, $z_n = 6X_{n-1}y_0 + Y_{n-1}z_0$, n = 1,2,3,…

[(X_{n-1}, Y_{n-1}) is the general solution of $Y^2 = 6X^2 + 1$]

(iii) (x_n, y_n, z_n), where $x_n = \frac{1}{3}\left\{[3^n + 2(-3)^n]x_0 + [2.3^n - 2(-3)^n]y_0\right\}$,

$y_n = \frac{1}{3}\left\{[3^n - (-3)^n]x_0 + [2.3^n + (-3)^n]y_0\right\}$ and $z_n = 3^n z_0$, n = 1,2,3,…

VII. Conclusion

One may search for other patterns of solutions and relations among the solutions, and also the relations between the solutions and polygonal numbers.

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