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On the vertex covering sets and vertex cover polynomials of square of paths

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Abstract: *Let G be a graph of order n with no isolated vertex. Let $\mathcal{C}(G,i)$ be the family of vertex covering sets in G with cardinality i and let c(G, i) = $|\mathcal{C}(G,i)|$. The polynomial C(G, x) =$\sum_{i=\beta(G)}^{n}$ c(G, i)x^i is called the vertex cover polynomial of G. In this paper, we obtain some properties of the polynomial $C(P_n^2,x)$ and its coefficients. Also ,we derive the reduction formula to calculate the vertex covering polynomial of square of path.*

Key word: *Square of path, vertex covering set, vertex covering number, vertex covering polynomial.*

I. Introduction:

Let G = (V,E) be a simple graph. For any vertex v ∈ V, the open neighborhood of v is the set N (v) = {u∈V/ uv ∈ E} and the closed neighbourhood of v is the set N[v] = N(v) U {v}. For a set S ⊆ V, the open neighborhood of S is N(S) = $\bigcup_{v\in S} N(v)$ and the closed neighborhood of S is N[S] = N(S) U S.

A set S ⊆ V is a vertex covering of G, if every edge uv ∈ E is adjacent to atleast one vertex in S. The vertex covering number, β(G), is the minimum cardinality of the minimum vertex covering sets in G. A vertex covering set with cardinality β(G) is called a β −set.

We use ⌊x⌋, for the largest integer less than or equal to x and ⌈x⌉, for the smallest integer greater than or equal to x.

Definition 1.1

The second power of a graph is a graph with the same set of vertices as G and it contains an edge between two vertices if and only if there is a path of length atmost two between them. The second power of a graph is also called its square.

Let P_n^2 be the square of the path P_n (2^{nd} power) with n vertices V (P_n^2) ={1,2,3,……………n} and E(P_n^2)={ (1,2) (2,3)……………..(n-1,n) , (1,3) (2,4)………………(n-2,n)}.

II. Vertex covering sets of square of the path

In this section we state the vertex covering number of the square of the path and some of its properties.

Definition 2.1

Let P_n^2 be the square of the path of order n with no isolated vertices. Let $\mathcal{C}(P_n^2, i)$ be the family of vertex covering sets of the graph P_n^2 with cardinality i and let c(P_n^2, i) = $|\mathcal{C}(P_n^2, i)|$. We call the polynomial C(P_n^2, x) = $\sum_{i=\beta(P_n^2)}^{n} c(P_n^2, i)\, x^i$ as the vertex covering polynomial of the graph P_n^2.

Lemma 2.2

Let P_n^2 be the square of the path P_n with n vertices, then its vertex covering number is β(P_n^2) =$\left\lfloor\frac{2n}{3}\right\rfloor$.

Lemma 2.3

Let P_n^2 , n ≥ 3 be the square of the path with n vertices, then c (P_n^2, i) =0 if i < $\left\lfloor\frac{2n}{3}\right\rfloor$ or i > n and c(P_n^2,i) > 0 if $\left\lfloor\frac{2n}{3}\right\rfloor$ ≤ i ≤ n.

Proof

If i < $\left\lfloor\frac{2n}{3}\right\rfloor$ or i > n, then there is no vertex covering set of cardinality i. Therefore C(P_n^2, i)=ϕ.

By Lemma 2.2, the cardinality of the minimum vertex covering set is $\left\lceil\frac{2n}{3}\right\rceil$. Therefore, c $(P_n^2, i)>0$ if $i \geq \left\lceil\frac{2n}{3}\right\rceil$ and $i \leq n$. Hence, we have $c(P_n^2, i) = 0$ if $i < \left\lceil\frac{2n}{3}\right\rceil$ or $i>n$, and $c(P_n^2, i)>0$, if $\left\lceil\frac{2n}{3}\right\rceil \leq i \leq n$.

Lemma 2.4

Let P_n^2, $n \geq 2$ be the square of the path with n vertices. Then

(i) If $\mathcal{C}(P_{n-1}^2, i-1) = \phi$ and $\mathcal{C}(P_{n-3}^2, i-2) = \phi$, then $\mathcal{C}(P_n^2, i) = \phi$.

(ii) If $\mathcal{C}(P_{n-1}^2, i-1) = \mathcal{C}(P_{n-2}^2, i-1) = \mathcal{C}(P_{n-3}^2, i-1) = \phi$, then $\mathcal{C}(P_n^2, i) = \phi$.

(iii) If $\mathcal{C}(P_{n-1}^2, i-1) = \phi$ and $\mathcal{C}(P_{n-3}^2, i-2) \neq \phi$, then $\mathcal{C}(P_n^2, i) \neq \phi$.

Proof

(i) By hypothesis, $i-1 < \left\lceil\frac{2(n-1)}{3}\right\rceil$ or $i-1 > n-1$ and $i-2 < \left\lceil\frac{2(n-3)}{3}\right\rceil$ or $i-2 > n-3$. Therefore, $i-1 < \left\lceil\frac{2n-2}{3}\right\rceil$ or $i-1 > n-1$ and $i-1 < \left\lceil\frac{2n-6}{3}\right\rceil + 1$ or $i-1 > n-2$. Therefore, $i-1 < \left\lceil\frac{2n-6}{3}\right\rceil + 1$ or $i-1 > n-1$. Therefore, $i < \left\lceil\frac{2n}{3}\right\rceil$ or $i > n$. Hence $\mathcal{C}(P_n^2, i) = \phi$.

(ii) By hypothesis, $i-1 < \left\lceil\frac{2(n-1)}{3}\right\rceil$ or $i-1 > n-1$ and $i-1 < \left\lceil\frac{2(n-2)}{3}\right\rceil$ or $i-1 > n-2$ and $i-1 < \left\lceil\frac{2(n-3)}{3}\right\rceil$ or $i-1 > n-3$. Therefore, $i-1 < \left\lceil\frac{2n-2}{3}\right\rceil$ or $i-1 > n-1$, and $i-1 < \left\lceil\frac{2n-4}{3}\right\rceil$ or $i-1 > n-2$ and $i-1 < \left\lceil\frac{2n-6}{3}\right\rceil$ or $i-1 > n-3$. Therefore, $i-1 < \left\lceil\frac{2n-6}{3}\right\rceil$ or $i-1 > n-1$. Therefore, $i < \left\lceil\frac{2n}{3}\right\rceil$ or $i > n$. Therefore, $\mathcal{C}(P_n^2, i) = \phi$.

(iii) By hypothesis, $i-1 < \left\lceil\frac{2n-2}{3}\right\rceil$ or $i-1 > n-1$ and $\left\lceil\frac{2n-6}{3}\right\rceil \leq i-2 \leq n-3$. Therefore, $\left\lceil\frac{2n-6}{3}\right\rceil + 1 \leq i-1 \leq n-2$ and $i-1 < \left\lceil\frac{2n-2}{3}\right\rceil$. Hence, $\left\lceil\frac{2n-6}{3}\right\rceil + 1 \leq i-1 < \left\lceil\frac{2n-2}{3}\right\rceil$. Therefore, $\left\lceil\frac{2n-6}{3}\right\rceil + 2 \leq i < \left\lceil\frac{2n-2}{3}\right\rceil + 1$. Therefore, $\left\lceil\frac{2n}{3}\right\rceil \leq i < \left\lceil\frac{2n-2}{3}\right\rceil + 1$. Therefore, $\mathcal{C}(P_n^2, i) \neq \phi$.

Theorem 2.5

Let P_n^2, $n \geq 2$ be the square of path P_n with n vertices. Suppose that $\mathcal{C}(P_n^2, i) \neq \phi$. Then we have

(i) $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$ and $\mathcal{C}(P_{n-2}^2, i-2) = \phi$ if and only if $n = 3k-1$ and $i = 2k-1$.

(ii) $\mathcal{C}(P_{n-2}^2, i-1) = \mathcal{C}(P_{n-3}^2, i-2) = \phi$ and $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$ if and only if $i = n$.

(iii) $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$, $\mathcal{C}(P_{n-2}^2, i-1) \neq \phi$ and $\mathcal{C}(P_{n-3}^2, i-1) = \phi$ if and only if $i = n-1$.

(iv) $\mathcal{C}(P_{n-1}^2, i-1) = \phi$, $\mathcal{C}(P_{n-2}^2, i-1) \neq \phi$ and $\mathcal{C}(P_{n-3}^2, i-1) \neq \phi$ if and only if $n = 3k+1$ and $i = 2k$ for some k.

(v) $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$, $\mathcal{C}(P_{n-2}^2, i-1) \neq \phi$, and $\mathcal{C}(P_{n-3}^2, i-1) \neq \phi$ if and only if $\left\lceil\frac{2(n-1)}{3}\right\rceil + 1 \leq i \leq n-2$

Proof

(i) Assume $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$ and $(P_{n-2}^2, i-2) = \phi$. Since $\mathcal{C}(P_{n-2}^2, i-2) = \phi$, we have $i-2 > n-2$ (or) $i-2 < \left\lceil\frac{2(n-2)}{3}\right\rceil$. Suppose $i-2 > n-2$, then $i > n$. Therefore, $\mathcal{C}(P_n^2, i) = \phi$, which is a contradiction.

If $i-2 < \left\lceil\frac{2(n-2)}{3}\right\rceil$ then $i < \left\lceil\frac{2n-4}{3}\right\rceil + 2$ (1)

Also, $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$. Therefore, $\left\lceil\frac{2(n-1)}{3}\right\rceil \leq i-1 \leq n-1$. That is $\left\lceil\frac{2n-2}{3}\right\rceil \leq i-1 \leq n-1$. Therefore, $\left\lceil\frac{2n-2}{3}\right\rceil + 1 \leq i \leq n$ (2)

From (1) and (2) we have $\left\lceil\frac{2n-2}{3}\right\rceil + 1 \leq i < \left\lceil\frac{2n-4}{3}\right\rceil + 2$ (3).

If $n \neq 3k-1$, then from (3), we obtain an inequality of the form $s \leq i < s$, which is not possible. when $n = 3k-1$, (3), holds and in this case we obtain $i = 2k-1$. Conversely, assume $n = 3k-1$ and $i = 2k-1$ then $2n = 6k-2$, $2n-4 = 6k-6$ and $\frac{2n-4}{3} = 2k-2$. Therefore, $\frac{2(n-2)}{3} = 2k-2$ (1)

Now $i = 2k-1$, $i-2 = 2k-3 < 2k-2 = \frac{2(n-2)}{3}$. Therefore, $i-2 < \frac{2(n-2)}{3}$. Therefore, $\mathcal{C}(P_{n-2}^2, i-2) = \phi$. Also $\left\lceil\frac{2(3k-2)}{3}\right\rceil \leq 2k-2 \leq 3k-2$. That is $\left\lceil\frac{2(n-1)}{3}\right\rceil \leq i-1 \leq n-1$. Hence $\mathcal{C}(P_{n-1}^2, i-1) > 0$ and hence, $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$.

(ii) Since $\mathcal{C}(P_{n-2}^2, i-1) = \phi = \mathcal{C}(P_{n-3}^2, i-2)$, we have $i-1 > n-2$ or i – 1 <$\left\lceil\frac{2(n-2)}{3}\right\rceil$and i - 2 < $\left\lceil\frac{2(n-3)}{3}\right\rceil$ or i – 2 > n – 3. Therefore, i -1 > n- 2 or i –1 < $\left\lceil\frac{2n-4}{3}\right\rceil$ and i-1 < $\left\lceil\frac{2n-6}{3}\right\rceil$ + 1 or i – 1 > n- 2. Therefore, i – 1 < $\left\lceil\frac{2n-4}{3}\right\rceil$ or i -1 > n-2. Suppose i – 1 < $\left\lceil\frac{2n-4}{3}\right\rceil$ then i - 1 < $\left\lceil\frac{2n-2}{3}\right\rceil$. Therefore,$\mathcal{C}(P_{n-1}^2, i-1) = \phi,$ which is a contradiction .

Therefore, i-1 > n- 2 , that is i > n-1 which implies that i $\geq n$.

Also, since $\mathcal{C}(P_n^2, i) \neq \phi$ i ≤ n. Combining these we get i= n .

Conversely if i= n then $\mathcal{C}(P_{n-2}^2, i-1) = \mathcal{C}(P_{n-2}^2, n-1) = \phi$.

$\mathcal{C}(P_{n-3}^2, i-2) = \mathcal{C}(P_{n-3}^2, n-2) = \phi$, and $\mathcal{C}(P_{n-1}^2, i-1) = \mathcal{C}(P_{n-1}^2, n-1) \neq \phi$.

(iii) Since $\mathcal{C}(P_{n-3}^2, i-1) = \phi$,we have i-1 > n- 3 or i-1 < $\left\lceil\frac{2(n-3)}{3}\right\rceil$ (1)

Since $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$,we have $\left\lceil\frac{2(n-1)}{3}\right\rceil \leq i-1 \leq n-1$ that is

$\left\lceil\frac{2n-2}{3}\right\rceil \leq i-1 \leq n-1$ (2)

Suppose i -1 <$\left\lceil\frac{2(n-3)}{3}\right\rceil$, then (2) does not hold. Therefore, our assumption is wrong .

Therefore, i-1 > n-3. Also,since $\mathcal{C}(P_{n-2}^2, i-1) \neq \phi$, $\left\lceil\frac{2(n-2)}{3}\right\rceil \leq i-1 \leq n-2.$

But i-1 > n – 3. Therefore, i-1 ≥n-2 (3)

From (2) and (3), we get i -1 = n – 2.Therefore, i= n-1 .

Conversely, if i = n-1 ,then $\mathcal{C}(P_{n-1}^2, i-1) = \mathcal{C}(P_{n-1}^2, n-2) \neq \phi$.

And $\mathcal{C}(P_{n-2}^2, i-1) = \mathcal{C}(P_{n-2}^2, n-2) \neq \phi$and

$\mathcal{C}(P_{n-3}^2, i-1) = \mathcal{C}(P_{n-3}^2, n-2),= \phi$. since, n-2 > n – 3. We have $\mathcal{C}(P_{n-3}^2, n-2) = \phi$. That is $\mathcal{C}(P_{n-3}^2, i-1) = \phi$.

(iv) Since $\mathcal{C}(P_{n-1}^2, i-1) = \phi$, by lemma 2.3 i-1> n-1 (or) i -1 < $\left\lceil\frac{2(n-1)}{3}\right\rceil$.

If i -1 > n-1 then i -1 > n – 2. Therefore, $\mathcal{C}(P_{n-2}^2, i-1) = \phi$ and $\mathcal{C}(P_{n-3}^2, i-1) = \phi$ which is a contradiction .

Therefore i-1 <$\left\lceil\frac{2(n-2)}{3}\right\rceil$ (1)

Since $\mathcal{C}(P_{n-2}^2, i-1) \neq \phi,$ we have $\left\lceil\frac{2(n-2)}{3}\right\rceil \leq$ i -1 ≤ n-2 (2)

and since $\mathcal{C}(P_{n-3}^2, i-1) \neq \phi$, we have $\left\lceil\frac{2(n-3)}{3}\right\rceil \leq$ i-1 ≤ n-3 (3).

Since $\mathcal{C}(P_n^2, i) \neq \phi$,$\left\lceil\frac{2n}{3}\right\rceil \leq i \leq$ n – 1,$\left\lceil\frac{2n}{3}\right\rceil - 1 \leq$ i–1 ≤ n-2 (4)

By combining all the above in equalities, we have $\left\lceil\frac{2n}{3}\right\rceil - 1 \leq$ i–1 < $\left\lceil\frac{2n-2}{3}\right\rceil$ (5)

When n ≠3k+1,we get an inequality of the form s≤i – 1 < s which is not possible.When n=3k+1, we have s≤ i-1 < s+1.Therefore, (5) holdshood. In this case i = 2k .

Conversely, assume n = 3k+1 and i = 2k. Therefore, n – 1 = 3k and i – 1 = 2k-1 ,

2k – 1 < 2k = $\frac{2(n-1)}{3}$, Therefore, i – 1 < $\left\lceil\frac{2(n-1)}{3}\right\rceil$, that is $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$. Also,

$\frac{2(3k-1)}{3}$ ≤ 2k-1 ≤ 3k-1. Therefore, $\left\lceil\frac{2(n-2)}{3}\right\rceil \leq$ i -1 ≤ n – 2.Therefore,

$\mathcal{C}(P_{n-2}^2, i-1) \neq \phi$. Also $\left\lceil\frac{2(3k-2)}{3}\right\rceil \leq$ 2k-1 ≤ 3k-2. That is $\left\lceil\frac{2(n-3)}{3}\right\rceil \leq$ i -1 ≤n– 3,

which implies $\mathcal{C}(P_{n-3}^2, i-1) \neq \phi$.

(v) Assume $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$, $\mathcal{C}(P_{n-2}^2, i-1) \neq \phi$ and $\mathcal{C}(\mathrm{P}_{n-3}^2, i-1) \neq \phi$. Then we have$\left\lceil\frac{2(n-1)}{3}\right\rceil \leq$ i - 1 ≤ n-1 and $\left\lceil\frac{2(n-2)}{3}\right\rceil \leq$ i -1 ≤ n-2 and $\left\lceil\frac{2(n-3)}{3}\right\rceil \leq$ i -1 ≤ n-3. Also, since $\mathcal{C}(P_n^2, i) \neq \phi$, we have$\left\lceil\frac{2n}{3}\right\rceil$ -1 ≤ i-1 ≤n-1 . Therefore, $\left\lceil\frac{2(n-1)}{3}\right\rceil$ +1 ≤ i ≤ n - 2 . Conversely, suppose$\left\lceil\frac{2(n-1)}{3}\right\rceil$ +1 ≤ i ≤n-2. Therefore, $\left\lceil\frac{2(n-1)}{3}\right\rceil \leq$ i-1 ≤ n-3 and $\left\lceil\frac{2(n-2)}{3}\right\rceil \leq$ i-1 ≤ n-2,$\left\lceil\frac{2(n-3)}{3}\right\rceil \leq$ i-1 ≤ n-3 and $\left\lceil\frac{2(n-1)}{3}\right\rceil \leq$ i-1 ≤ n-1.

From these, we obtain $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$ and $\mathcal{C}(\mathrm{P}_{n-2}^2, i-1) \neq \phi$ and $\mathcal{C}(\mathrm{P}_{n-3}^2, i-1) \neq \phi$.

Theorem 2.6

For every n ≥ 3 and i > $\left\lceil\frac{2n}{3}\right\rceil$, we have

(i) If $\mathcal{C}(\mathrm{P}_{n-1}^2, i-1) \neq \phi$ and $\mathcal{C}(\mathrm{P}_{n-3}^2, i-2) = \phi$, then $\mathcal{C}(\mathrm{P}_n^2, i)$= {{1,2,3…………n}}.

(ii) If $(P_{n-1}^2, i-1) = \phi$ and $\mathcal{C}(\mathrm{P}_{n-3}^2, i-2) \neq \phi$, Then $\mathcal{C}(\mathrm{P}_n^2, i)$= {X U {n-1, n-2}/ X ∈ $\mathcal{C}(\mathrm{P}_{n-3}^2, i-2)$}

(iii) If $\mathcal{C}(\mathrm{P}_{n-1}^2, i-1) \neq \phi$ and $\mathcal{C}(\mathrm{P}_{n-3}^2, i-2) \neq \phi$ then

$\mathcal{C}(P_n^2, i)$ = { X U {n-1, n–2} / X ∈ $\mathcal{C}(P_{n-3}^2, i-2)$} ∪ $\{Y \cup \{n\}$/Y∈ $\mathcal{C}(P_{n-1}^2, i-1)\}$

Proof

(i) Since $\mathcal{C}(P_{n-3}^2, i-2) = \phi$ and $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$, by lemma 2.5 (ii) i =n.
Therefore, $\mathcal{C}(P_n^2, i)$ =$\mathcal{C}(P_n^2, n)$ ={1,2,3…………n}.

(ii) Let $\mathcal{C}(P_{n-1}^2, i-1) = \phi$. Let X ∈ $\mathcal{C}(P_{n-3}^2, i-2)$. Then X = X_1 U {n,n-1} / X_1 ∈ ($\mathcal{C}(P_{n-1}^2, i)$.
Therefore X U {n-1,n-2}∈ $\mathcal{C}(P_n^2, i)$ (1).
Therefore {X U {n-1,n-2} / X ∈ $\mathcal{C}(P_{n-3}^2, i-2) \subseteq \mathcal{C}(P_n^2, i)$.
Conversely, assume X∈ $\mathcal{C}(P_n^2, i)$. Then, X is a vertex covering set of P_n^2 with cardinality i. Elements of $\mathcal{C}(P_n^2, i)$ end with n-1 , n or n-1 , n-2 or n-2, n. Suppose it ends with n-1,n. Then, X-{n} ∈ $\mathcal{C}(P_{n-1}^2, i-1)$,which is a contradiction. Suppose it ends with n-2 , n then X-{n} ∈ $\mathcal{C}(P_{n-1}^2, i-1)$, which is also a contradiction. Therefore, the only possibility is that it ends with n-1, n-2. Therefore, we can write X= X_1 U {n-1, n-2} where $X_1 \in \mathcal{C}(P_{n-3}^2, i-2)$. This implies $\mathcal{C}(P_n^2, i) \subseteq$ {XU {n-1, n-2} / X∈ $\mathcal{C}(P_{n-3}^2, i-2)$}. Therefore, $\mathcal{C}(P_n^2, i)$ = {X U {n-1, n-2} / X ∈ $\mathcal{C}(P_{n-3}^2, i-2)$ }.

(iii) The construction of $\mathcal{C}(P_n^2, i)$from $\mathcal{C}(P_{n-1}^2, i-1)$ } and $\mathcal{C}(P_{n-3}^2, i-2)$ is as follows. Let X be an vertex covering set of P_{n-3}^2 with cardinality i-2 . All the elements of $\mathcal{C}(P_{n-3}^2, i-2)$ end with n-3 or n-4. Now adjoin n-1 and n-2 with X. Then X U {n-1,n-2} is a vertex covering set of $\mathcal{C}(P_n^2, i)$. Therefore, {X U {n-1, n-2}/ X∈ $\mathcal{C}(P_{n-3}^2, i-2)\} \subset \mathcal{C}(P_n^2, i)$.
Now, let us consider $\mathcal{C}(P_{n-1}^2, i-1)$. All the elements of $\mathcal{C}(P_{n-1}^2, i-1)$ end with n-1 or n-2. Let Y be a vertex covering set of P_{n-1}^2 with cardinality i-1. Now adjoin {n} with Y. Then Y ∪ {n} ∈ $\mathcal{C}(P_n^2, i)$. Therefore, $C(P_n^2, i) \subseteq$ { X ∪ {n-1,n-2}/
X∈ $\mathcal{C}(P_{n-3}^2, i-2)$}∪{Y ∪{n}/Y ∈ $\mathcal{C}(P_{n-1}^2, i-1)$} (1)
Conversely, let us assume $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$ and $\mathcal{C}(P_{n-3}^2, i-2) \neq \phi$.
Let X ∈ $\mathcal{C}(P_{n-3}^2, i-2)$. Then n-3, n-4 or n-3, n-5 or n-4,n-5 is in X.
If n-3 , n-4, or n-3 , n-5 or n-4 , n-5 ∈ X, then X ∪{n-1,n-2} ∈ $\mathcal{C}(P_n^2, i)$.
Let $Y \in \mathcal{C}(P_{n-1}^2, i-1)$, then atleast one vertex labeled n-1, n-3 or n-2 , n-3 is in Y .
If n-1 , n-3 or n-2 , n-3 ∈Y,then Y∪{n} ∈ $\mathcal{C}(P_n^2, i)$. Thus we have
{X∪ {n-1, n-2} / X ∈ $\mathcal{C}(P_{n-3}^2, i-2)$} ∪{Y ∪{n}/ Y ∈ $\mathcal{C}(P_{n-1}^2, i-1)\} \subseteq \mathcal{C}(P_n^2, i)$ (2)
Combining (1) and (2), we get
$\mathcal{C}(P_n^2, i)$ = { X ∪{n-1,n-2} / X ∈ $\mathcal{C}(P_{n-3}^2, i-2)$} ∪{Y ∪{n}/Y ∈ $\mathcal{C}(P_{n-1}^2, i-1)$}

Theorem 2.7

If $\mathcal{C}(P_n^2, i)$ is the family of vertex covering sets of P_n^2 with cardinality i, where i > $\left\lfloor \frac{2n}{3} \right\rfloor$, then $|\mathcal{C}(P_n^2, i)| = |\mathcal{C}(P_{n-1}^2, i-1)| + |\mathcal{C}(P_{n-3}^2, i-2)|$.

Proof

From theorem 2.6, we consider the following three cases where i≥ $\left\lfloor \frac{2n}{3} \right\rfloor$. If $\mathcal{C}(P_{n-1}^2, i-1) = \phi$ and $\mathcal{C}(P_{n-3}^2, i-2) = \phi$. Then $\mathcal{C}(P_n^2, i) = \Phi$.

(i) If $\mathcal{C}(P_{n-1}^2, i-1) = \phi$ and $\mathcal{C}(P_{n-3}^2, i-2) \neq \phi$ then $\mathcal{C}(P_n^2, i)$ = { X U {n-1, n-2}/ X∈ $\mathcal{C}(P_{n-3}^2, i-2)$}.

(ii) If $\mathcal{C}(P_{n-1}^2, i-1) \neq \phi$ and $\mathcal{C}(P_{n-3}^2, i-2) \neq \phi$, then
$\mathcal{C}(P_n^2, i)$ = { X U {n-1, n-2}/ X∈ $\mathcal{C}(P_{n-3}^2, i-2$)} ∪ $\{Y \cup \{n\}$/Y∈ $\mathcal{C}(P_{n-1}^2, i-1)\}$
From the above construction, in each case , we obtain that
$|\mathcal{C}(P_n^2, i)| = |\mathcal{C}(P_{n-1}^2, i-1)| + |\mathcal{C}(P_{n-3}^2, i-2)|$.

III. Vertex covering polynomial of P_n^2

Let C(P_n^2,x) = $\sum_{i=\left\lfloor \frac{2n}{3} \right\rfloor}^{n} c(P_n^2, i)x^i$ be the vertex covering polynomial of path P_n^2 . In this section, we derive the expression for C(P_n^2,x) .

Theorem 3.1

For every n ≥ 4,C(P_n^2,x) = xC(P_{n-1}^2,x) + x^2 C(P_{n-3}^2,x) with initial values $C(P_2^2, x) = x$, C(P_3^2 ,x) = $3x^2+x^3$.

Proof

We have $c(P_n^2, i) = c(P_{n-1}^2, i-1) + c(P_{n-3}^2, i-2)$.
Therefore,$c(P_n^2, i)x^i = c(P_{n-1}^2, i-1)x^i + c(P_{n-3}^2, i-2)x^i$.
$\sum c(P_n^2, i)x^i = \sum c(P_{n-1}^2, i-1)x^i + \sum c(P_{n-3}^2, i-2)x^i$.

$\sum c(P_n^2, i)x^i = x\sum c(P_{n-1}^2, i-1)x^{i-1} + x^2\sum c(P_{n-3}^2, i-2)x^{i-2}$.
$C(P_n^2, x) = x\, C(P_{n-1}^2, x) + x^2\, C(P_{n-3}^2, x)$, With initial values $C(P_2^2, x) = x$, $C(P_3^2, x) = 3x^2 + x^3$
We obtain $c(P_n^2, i)$ for $1 \le n \le 15$ as shown in the table 1.

Table 1

$c(P_n^2, i)$ the numbers of vertex covering set of P_n^2 with cardinality i.

P_n^2	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	3	1													
4	1	4	1												
5	0	3	5	1											
6	0	0	6	6	1										
7	0	0	1	10	7	1									
8	0	0	0	4	15	8	1								
9	0	0	0	0	10	21	9	1							
10	0	0	0	0	1	20	28	10	1						
11	0	0	0	0	0	5	35	36	11	1					
12	0	0	0	0	0	0	15	56	45	12	1				
13	0	0	0	0	0	0	1	35	84	55	13	1			
14	0	0	0	0	0	0	0	6	70	120	66	14	1		
15	0	0	0	0	0	0	0	0	21	126	165	78	15	1	
16	0	0	0	0	0	0	0	0	1	56	210	220	91	16	1

In the following theorem, we obtain some properties of $c(P_n^2, i)$

Theorem 3.2

The following properties hold for the coefficients of $c(P_n^2, x)$:

I. $c(P_n^2, n) = 1$ for every $n \in N$.
II. $c(P_n^2, n-1) = n$ for every $n \in N$.
III. $c(P_n^2, n-2) = \frac{1}{2}[n^2 - 5n + 6]$, $n \ge 4$.
IV. $c(P_n^2, n-3) = (n-4)C_3$, for every $n \ge 7$.
V. $c(P_{3n+1}^2, 2n) = 1$, for every $n \in N$.
VI. $c(P_{3n-1}^2, 2n-1) = n+1$, for $n \ge 2$.
VII. $c(P_{3n+2}^2, 2n+1) = n+2$, for every $n \in N$.

Proof

(i) Since for any graph with n vertices $\mathcal{C}(G, n) = 1$, we have $c(P_n^2, n) = 1$.

(ii) Since $\mathcal{C}(P_n^2, n-1) = \{[n] - \{x\} / x \in [n]\}$, we have $c(P_n^2, n-1) = n$.

(iii) To prove $c(P_n^2, n-2) = \frac{1}{2}[n^2 - 5n + 6]$.

We apply induction on n. When n = 4.
L.H.S = $c(P_n^2, n-2) = c(p_4^2, 4-2) = c(p_4^2, 2) = 1$ (From the table)
and R.H.S= $\frac{1}{2}[n^2 - 5n + 6] = \frac{1}{2}[16 - 20 + 6] = 1$

Therefore, the result is true for n=4. Now suppose that the result is true for all numbers less than 'n' and we prove it for n. By theorem 3.1, we have

$c(P_n^2, n-2) = c(P_{n-1}^2, n-3) + c(P_{n-3}^2, n-4)$
$= \frac{1}{2}[(n-1)^2 - 5(n-1) + 6]$ + n-3
$= \frac{1}{2}[n^2 + 1 - 2n - 5n + 5 + 6 + 2n - 6]$
$= \frac{1}{2}[n^2 - 5n + 6]$. Hence, the result is true for all n.

(iv) To prove $c(P_n^2, n-3) = (n-4)\, C_3$ for $n \ge 7$.

We apply induction on n. when n=7,

L.H.S = c(P_n^2, n-3)= c(P_7^2,7-3) = c(P_7^2,4-2) =1 (From the table) and
R.H.S = (n-4) C_3=(7-4)C_3=3C_3 =1. Hence, the result is true for n=7. Now suppose that the result is true for all numbers less than 'n' . Therefore, c(P_m^2,m-3)=(m-4)C_3, $7 \leq m \leq$n-1.
Now to prove the result is true for n.
From Theorem 2.7, c(P_n^2,n-3) = c(P_{n-1}^2,n-4)+ c(P_{n-3}^2,n-5) = (n-5)C_3+ c(P_{n-3}^2,n-5)

$$= \frac{(n-5)(n-6)(n-7)}{1*2*3} + \frac{1}{2}[(n-3)^2 - 5(n-3) + 6]$$

$$= \frac{n^3 - 15n^2 + 74n - 120}{6}$$

c(P_n^2,n-3) = (n-4) C_3. Therefore, the result is true for all n.
(v)To prove c(P_{3n+1}^2,2n) =1 ,for every n∈ N. We apply induction on n.
Suppose n=1. c(P_{3n+1}^2,2n) = c(P_4^2 ,2) =1 (From the table). Assume the result true for all natures numbers less than n . c(P_{3m+1}^2,2m) =1 for all m less than n. Now we prove that the result is true for n .
c(P_{3n+1}^2,2n) = c(P_{3n}^2,2n-1)+ c(P_{3n-2}^2,2n-2) = c(P_{3n}^2,2n-1)+ c($P_{3(n-1)+1}^2$,2(n-1))=0+1=1
Therefore,c(P_{3n+1}^2,2n) = 1 for all n ∈N.
(vi) To prove c(P_{3n-1}^2,2n-1) =n+1 ,n≥2 . we apply induction on n.
Put n=2 .L.H.S = c(P_{3n-1}^2,2n-1) = c(P_5^2,3) =3 =2+1=n+1 (R.H.S). Hence the result is true for all natural numbers less than n .c (P_{3m-1}^2,2m-1) = m+1, m<n . We prove that the result is true for n.
c(P_{3n-1}^2,2n-1) = c(P_{3n-2}^2,2n-2)+ c(P_{3n-4}^2,2n-3) = c($P_{3(n-1)+1}^2$,2(n-1) *)* +c(P_{3n-4}^2,2n-3)
= 1+ c($P_{3(n-1)-1}^2$,2(n-1)-1)
=1+(n-1)+1=n+1. Hence, the result is true for n .
Therefore , c(P_{3n-1}^2,2n-1)=n+1 for all n≥ 2.
(vii) To prove c (P_{3n+2}^2,2n+1) =n+2 , n≥2 . We apply induction on n.
Put n=1.L.H.S = c(P_{3n+2}^2,2n+1) = c(P_5^2,3) =3 =1+2=n+2 (R.H.S)
Hence, the result is true for all natural numbers less than n.
Therefore, c(P_{3m+2}^2,2m+1) =m+2, m<n . To prove that the result is true for n .
c(P_{3n+2}^2,2n+1) = c(P_{3n+1}^2,2n)+ c(P_{3n-1}^2,2n-1) =1+n+1=n+2
Hence, the result is true for n . Therefore, c(P_{3n+2}^2,2n+1)=n+2 for all n∈N.

IV. Conclusion

In this paper the vertex cover polynomial of square of path has been derived by identifying its vertex covering sets. It also helps us to characterize the vertex covering sets and to find the number of vertex covering sets of cardinality i . We can generalize this study to any power of the path and some interesting properties can be obtained via the roots of the vertex cover polynomial of P_n^k .

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