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Fuzzy 'useful' entropy measures and Holder's inequality

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Abstract: *In this communication, we propose useful fuzzy entropy measure and study its particular cases. Some coding theorems have been proved for decipherable codes using Holder's inequalities. Some known results are the particular cases of our proposed measure.*
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I. Introduction.

The concept of Fuzzy Logic (FL) was conceived by Lotfi Zadeh [17] and presented not as a control methodology, but as a way of processing data by allowing partial set membership rather than crisp set membership or non-membership. This approach to set theory was not applied to control systems until the 70's due to insufficient small-computer capability prior to that time. Zadeh reasoned that people do not require precise, numerical information input, and yet they are capable of highly adaptive control. FL was conceived as a better method for sorting and handling data but has proven to be an excellent choice for many control system applications since it mimics human control logic. It can be built into anything from small, hand-held products to large computerized process control systems. It uses an imprecise but very descriptive language to deal with input data more like a human operator. It is very robust and forgiving of operator and data input and often works when first implemented with little or no tuning. There is a unique membership function associated with each input parameter.

Fuzzy Logic (FL) plays an important role in the context of Information theory. Klir G.J and B. Parviz [9] first made an attempt to apply Fuzzy set and Fuzzy logic in information theory, later on various researchers applied the concept of Fuzzy in information theoretic entropy function. Besides above applications of fuzzy logic in information theory there is a numerous literature present on the application of fuzzy logic in information theory.

A fuzzy set is represented as

$$A = \{x_i/\mu_A(x_i) : i = 1,2,\ldots,n\},$$

where $\mu_A(x_i)$ gives the degree of belongingness of the element 'x_i' to the set 'A'. If every element of the set 'A' is '0' or '1', there is no uncertainty about it and a set is said to be a crisp set. On the other hand, a fuzzy set 'A' is defined by a characteristic function

$$\mu_A(x_i) = \{x_1, x_2, \ldots, x_n\} \rightarrow [0,1].$$

The function $\mu_A(x_i)$ associates with each $x_i \in R^n$ grade of membership function.

A fuzzy set A^* is called a sharpened version of fuzzy set A if the following conditions are satisfied:

$$\mu_{A^*}(x_i) \leq \mu_A(x_i), \quad if \;\; \mu_A(x_i) \leq 0.5 \;\; for\ all\ i = 1,2,\ldots,n$$

and $$\mu_{A^*}(x_i) \geq \mu_A(x_i), \quad if \;\; \mu_A(x_i) \geq 0.5 \; for\ all\ i = 1,2,\ldots,n$$

De Luca and Termini [13] formulated a set of properties and these properties are widely accepted as criterion for defining any fuzzy entropy. In fuzzy set theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity in making a decision whether an element belong to a set or not. So, a measure of average fuzziness is fuzzy set H(A) should have the following properties to be a valid entropy.

i. (Sharpness): $H(A)$ is minimum if and only if A is a crisp set
i.e., $\mu_A(x_i) = 0 \; or \; 1; \; \forall_i$

ii. (Maximality): $H(A)$ is maximum if and only if A is most fuzzy set
i.e., $\mu_A(x_i) = \frac{1}{2} \;\; \forall_i$

iii.(Resolution): $H(A^*) \leq H(A)$ where A^* is sharpened version of A.

iv. (Symmetry): $H(A) = H(\overline{A})$, where $\overline{A}$ is the complement of set A
i.e. $\overline{\mu}_A(x_i) = 1 - \mu_A(x_i)$

The importance of fuzzy set comes from the fact that it can deal with imprecise and inexact information. Its application areas span from design of fuzzy controller to robotics and artificial intelligence.

II. Basic Concepts

Let X be discrete random variable taking on a finite number of possible values $X = (x_1, x_2, \ldots, x_n)$ with respective membership function $A = \{\mu_A(x_1), \mu_A(x_2), \ldots \mu_A(x_n)\} \rightarrow [0,1]$, $\mu_A(x_i)$ gives the elements the degree of belongingness x_i to the set A.The function $\mu_A(x_i)$ associates with each $x_i \in R^n$ a grade of membership to the set A and is known as membership function.
Denote

$$X = \begin{bmatrix} x_1 & x_2 & \ldots & x_2 \\ \mu_A(x_1) & \mu_A(x_2) & \cdots & \mu_A(x_n) \end{bmatrix} \tag{2.1}$$

We call the scheme (2.1) as a finite fuzzy information scheme. Every finite scheme describes a state of uncertainty. De Luca and termini [13] introduced a quantity which, in a reasonable way to measures the amount of uncertainty (fuzzy entropy) associated with a given finite scheme. This measure is given by

$$H(A) = -\sum_{i}^{n}\left[\mu_A(x_i)\, log\, \mu_A(x_i) + \left(1 - \mu_A(x_i)\right) log\left(1 - \mu_A(x_i)\right)\right] \tag{2.2}$$

The measure (2.2) serve as a very suitable measure of fuzzy entropy of the finite information scheme (2.1).

Let a finite source of n source symbols $X = (x_1, x_2, \ldots, x_n)$ be encoded using alphabet of D symbols, then it has been shown by Feinstein [4] that there is a uniquely decipherable/ instantaneous code with lengths $l_1, l_2 \ldots, l_n$ iff the following Kraft [10] inequality is satisfied

$$\textstyle\sum_i^n D^{-l_i} \leq 1 \tag{2.3}$$

Belis and Guiasu [2] observed that a source is not completely specified by the probability distribution P over the source alphabet X in the absence of qualitative character. So it can be assumed (Belis and Guiasu [2]) that the source alphabet letters are assigned weights according to their importance or utilities in view of the experimenter.

Let $U = \left(u_1, u_{2,} \ldots, u_n\right)$ be the set of positive real numbers, u_i is the utility or importance of x_i. The utility, in general, is independent of probability of encoding of source symbol x_i, i.e. p_i. The information source is thus given by

$$X = \begin{bmatrix} X_1 & X_2 \ldots & X_n \\ p_1 & p_2 \ldots & p_n \\ u_1 & u_2 \ldots & u_n \end{bmatrix}, \qquad u_i > 0\, p_i \geq 0, \sum_i^n p_i = 1 \tag{2.4}$$

Belis and Guiasu [2] introduced the following quantitative- qualitative measure of information

$$H(P,U) = -\sum_i^n u_i p_i\, log\, p_i \tag{2.5}$$

which is a measure for the average of quantity of 'variable' or 'useful' information provided by the information source (2.4).

Guiasu and Picard [5] considered the problem of encoding the letter output by the source (2.4) by means of a single letter prefix code whose codeword's $c_1, c_2, \ldots, c_n$ are of lengths $l_1, l_2, \ldots, l_n$ respectively and satisfy the Kraft's inequality(2.3), they included the following 'useful' mean length of the code

$$L(U) = \frac{\sum_i^n u_i p_i l_i}{\sum_i^n u_i p_i} \tag{2.6}$$

Further they derived a lower bound for(2.6).

Now, corresponding to (2.5) and (2.6), we have the following fuzzy measures $H(A,U) = -\sum_{i=1}^n u_i\left\{\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right\} log\left\{\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right\}$ (2.7)

and

$$L(U) = \frac{\sum_{i=1}^n u_i\left\{\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right\} l_i}{\sum_{i=1}^n u_i\left\{\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right\}} \tag{2.8}$$

respectively.

In the next section, fuzzy coding theorem have been obtained by considering a new parametric fuzzy entropy function involving utilities and generalized useful fuzzy code word mean length. The result obtained here are not only new, but also generalized some well known results available in the literature of information theory

III. Coding Theorems:

We define 'useful' information of order α for the power distribution $\mu_A^\beta(x_i)$ as such

$$\alpha^H\left(A^\beta; U\right) = 1/(1-\alpha)\, log \sum_{i=1}^n \left(\frac{\mu_A^{\alpha\beta}(x_i) u_i + \left(1 - \mu_A(x_i)\right)^{\alpha\beta} u_i}{\sum_{i=1}^n \left(\mu_A^\beta(x_i) u_i + \left(1 - \mu_A(x_i)\right)^\beta u_i\right)}\right),$$
$$\alpha \neq 0, \beta \neq 0, \beta > 1. \alpha \neq 1. \tag{3.1}$$

It is obvious that (3.1) is a generalization of fuzzy 'useful' information of order α, considered by Gurdial and Pessoa [6] as when $\beta = 1$, (3.1) becomes

$$\alpha^{H}(A;U) = 1/(1-\alpha)\,log\sum_{i=1}^{n}\left(\frac{\mu_A^{\alpha}(x_i)u_i + \left(1-\mu_A(x_i)\right)^{\alpha}u_i}{\sum_{i=1}^{n}(\mu_A(x_i)u_i + (1-\mu_A(x_i)u_i)}\right), \alpha > 0, \alpha \neq 1 \qquad (3.2)$$

which is the generalization of Renyi's [15] entropy of order α, when $u_i = 1$ for all i.

$$\lim_{\alpha\to 1}\alpha^{H}(A;U) = -\frac{\sum_{i=1}^{n}\{u_i\mu_A(x_i)\,log\,\mu_A(x_i) + u_i(1-\mu_A(x_i)\,log(1-\mu_A(x_i))\}}{\sum_{i=1}^{n}(\mu_A(x_i)u_i + (1-\mu_A(x_i)u_i)} \qquad (3.3)$$

which is Guiasi's [5] fuzzy weighed entropy.

Shannon's [16] inequality plays a vital role in characterizing entropy and proving coding theorem which can be written as

$$-\sum_{i=1}^{n}[\mu_A(x_i)\,log\,\mu_A(x_i) + (1-\mu_A(x_i)\,log(1-\mu_A(x_i))] \le -\sum_{i=1}^{n}[\mu_A(x_i)\,log\,\mu_B(x_i) + (1-\mu_A(x_i)\,log(1-\mu_B(x_i))] \qquad (3.4)$$

Aczel [1], proved that every uniquely decipherable code, the average length of code words satisfies

$$\sum_{i=1}^{n}[\mu_A(x_i)\,l_i + (1-\mu_A(x_i)]l_i \ge -\frac{\sum_{i=1}^{n}[\mu_A(x_i)\,log\,\mu_A(x_i) + (1-\mu_A(x_i))\,log(1-\mu_A(x_i))]}{log_2 D} \qquad (3.5)$$

or

$$-log_D\left(\frac{\mu_A^{\beta\alpha}(x_i)u_i + \left(1-\mu_A(x_i)\right)^{\beta\alpha}u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta\alpha}(x_i)u_i + \left(1-\mu_A(x_i)\right)^{\beta\alpha}u_i\right)}\right) \le -log_D\left(\frac{\mu_A^{\beta\alpha}(x_i)u_i + \left(1-\mu_A(x_i)\right)^{\beta\alpha}u_i}{\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)u_i + \left(1-\mu_A(x_i)\right)^{\alpha\beta}u_i\right)}\right) + 1,$$

where, $\sum_{i=1}^{n} D^{-l_i} \le 1, D \ge 2, l$ integers i=1,2,..,n.

Campbell [3] utilized Holder's inequality [6]

$$\sum_{i=1}^{n} X_i Y_i \ge \left(\sum_{i=1}^{n} X_i^{p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} Y_i^{q}\right)^{\frac{1}{q}}; \quad \text{if } \frac{1}{p}+\frac{1}{q} = 1, 0 < p < 1, q < 0 \text{ or } 0 < q < 1, p < 0$$

$$\le \left(\sum_{i=1}^{n} X_i^{p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} Y_i^{q}\right)^{\frac{1}{q}}; \quad \text{if } \frac{1}{p}+\frac{1}{q} = 1, p \ge 1, q > 1 \text{ and } X_i, Y_i > 0 \qquad (3.6)$$

to prove

$$\alpha/(1-\alpha)\,log_D\left(\sum_{i=1}^{n}\{\mu_A(x_i) + (1-\mu_A(x_i)\}D^{l_i(1-\alpha)/\alpha}\right) \ge \frac{\alpha^{H_N}(\mu_A(x_1), \mu_A(x_2), .., \mu_A(x_n))}{log_2 D} \qquad (3.7)$$

where

$$\alpha^{H_N}(\mu_A(x_1), \mu_A(x_2), .., \mu_A(x_n)) = 1/(1-\alpha)\,log\left(\sum_{i=1}^{n}\left\{\mu_A^{\alpha}(x_i) + \left(1-\mu_A(x_i)\right)^{\alpha}\right\}\right)$$

and also

$$\alpha/(1-\alpha)\,log_D\left(\sum_{i=1}^{n}\{\mu_A(x_i) + (1-\mu_A(x_i)\}D^{l_i(1-\alpha)/\alpha}\right) < \frac{\alpha^{H_N}(\mu_A(x_1), \mu_A(x_2), .., \mu_A(x_n))}{log_2 D} + 1 \qquad (3.8)$$

We consider the power distribution due to the fact that the probabilities repeat more than once in many of the experiments. Utilizing Holder's inequality, we prove some important noiseless coding theorems for sources having utilities. These theorems generalize Longo [12], Campbell [3], Aczel [1] and Gurdial et al. [6] results and provide a tool for power distributions.

IV. Information of order α for power distribution and exponential mean length

Let us consider the function defined earlier by (3.1)

$$\alpha^{H}\left(A^{\beta};U\right)=1/(1-\alpha)\,log\sum_{i=1}^{n}\left(\frac{\mu_A^{\alpha\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\alpha\beta}u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)}\right),$$

and the average code word length corresponding to (3.1) should be

$$t^{L_u}=\frac{\alpha}{1-\alpha}\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)\left(\frac{u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)}\right)^{1/\alpha}D^{((1-\alpha)/\alpha)l_i}\tag{4.1}$$

= L.H.S.

which is called fuzzy exponential 'useful' mean lengths of code words weighted with the function of power probabilities and utilities?

Now we prove a relation between (3.1) and (4.1) with the help of Holder's inequality (3.7). The statement of the relation in terms of theorem; is stated as such:

Theorem 1. For every uniquely decipherable code, the 'useful' α-average length of codewords satisfies

$$t^{L_u}\geq\frac{\alpha^{H}(A^{\beta};U)}{logD},$$

$$1/(1-\alpha)\,log_D\left(\sum_{i=1}^{n}\frac{\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i^{1/t}D^{l_i(1-\alpha)/\alpha}}{\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)\right)^{1/t}}\right)$$

$$\geq\frac{1/(1-\alpha)\,log_2\sum_{i=1}^{n}\left(\mu_A^{\beta\alpha}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta\alpha}u_i/\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)\right)}{log_2D}$$

$$=\frac{\alpha^{H}\left(A^{\beta};U\right)}{logD},\tag{4.2}$$

where $\alpha>o,D\geq 2,l_i$, integers, $i=1,2\ldots,n$ and

$$\sum_{i=1}^{n}D^{-l_i}\leq 1,\ \sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)=1.$$

Proof. Using Holder's inequality (3.6), i.e.

$$\sum_{i=1}^{n}X_iY_i\geq\left(\sum_{i=1}^{n}X_i^{p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}Y_i^{q}\right)^{\frac{1}{q}}\quad \text{if } \frac{1}{p}+\frac{1}{q}=1, and\ if\ p_i<1,p\neq 0,$$

there is equality in (3.6) if there exists a positive number c such that $X_i^{p}=cY^{q}$. Setting in (3.6)

$$X_i=\left\{\mu_A^{\beta/t}(x_i)+\left(1-\mu_A(x_i)\right)^{-\beta/t}\right\}\left\{\frac{u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)}\right\}^{-(t+1)/t}D^{-l_i}\tag{4.3}$$

$$Y_i=\left\{\mu_A^{+\beta/t}(x_i)+\left(1-\mu_A(x_i)\right)^{+\beta/t}\right\}\left\{\frac{u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)}\right\}^{(t+1)/t}\tag{4.4}$$

Take $p=-t,q=t/(t+1)\,,\ t>-1,t\neq 0.$

Since codes are uniquely decipherable, therefore, we have

$$\sum_{i=1}^{n}D^{-l_i}\ \leq 1\tag{4.5}$$

Using (4.3) and (4.4) in (3.6) and applying (4.5), we get

$$1\geq\sum_{i=1}^{n}D^{-l_i}$$

$$\geq\left\{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)\quad\left(\frac{u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)}\right)^{t+1}D^{tl_i}\right\}^{-1/t}$$

$$\left\{\sum_{i=1}^{n}\frac{\left(\mu_A^{\beta/(t+1)}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta/(t+1)}u_i\right)}{\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)\right)}\right\}^{(t+1)/t}$$

or

$$\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)\left(\frac{u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)}\right)^{t+1} D^{tl_i}\right)^{-1/t}$$
$$\geq \left(\sum_{i=1}^{n}\frac{\left(\mu_A^{\beta/(t+1)}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta/(t+1)}u_i\right)}{\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)\right)}\right)^{(t+1)/t}$$

or

$$(1/t)log_D\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)\left(\frac{u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)}\right)\right)^{t+1} D^{tl_i}$$
$$\geq \frac{t+1}{t}log_D\left(\sum_{i=1}^{n}\frac{\left(\mu_A^{\beta/(t+1)}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta/(t+1)}u_i\right)}{\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)\right)}\right)$$

Putting $\alpha = 1/(1+t), \alpha > 0,\ \ \alpha \neq 1, we\ have$

$$\alpha\Big/(1-\alpha)log_D\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)\left(\frac{u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)}\right)^{1/\alpha} D^{l_i(1-\alpha)/\alpha}\right)$$
$$\geq \frac{\sum_{i=1}^{\alpha/(1-\alpha)log_2\left(\sum_{i=1}^{n}\left(\mu_A^{\beta\alpha}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta\alpha}u_i\right)/n\right)}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)}{log_2 D}.$$

This proves the result (4.1).

4.1. Particular cases

Case 1. When $\beta = 1$, i.e., when power distribution is changed into ordinary probability distribution (4.1) reduces to Gurdial and Pessoa's [6].

Case 2. When utilities are ignored and $\beta = 1$, (4.1) reduces to

$$\alpha/(1-\alpha)\ log_D\left(\sum_{i=1}^{n}\left\{\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right\}D^{l_i(1-\alpha)/\alpha}\right)$$
$$\geq \frac{\alpha^{H_N}(p_1,p_2,..,p_n)}{logD} \tag{4.6}$$

Theorem 2. Choosing the lengths L = $l_1, l_2, \dots, l_n$, properly, in the codewords of Theorem 1, t^{L_u} can be made to satisfy the following inequality:

$$\alpha/(1-\alpha)\ log_D\left(\sum_{i=1}^{n}\frac{\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i^{1/\alpha}D^{[l_i(1-\alpha)/\alpha]}}{\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i\right)^{1/\alpha}}\right)$$
$$< \frac{1/(1-\alpha)log_2\sum_{i=1}^{n}\left(\mu_A^{\beta\alpha}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta\alpha}u_i/\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)u_i+\left(1-\mu_A(x_i)\right)^{\beta}u_i\right)\right)}{log_2 D}+1 \tag{4.7}$$

Proof. Choose the codeword lengths l_i as the unique integers satisfying

$$-log_D\left(\frac{\left(\mu_A^{\beta\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta\alpha}\right)u_i}{\left(\sum_{i=1}^{n}\left(\mu_A^{\beta\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta\alpha}\right)u_i\right)}\right)$$
$$\leq -log_D\left(\frac{\left(\mu_A^{\beta\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta\alpha}\right)u_i}{\left(\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha\beta}\right)u_i\right)}\right)+1 \tag{4.12}$$

From the left of inequality (4.12), we have

$$D^{-l_i} \leq \frac{\left(\mu_A^{\alpha\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha\beta}\right)u_i}{\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha\beta}\right)u_i}, \quad i = 1,2,..,n. \tag{4.13}$$

Therefore,

$$\sum_{i=1}^{n}D^{-l_i} \leq 1,$$

which is Kraft's inequality. Therefore, there exists indeed uniquely decipherable codes with the codeword lengths determined by (4.12). It is also noteworthy that the 'useful' α-average length with power distribution is increasing with each k_i and translator. Hence with power inequality in (4.12) implies

$$\alpha/(1-\alpha)\,log_D\left[\sum_{i=1}^{n}\frac{\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i^{1/\alpha}D^{[l_i(1-\alpha)/\alpha]}}{\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i\right)^{1/\alpha}}\right]$$

$$< \alpha/(1-\alpha)\,log_D$$

$$\left(\sum_{i=1}^{n}\frac{\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i^{1/\alpha}}{\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i\right)^{1/\alpha}}D^{[l_i(1-\alpha)/\alpha]}\,log_2 D\,\frac{\left(\mu_A^{\alpha\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha\beta}\right)u_i}{\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha\beta}\right)u_i}\right)+1$$

$$= \alpha/(1-\alpha)\,log_D\left(\sum_{i=1}^{n}\frac{\left(\mu_A^{\alpha\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha\beta}\right)u_i\left(\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha\beta}\right)u_i\right)^{(1-\alpha)/\alpha}}{\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i\right)^{1/\alpha}}\right)+1$$

$$= \alpha/(1-\alpha)\,log_D\left[\left(\sum_{i=1}^{n}\frac{\left(\mu_A^{\alpha\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha\beta}\right)u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i}\right)\left(\frac{\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha\beta}\right)u_i}{\left(\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i\right)}\right)^{(1-\alpha)}\right]$$

$$+1$$

$$= \alpha/(1-\alpha)\,log_D\left[\left(\sum_{i=1}^{n}\frac{\left(\mu_A^{\alpha\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha\beta}\right)u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i}\right)\right]^{1/\alpha}+1$$

$$= 1/(1-\alpha)\,log_D\left[\sum_{i=1}^{n}\left(\frac{\left(\mu_A^{\alpha\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha\beta}\right)u_i}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)u_i}\right)\right]+1.$$

Hence the theorem 2.

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